CHAPTER 1

Basics of Geometry

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Reasoning and Proof

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Perpendicular and Parallel Lines

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Congruent Triangles

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CHAPTER 5

Properties of Triangles

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CHAPTER 6

Quadrilaterals

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Transformations

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8

Similarity

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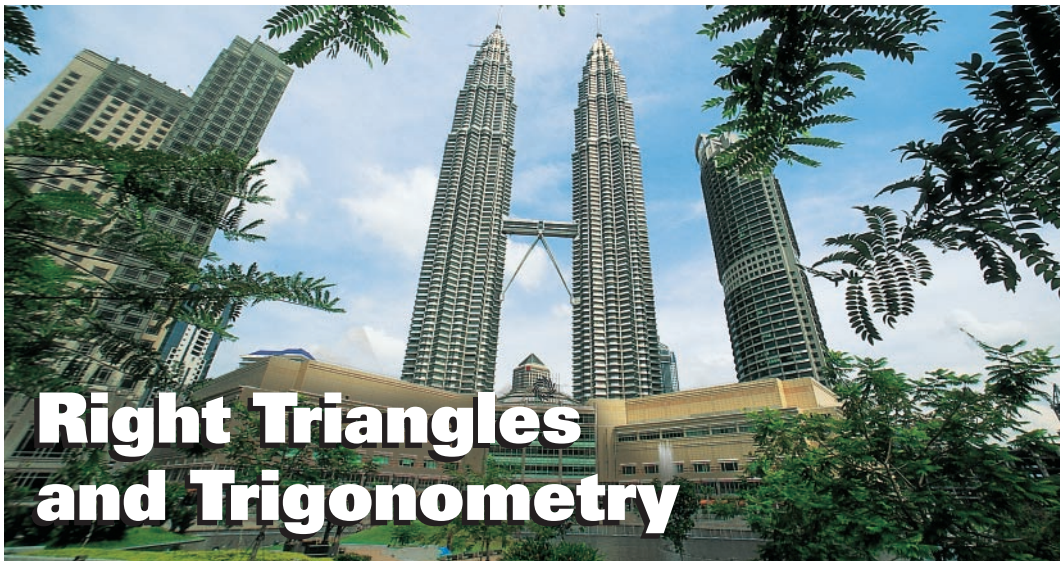
CHAPTER 9



Right Triangles and Trigonometry

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Circles

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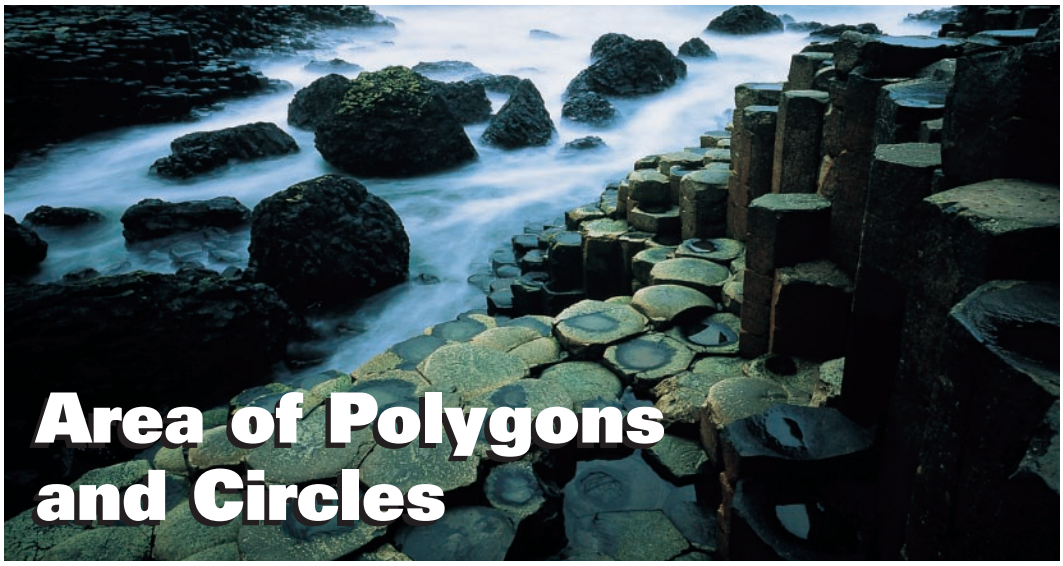
CHAPTER 11



Area of Polygons and Circles

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CHAPTER 12

Surface Area and Volume

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READING, WRITING, AND NOTETAKING

Student Resources

1.1 Patterns and Inductive Reasoning

What you should learn

GOAL 1 Find and describe patterns.

GOAL 2 Use inductive reasoning to make **real-life** conjectures, as in **Ex. 42**.

Why you should learn it

▼ To make predictions based on observations, such as predicting full moons in **Example 6**.

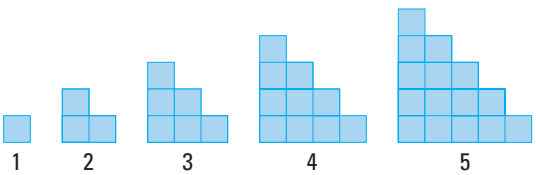
REAL LIFE

GOAL 1 FINDING AND DESCRIBING PATTERNS

Geometry, like much of mathematics and science, developed when people began recognizing and describing patterns. In this course, you will study many amazing patterns that were discovered by people throughout history and all around the world. You will also learn to recognize and describe patterns of your own. Sometimes, patterns allow you to make accurate predictions.

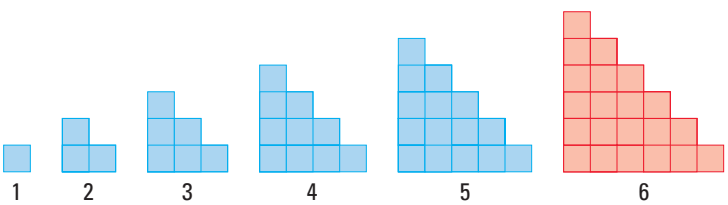
EXAMPLE 1 *Describing a Visual Pattern*

Sketch the next figure in the pattern.



SOLUTION

Each figure in the pattern looks like the previous figure with another row of squares added to the bottom. Each figure looks like a stairway.



▶ The sixth figure in the pattern has six squares in the bottom row.

EXAMPLE 2 *Describing a Number Pattern*

Describe a pattern in the sequence of numbers. Predict the next number.

a. 1, 4, 16, 64, . . . **b.** −5, −2, 4, 13, . . .

SOLUTION

a. Each number is four times the previous number. The next number is 256.

b. You add 3 to get the second number, then add 6 to get the third number, then add 9 to get the fourth number. To find the fifth number, add the next multiple of 3, which is 12.

▶ So, the next number is 13 + 12, or 25.

GOAL 2 USING INDUCTIVE REASONING

Much of the reasoning in geometry consists of three stages.

1. **Look for a Pattern** Look at several examples. Use diagrams and tables to help discover a pattern.
2. **Make a Conjecture** Use the examples to make a general *conjecture*. A **conjecture** is an unproven statement that is based on observations. Discuss the conjecture with others. Modify the conjecture, if necessary.
3. **Verify the Conjecture** Use logical reasoning to verify that the conjecture is true in all cases. (You will do this in Chapter 2 and throughout this book.)

Looking for patterns and making conjectures is part of a process called **inductive reasoning**.

EXAMPLE 3 *Making a Conjecture*

Logical Reasoning

Complete the conjecture.

Conjecture: The sum of the first n odd positive integers is __?__.

SOLUTION

List some specific examples and look for a pattern.

Examples:

first odd positive integer:	$1 = \mathbf{1}^2$
sum of first **two** odd positive integers:	$1 + 3 = 4 = \mathbf{2}^2$
sum of first **three** odd positive integers:	$1 + 3 + 5 = 9 = \mathbf{3}^2$
sum of first **four** odd positive integers:	$1 + 3 + 5 + 7 = 16 = \mathbf{4}^2$

Conjecture: The sum of the first $\boldsymbol{n}$ odd positive integers is $\boldsymbol{n}^2$.

. . . . . .

To prove that a conjecture is true, you need to prove it is true in *all* cases. To prove that a conjecture is false, you need to provide a single *counterexample*. A **counterexample** is an example that shows a conjecture is false.

EXAMPLE 4 *Finding a Counterexample*

Show the conjecture is false by finding a counterexample.

Conjecture: For all real numbers x, the expression x^2 is greater than or equal to x.

SOLUTION

The conjecture is false. Here is a counterexample: $(0.5)^2 = 0.25$, and 0.25 is *not* greater than or equal to 0.5. In fact, any number between 0 and 1 is a counterexample.

Not every conjecture is known to be true or false. Conjectures that are not known to be true or false are called *unproven* or *undecided*.

EXAMPLE 5 *Examining an Unproven Conjecture*

In the early 1700s a Prussian mathematician named Goldbach noticed that many even numbers greater than 2 can be written as the sum of two primes.

Specific Cases:

$4 = 2 + 2$	$10 = 3 + 7$	$16 = 3 + 13$
$6 = 3 + 3$	$12 = 5 + 7$	$18 = 5 + 13$
$8 = 3 + 5$	$14 = 3 + 11$	$20 = 3 + 17$

Conjecture: Every even number greater than 2 can be written as the sum of two primes.

This is called *Goldbach's Conjecture*. No one has ever proved that this conjecture is true or found a counterexample to show that it is false. As of the writing of this book, it is unknown whether this conjecture is true or false. It is known, however, that all even numbers up to 4×10^{14} confirm Goldbach's Conjecture.

EXAMPLE 6 *Using Inductive Reasoning in Real Life*

FOCUS ON APPLICATIONS

REAL LIFE **CALENDARS** The earliest calendars were based on lunar or seasonal patterns. The ancient Egyptians were the first to introduce a calendar based on the solar year.

MOON CYCLES A full moon occurs when the moon is on the opposite side of Earth from the sun. During a full moon, the moon appears as a complete circle.

Use inductive reasoning and the information below to make a conjecture about how often a full moon occurs.

Specific Cases: In 2005, the first six full moons occur on January 25, February 24, March 25, April 24, May 23, and June 22.

SOLUTION

Conjecture: A full moon occurs every 29 or 30 days.

This conjecture is true. The moon revolves around Earth once approximately every 29.5 days.

. . . . . .

Inductive reasoning is important to the study of mathematics: you look for a pattern in specific cases and then you write a conjecture that you think describes the general case. Remember, though, that just because something is true for several specific cases does not *prove* that it is true in general.

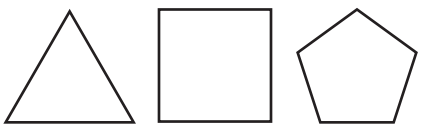
GUIDED PRACTICE

Vocabulary Check ✓ 1. Explain what a *conjecture* is.

Concept Check ✓ 2. How can you prove that a conjecture is false?

Skill Check ✓ **Sketch the next figure in the pattern.**

3.



4.

Describe a pattern in the sequence of numbers. Predict the next number.

5. 2, 6, 18, 54, . . .

6. 0, 1, 4, 9, . . .

7. 256, 64, 16, 4, . . .

8. 3, 0, −3, 0, 3, 0, . . .

9. 7.0, 7.5, 8.0, 8.5, . . .

10. 13, 7, 1, −5, . . .

11. Complete the conjecture based on the pattern you observe.

$3 + 4 + 5 = 4 \cdot 3$	$6 + 7 + 8 = 7 \cdot 3$	$9 + 10 + 11 = 10 \cdot 3$
$4 + 5 + 6 = 5 \cdot 3$	$7 + 8 + 9 = 8 \cdot 3$	$10 + 11 + 12 = 11 \cdot 3$
$5 + 6 + 7 = 6 \cdot 3$	$8 + 9 + 10 = 9 \cdot 3$	$11 + 12 + 13 = 12 \cdot 3$

Conjecture: The sum of any three consecutive integers is ___?___.

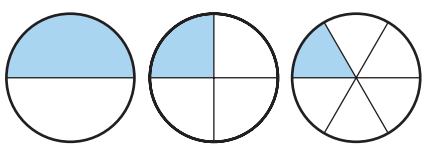
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 803.

SKETCHING VISUAL PATTERNS **Sketch the next figure in the pattern.**

12.



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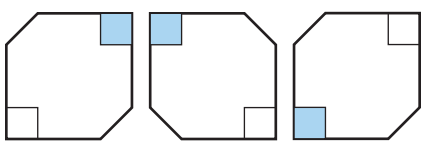

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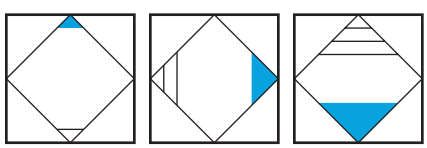

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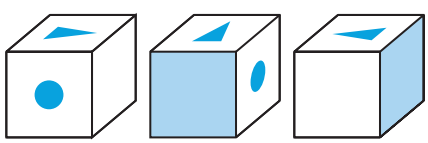


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 12–15, 24, 25
Example 2: Exs. 16–23, 26–28
Example 3: Exs. 29–33
Example 4: Exs. 34–39
Example 5: Exs. 40, 41
Example 6: Exs. 42, 43

DESCRIBING NUMBER PATTERNS **Describe a pattern in the sequence of numbers. Predict the next number.**

16. 1, 4, 7, 10, . . .

17. 10, 5, 2.5, 1.25, . . .

18. 1, 11, 121, 1331, . . .

19. 5, 0, −5, −10, . . .

20. 7, 9, 13, 19, 27, . . .

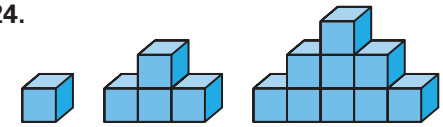
21. 1, 3, 6, 10, 15, . . .

22. 256, 16, 4, 2, . . .

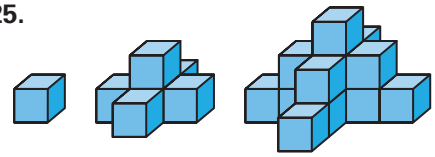
23. 1.1, 1.01, 1.001, 1.0001, . . .

VISUALIZING PATTERNS **The first three objects in a pattern are shown. How many blocks are in the next object?**

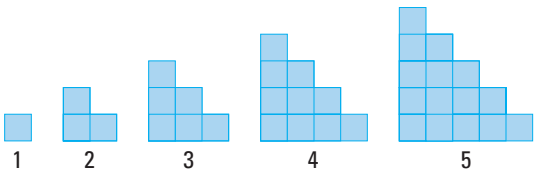
24.



25.



MAKING PREDICTIONS **In Exercises 26–28, use the pattern from Example 1 shown below. Each square is 1 unit × 1 unit.**



26. Find the distance around each figure. Organize your results in a table.

27. Use your table to describe a pattern in the distances.

28. Predict the distance around the twentieth figure in this pattern.

STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 29–31.

MAKING CONJECTURES **Complete the conjecture based on the pattern you observe in the specific cases.**

29. Conjecture: The sum of any two odd numbers is ___?___.

$1 + 1 = 2$ $\qquad$ $7 + 11 = 18$

$1 + 3 = 4$ $\qquad$ $13 + 19 = 32$

$3 + 5 = 8$ $\qquad$ $201 + 305 = 506$

30. Conjecture: The product of any two odd numbers is ___?___.

$1 \times 1 = 1$ $\qquad$ $7 \times 11 = 77$

$1 \times 3 = 3$ $\qquad$ $13 \times 19 = 247$

$3 \times 5 = 15$ $\qquad$ $201 \times 305 = 61{,}305$

31. Conjecture: The product of a number $(n - 1)$ and the number $(n + 1)$ is always equal to ___?___.

$3 \cdot 5 = 4^2 - 1$ $\qquad$ $6 \cdot 8 = 7^2 - 1$

$4 \cdot 6 = 5^2 - 1$ $\qquad$ $7 \cdot 9 = 8^2 - 1$

$5 \cdot 7 = 6^2 - 1$ $\qquad$ $8 \cdot 10 = 9^2 - 1$

CALCULATOR **Use a calculator to explore the pattern. Write a conjecture based on what you observe.**

32. $101 \times 34 =$ ___?___

$101 \times 25 =$ ___?___

$101 \times 97 =$ ___?___

$101 \times 49 =$ ___?___

33. $11 \times 11 =$ ___?___

$111 \times 111 =$ ___?___

$1111 \times 1111 =$ ___?___

$11{,}111 \times 11{,}111 =$ ___?___

FINDING COUNTEREXAMPLES **Show the conjecture is false by finding a counterexample.**

34. All prime numbers are odd.

35. The sum of two numbers is always greater than the larger number.

36. If the product of two numbers is even, then the two numbers must be even.

37. If the product of two numbers is positive, then the two numbers must both be positive.

38. The square root of a number x is always less than x.

39. If m is a nonzero integer, then $\frac{m+1}{m}$ is always greater than 1.

GOLDBACH'S CONJECTURE **In Exercises 40 and 41, use the list of the first prime numbers given below.**

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \ldots\}$$

40. Show that Goldbach's Conjecture (see page 5) is true for the even numbers from 20 to 40 by writing each even number as a sum of two primes.

41. Show that the following conjecture is not true by finding a counterexample.

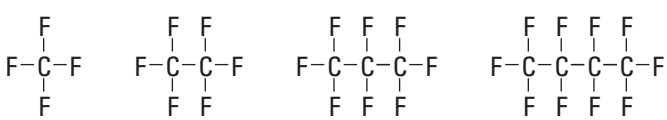
Conjecture: All *odd* numbers can be expressed as the sum of two primes.

42. **BACTERIA GROWTH** Suppose you are studying bacteria in biology class. The table shows the number of bacteria after n doubling periods.

n (periods)	0	1	2	3	4	5
Billions of bacteria	3	6	12	24	48	96

Your teacher asks you to predict the number of bacteria after 8 doubling periods. What would your prediction be?

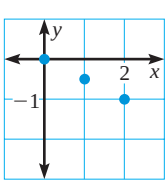
43. **SCIENCE CONNECTION** Diagrams and formulas for four molecular compounds are shown. Draw a diagram and write the formula for the next two compounds in the pattern.



CF_4 C_2F_6 C_3F_8 C_4F_{10}

USING ALGEBRA **Find a pattern in the coordinates of the points. Then use the pattern to find the y-coordinate of the point (3, ?).**

44.



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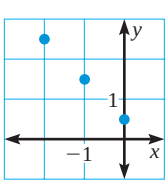

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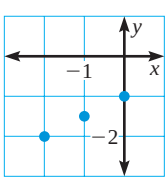

FOCUS ON CAREERS

REAL LIFE LABORATORY TECHNOLOGIST
Laboratory technologists study microscopic cells, such as bacteria. The time it takes for a population of bacteria to double (the *doubling period*) may be as short as 20 min.

INTERNET CAREER LINK
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Test Preparation

47. MULTIPLE CHOICE Which number is next in the sequence?

45, 90, 135, 180, . . .

Ⓐ 205 Ⓑ 210 Ⓒ 215 Ⓓ 220 Ⓔ 225

48. MULTIPLE CHOICE What is the next figure in the pattern?

Ⓐ 

Ⓑ 

Ⓒ 

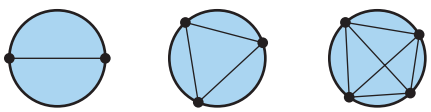
Ⓓ 

Ⓔ 

★ Challenge

DIVIDING A CIRCLE In Exercises 49–51, use the information about regions in a circle formed by connecting points on the circle.

If you draw points on a circle and then connect every pair of points, the circle is divided into a number of regions, as shown.



2 regions 4 regions ?

49. Copy and complete the table for the case of 4 and 5 points.

Number of points on circle	2	3	4	5	6
Maximum number of regions	2	4	?	?	?

50. Make a conjecture about the relationship between the number of points on the circle and number of regions in the circle.

51. Test your conjecture for the case of 6 points. What do you notice?

EXTRA CHALLENGE
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MIXED REVIEW

PLOTTING POINTS Plot in a coordinate plane. (Skills Review, p. 792, for 1.2)

52. (5, 2) **53.** (3, −8) **54.** (−4, −6) **55.** (1, −10)

56. (−2, 7) **57.** (−3, 8) **58.** (4, −1) **59.** (−2, −6)

EVALUATING EXPRESSIONS Evaluate the expression. (Skills Review, p. 786)

60. 3^2 **61.** 5^2 **62.** $(-4)^2$ **63.** -7^2

64. $3^2 + 4^2$ **65.** $5^2 + 12^2$ **66.** $(-2)^2 + 2^2$ **67.** $(-10)^2 + (-5)^2$

FINDING A PATTERN Write the next number in the sequence. (Review 1.1)

68. 1, 5, 25, 125, . . . **69.** 4.4, 40.4, 400.4, 4000.4, . . .

70. 3, 7, 11, 15, . . . **71.** −1, +1, −2, +2, −3, . . .

1.2 Points, Lines, and Planes

What you should learn

GOAL 1 Understand and use the basic undefined terms and defined terms of geometry.

GOAL 2 Sketch the intersections of lines and planes.

Why you should learn it

▼ To name and draw the basic elements of geometry, including lines that intersect, as in the perspective drawing in **Exs. 68–72**.

REAL LIFE

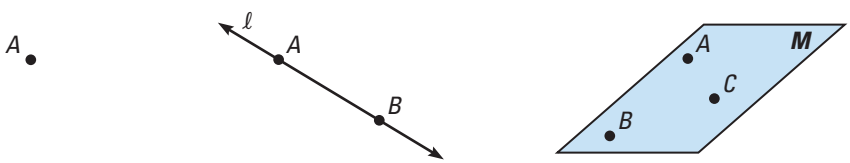
GOAL 1 USING UNDEFINED TERMS AND DEFINITIONS

A **definition** uses known words to describe a new word. In geometry, some words, such as *point*, *line*, and *plane*, are **undefined terms**. Although these words are not formally defined, it is important to have general agreement about what each word means.

A **point** has no dimension. It is usually represented by a small dot.

A **line** extends in one dimension. It is usually represented by a straight line with two arrowheads to indicate that the line extends without end in two directions. In this book, lines are always straight lines.

A **plane** extends in two dimensions. It is usually represented by a shape that looks like a tabletop or wall. You must imagine that the plane extends without end, even though the drawing of a plane appears to have edges.



Point *A* **Line ℓ or $\overleftrightarrow{AB}$** **Plane *M* or plane *ABC***

A few basic concepts in geometry must also be commonly understood without being defined. One such concept is the idea that a point *lies on* a line or a plane.

Collinear points are points that lie on the same line.

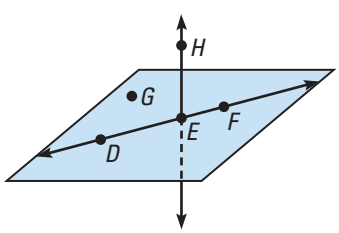
Coplanar points are points that lie on the same plane.

EXAMPLE 1 *Naming Collinear and Coplanar Points*

a. Name three points that are collinear.

b. Name four points that are coplanar.

c. Name three points that are not collinear.



SOLUTION

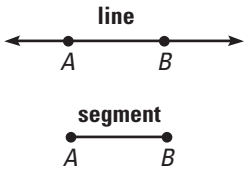
a. Points *D*, *E*, and *F* lie on the same line, so they are collinear.

b. Points *D*, *E*, *F*, and *G* lie on the same plane, so they are coplanar. Also, *D*, *E*, *F*, and *H* are coplanar, although the plane containing them is not drawn.

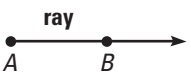
c. There are many correct answers. For instance, points *H*, *E*, and *G* do not lie on the same line.

Another undefined concept in geometry is the idea that a point on a line *is between* two other points on the line. You can use this idea to define other important terms in geometry.

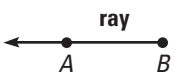
Consider the **line** AB (symbolized by $\overleftrightarrow{AB}$).
The **line segment** or **segment** AB (symbolized by $\overline{AB}$) consists of the **endpoints** A and B, and all points on $\overleftrightarrow{AB}$ that are between A and B.



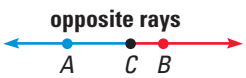
The **ray** AB (symbolized by $\overrightarrow{AB}$) consists of the **initial point** A and all points on $\overleftrightarrow{AB}$ that lie on the same side of A as point B.



Note that $\overleftrightarrow{AB}$ is the same as $\overleftrightarrow{BA}$, and $\overline{AB}$ is the same as $\overline{BA}$. However, $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are *not* the same. They have different initial points and extend in different directions.



If C is between A and B, then $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are **opposite rays**.

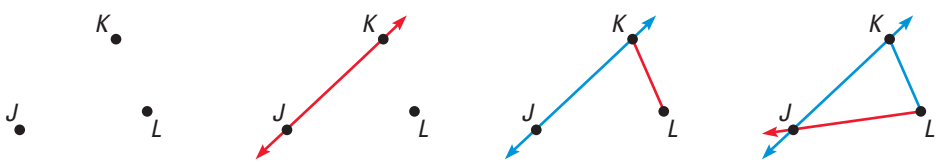


Like points, segments and rays are collinear if they lie on the same line. So, any two opposite rays are collinear. Segments, rays, and lines are coplanar if they lie on the same plane.

EXAMPLE 2 *Drawing Lines, Segments, and Rays*

Draw three noncollinear points, J, K, and L. Then draw $\overleftrightarrow{JK}$, $\overline{KL}$ and $\overrightarrow{LJ}$.

SOLUTION

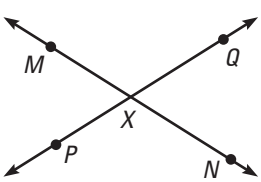


1. Draw J, K, and L.
2. Draw $\overleftrightarrow{JK}$.
3. Draw $\overline{KL}$.
4. Draw $\overrightarrow{LJ}$.

EXAMPLE 3 *Drawing Opposite Rays*

Draw two lines. Label points on the lines and name two pairs of opposite rays.

SOLUTION



Points M, N, and X are collinear and X is between M and N. So, $\overrightarrow{XM}$ and $\overrightarrow{XN}$ are opposite rays.

Points P, Q, and X are collinear and X is between P and Q. So, $\overrightarrow{XP}$ and $\overrightarrow{XQ}$ are opposite rays.

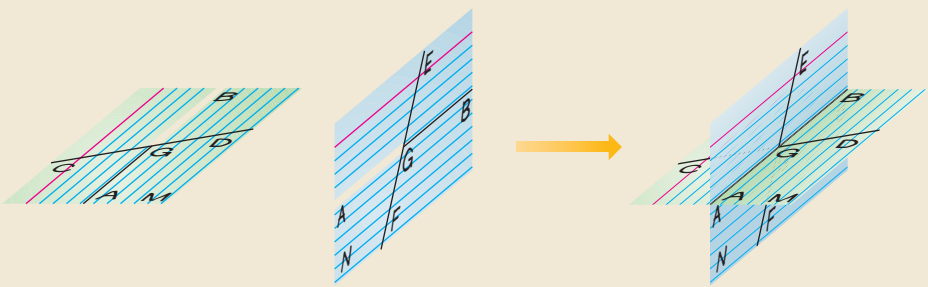
GOAL 2 SKETCHING INTERSECTIONS OF LINES AND PLANES

Two or more geometric figures **intersect** if they have one or more points in common. The **intersection** of the figures is the set of points the figures have in common.

ACTIVITY Developing Concepts

Modeling Intersections

Use two index cards. Label them as shown and cut slots halfway along each card.



1. What is the intersection of $\overline{AB}$ and $\overline{CD}$? of $\overline{AB}$ and $\overline{EF}$?
2. Slide the cards together. What is the intersection of $\overline{CD}$ and $\overline{EF}$?
3. What is the intersection of planes M and N?
4. Are $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ coplanar? Explain.

EXAMPLE 4 *Sketching Intersections*

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

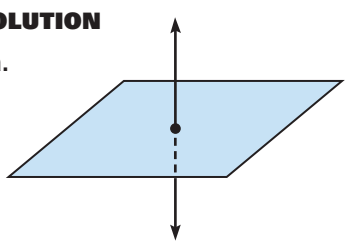
Sketch the figure described.

a. a line that intersects a plane in one point

b. two planes that intersect in a line

SOLUTION

a.

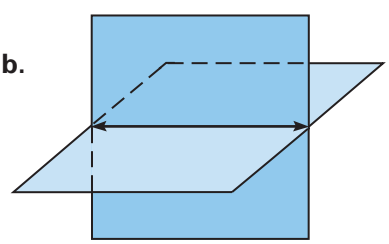


Draw a plane and a line.

Emphasize the point where they meet.

Dashes indicate where the line is hidden by the plane.

b.



Draw two planes.

Emphasize the line where they meet.

Dashes indicate where one plane is hidden by the other plane.

GUIDED PRACTICE

Vocabulary Check ✓

1. Describe what each of these symbols means: $\overline{PQ}$, $\overrightarrow{PQ}$, $\overleftrightarrow{PQ}$, $\overrightarrow{QP}$.

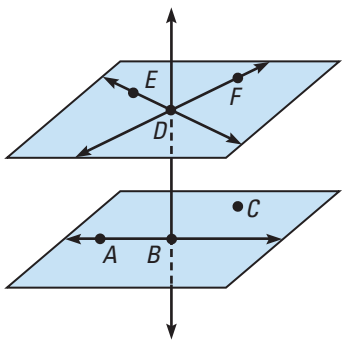
Concept Check ✓

2. Sketch a line that contains point R between points S and T. Which of the following are true?

A. $\overrightarrow{SR}$ is the same as $\overrightarrow{ST}$.
B. $\overleftrightarrow{SR}$ is the same as $\overleftrightarrow{RT}$.
C. $\overrightarrow{RS}$ is the same as $\overrightarrow{TS}$.
D. $\overrightarrow{RS}$ and $\overrightarrow{RT}$ are opposite rays.
E. $\overline{ST}$ is the same as $\overline{TS}$.
F. $\overrightarrow{ST}$ is the same as $\overrightarrow{TS}$.

Skill Check ✓

Decide whether the statement is *true* or *false*.



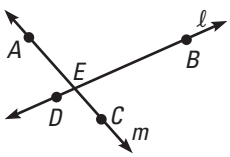
3. Points A, B, and C are collinear.
4. Points A, B, and C are coplanar.
5. Point F lies on $\overleftrightarrow{DE}$.
6. $\overleftrightarrow{DE}$ lies on plane DEF.
7. $\overleftrightarrow{BD}$ and $\overleftrightarrow{DE}$ intersect.
8. $\overleftrightarrow{BD}$ is the intersection of plane ABC and plane DEF.

PRACTICE AND APPLICATIONS

STUDENT HELP

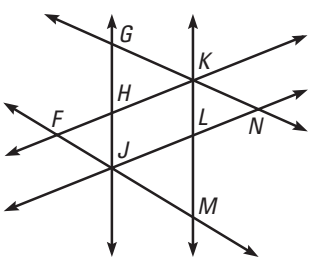
Extra Practice to help you master skills is on p. 803.

EVALUATING STATEMENTS **Decide whether the statement is *true* or *false*.**



9. Point A lies on line ℓ.
10. A, B, and C are collinear.
11. Point B lies on line ℓ.
12. A, B, and C are coplanar.
13. Point C lies on line m.
14. D, E, and B are collinear.
15. Point D lies on line m.
16. D, E, and B are coplanar.

NAMING COLLINEAR POINTS **Name a point that is collinear with the given points.**



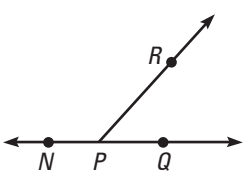
17. F and H
18. G and K
19. K and L
20. M and J
21. J and N
22. K and H
23. H and G
24. J and F

NAMING NONCOLLINEAR POINTS **Name three points in the diagram that are not collinear.**

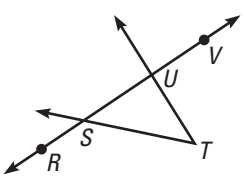
STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 9–43
Example 2: Exs. 44–49
Example 3: Exs. 50, 51
Example 4: Exs. 52–67

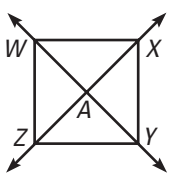
25.



26.



27.



NAMING COPLANAR POINTS **Name a point that is coplanar with the given points.**

28. A, B, and C

29. D, C, and F

30. G, A, and D

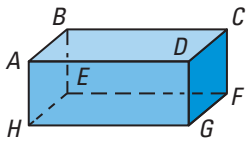
31. E, F, and G

32. A, B, and H

33. B, C, and F

34. A, B, and F

35. B, C, and G



NAMING NONCOPLANAR POINTS **Name all the points that are not coplanar with the given points.**

36. N, K, and L

37. S, P, and M

38. P, Q, and N

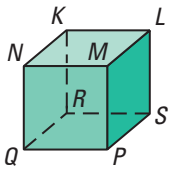
39. R, S, and L

40. P, Q, and R

41. R, K, and N

42. P, S, and K

43. Q, K, and L



COMPLETING DEFINITIONS **Complete the sentence.**

44. $\overline{AB}$ consists of the endpoints A and B and all the points on the line $\overleftrightarrow{AB}$ that lie __?__.

45. $\overrightarrow{CD}$ consists of the initial point C and all points on the line $\overleftrightarrow{CD}$ that lie __?__.

46. Two rays or segments are collinear if they __?__.

47. $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are opposite rays if __?__.

SKETCHING FIGURES **Sketch the lines, segments, and rays.**

48. Draw four points J, K, L, and M, no three of which are collinear. Then sketch $\overrightarrow{JK}$, $\overline{KL}$, $\overleftrightarrow{LM}$, and $\overrightarrow{MJ}$.

49. Draw five points P, Q, R, S, and T, no three of which are collinear. Then sketch $\overleftrightarrow{PQ}$, $\overleftrightarrow{RS}$, $\overline{QR}$, $\overline{ST}$, and $\overrightarrow{TP}$.

50. Draw two points, X and Y. Then sketch $\overleftrightarrow{XY}$. Add a point W between X and Y so that $\overrightarrow{WX}$ and $\overrightarrow{WY}$ are opposite rays.

51. Draw two points, A and B. Then sketch $\overrightarrow{AB}$. Add a point C on the ray so that B is between A and C.

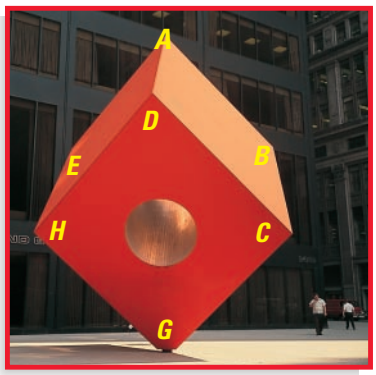
EVERYDAY INTERSECTIONS **What kind of geometric intersection does the photograph suggest?**

52.

53.

54.

COMPLETING SENTENCES **Fill in each blank with the appropriate response based on the points labeled in the photograph.**



Red Cube, by sculptor Isamu Noguchi

55. $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ intersect at __?__.

56. $\overleftrightarrow{AD}$ and $\overleftrightarrow{AE}$ intersect at __?__.

57. $\overleftrightarrow{HG}$ and $\overleftrightarrow{DH}$ intersect at __?__.

58. Plane *ABC* and plane *DCG* intersect at __?__.

59. Plane *GHD* and plane *DHE* intersect at __?__.

60. Plane *EAD* and plane *BCD* intersect at __?__.

SKETCHING FIGURES **Sketch the figure described.**

61. Three points that are coplanar but not collinear.

62. Two lines that lie in a plane but do not intersect.

63. Three lines that intersect in a point and all lie in the same plane.

64. Three lines that intersect in a point but do not all lie in the same plane.

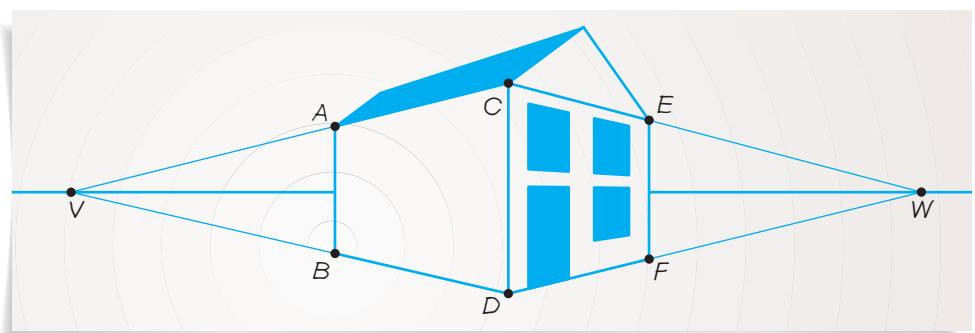
65. Two lines that intersect and another line that does not intersect either one.

66. Two planes that do not intersect.

67. Three planes that intersect in a line.

TWO-POINT PERSPECTIVE **In Exercises 68–72, use the information and diagram below.**

In *perspective drawing*, lines that do not intersect in real life are represented in a drawing by lines that appear to intersect at a point far away on the horizon. This point is called a *vanishing point*.



The diagram shows a drawing of a house with two vanishing points. You can use the vanishing points to draw the hidden parts of the house.

68. Name two lines that intersect at vanishing point *V*.

69. Name two lines that intersect at vanishing point *W*.

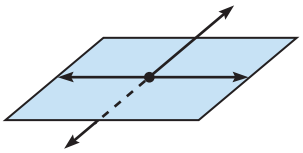
70. Trace the diagram. Draw $\overleftrightarrow{EV}$ and $\overleftrightarrow{AW}$. Label their intersection as *G*.

71. Draw $\overleftrightarrow{FV}$ and $\overleftrightarrow{BW}$. Label their intersection as *H*.

72. Draw the hidden edges of the house: $\overline{AG}$, $\overline{EG}$, $\overline{BH}$, $\overline{FH}$, and $\overline{GH}$.

Test Preparation

73. MULTIPLE CHOICE Which statement(s) are true about the two lines shown in the drawing to the right?



I. The lines intersect in one point.

II. The lines do not intersect.

III. The lines are coplanar.

(A) I only (B) I and II only (C) I and III only

(D) II and III only (E) I, II, and III

74. MULTIPLE CHOICE What is the intersection of $\overrightarrow{PQ}$ and $\overrightarrow{QP}$?

(A) $\overleftrightarrow{PQ}$ (B) $\overline{PQ}$ (C) P and Q (D) P only (E) Q only

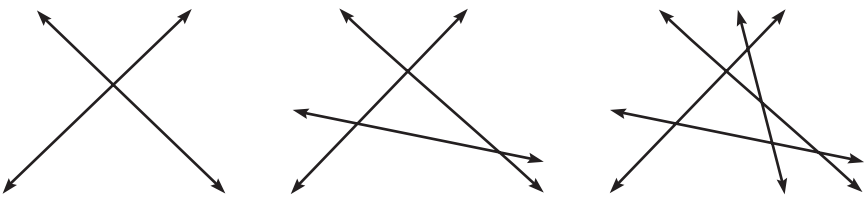
75. MULTIPLE CHOICE Points K, L, M, and N are not coplanar. What is the intersection of plane KLM and plane KLN?

(A) K and L (B) M and N (C) $\overline{KL}$ (D) $\overleftrightarrow{KL}$

(E) The planes do not intersect.

★ Challenge

76. INTERSECTING LINES In each diagram below, every line intersects all the other lines, but only two lines pass through each intersection point.



EXTRA CHALLENGE

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Can you draw 5 lines that intersect in this way? 6 lines? Is there a pattern to the number of intersection points?

MIXED REVIEW

DESCRIBING PATTERNS Describe a pattern in the sequence of numbers. Predict the next number. (Review 1.1)

77. 1, 6, 36, 216, . . . **78.** 2, −2, 2, −2, 2, . . .

79. 8.1, 88.11, 888.111, 8888.1111, . . . **80.** 0, 3, 9, 18, 30, . . .

OPERATIONS WITH INTEGERS Simplify the expression. (Skills Review, p.785)

81. $0 - 2$ **82.** $3 - 9$ **83.** $9 - (-4)$ **84.** $-5 - (-2)$

85. $5 - 0$ **86.** $4 - 7$ **87.** $3 - (-8)$ **88.** $-7 - (-5)$

RADICAL EXPRESSIONS Simplify the expression. Round your answer to two decimal places. (Skills Review, p. 799, for 1.3)

89. $\sqrt{21 + 100}$ **90.** $\sqrt{40 + 60}$ **91.** $\sqrt{25 + 144}$ **92.** $\sqrt{9 + 16}$

93. $\sqrt{5^2 + 7^2}$ **94.** $\sqrt{3^2 + (-2)^2}$ **95.** $\sqrt{(-3)^2 + 3^2}$ **96.** $\sqrt{(-5)^2 + 10^2}$

1.3 Segments and Their Measures

What you should learn

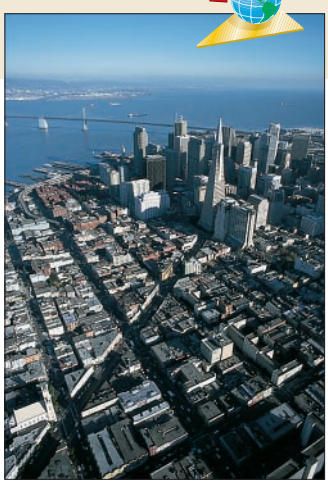
GOAL 1 Use segment postulates.

GOAL 2 Use the Distance Formula to measure distances, as applied in **Exs. 45–54**.

Why you should learn it

▼ To solve **real-life** problems, such as finding distances along a diagonal city street in **Example 4**.

REAL LIFE



GOAL 1 USING SEGMENT POSTULATES

In geometry, rules that are accepted without proof are called **postulates** or **axioms**. Rules that are proved are called *theorems*. In this lesson, you will study two postulates about the lengths of segments.

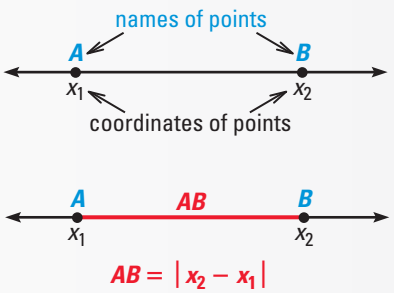
POSTULATE

POSTULATE 1 ***Ruler Postulate***

The points on a line can be matched one to one with the real numbers. The real number that corresponds to a point is the **coordinate** of the point.

The **distance** between points A and B, written as AB, is the absolute value of the difference between the coordinates of A and B.

AB is also called the **length** of $\overline{AB}$.

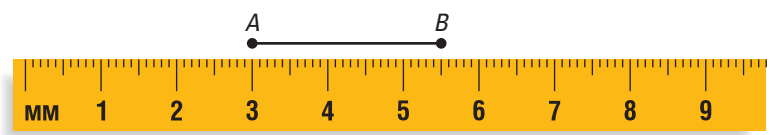


EXAMPLE 1 *Finding the Distance Between Two Points*

Measure the length of the segment to the nearest millimeter.

SOLUTION

Use a metric ruler. Align one mark of the ruler with A. Then estimate the coordinate of B. For example, if you align A with 3, B appears to align with 5.5.



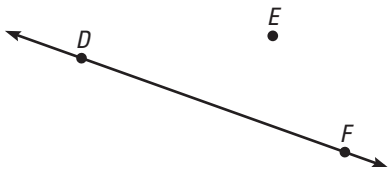
$$AB = |5.5 - 3| = |2.5| = 2.5$$

▶ The distance between A and B is about 2.5 cm.

It doesn't matter how you place the ruler. For example, if the ruler in Example 1 is placed so that A is aligned with 4, then B aligns with 6.5. The difference in the coordinates is the same.

When three points lie on a line, you can say that one of them is **between** the other two. This concept applies to collinear points only. For instance, in the figures below, point B is between points A and C, but point E is not between points D and F.

Point *B* is between points *A* and *C*.



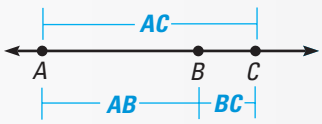
Point *E* is not between points *D* and *F*.

POSTULATE

POSTULATE 2 ***Segment Addition Postulate***

If B is between A and C, then $AB + BC = AC$.

If $AB + BC = AC$, then B is between A and C.



EXAMPLE 2 *Finding Distances on a Map*

MAP READING Use the map to find the distances between the three cities that lie on a line.

SOLUTION

Using the scale on the map, you can estimate that the distance between Athens and Macon is

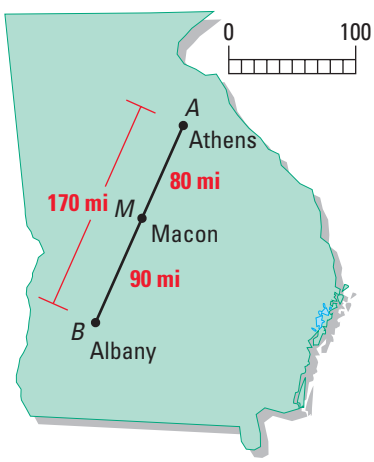
$AM = 80$ miles.

The distance between Macon and Albany is

$MB = 90$ miles.

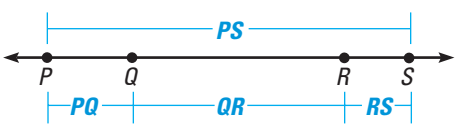
Knowing that Athens, Macon, and Albany lie on the same line, you can use the Segment Addition Postulate to conclude that the distance between Athens and Albany is

$AB = AM + MB = 80 + 90 = 170$ miles.



The Segment Addition Postulate can be generalized to three or more segments, as long as the segments lie on a line. If P, Q, R, and S lie on a line as shown, then

$PS = PQ + QR + RS$.



GOAL 2 USING THE DISTANCE FORMULA

STUDENT HELP

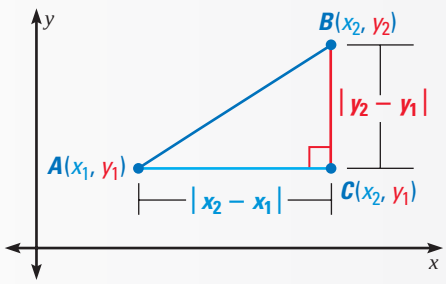
Study Tip
The small numbers in x_1 and x_2 are called *subscripts*. You read them as "x sub 1" and "x sub 2."

The **Distance Formula** is a formula for computing the distance between two points in a *coordinate* plane.

THE DISTANCE FORMULA

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are points in a coordinate plane, then the distance between A and B is

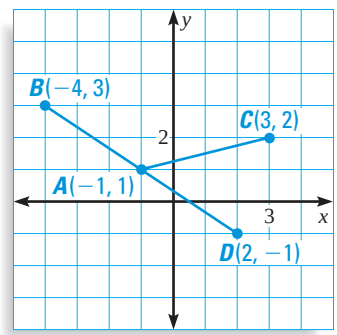
$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



EXAMPLE 3 *Using the Distance Formula*

Using Algebra

Find the lengths of the segments. Tell whether any of the segments have the same length.



SOLUTION

Use the Distance Formula.

$$AB = \sqrt{[(-4) - (-1)]^2 + (3 - 1)^2}$$
$$= \sqrt{(-3)^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13}$$

$$AC = \sqrt{[3 - (-1)]^2 + (2 - 1)^2}$$
$$= \sqrt{4^2 + 1^2} = \sqrt{16 + 1} = \sqrt{17}$$

$$AD = \sqrt{[2 - (-1)]^2 + (-1 - 1)^2}$$
$$= \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}$$

▶ So, $\overline{AB}$ and $\overline{AD}$ have the same length, but $\overline{AC}$ has a different length.

Segments that have the same length are called **congruent segments**. For instance, in Example 3, $\overline{AB}$ and $\overline{AD}$ are congruent because each has a length of $\sqrt{13}$. There is a special symbol, $\cong$, for indicating *congruence*.

LENGTHS ARE EQUAL.	SEGMENTS ARE CONGRUENT.
$AB = AD$	$\overline{AB} \cong \overline{AD}$
"is equal to"	"is congruent to"

The Distance Formula is based on the *Pythagorean Theorem,* which you will see again when you work with right triangles in Chapter 9.

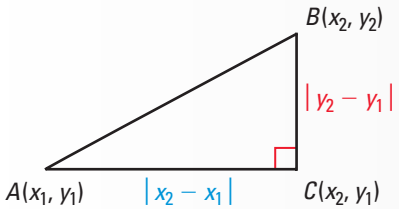
STUDENT HELP

Study Tip
The red mark at one corner of each triangle indicates a right angle.

CONCEPT SUMMARY — **DISTANCE FORMULA AND PYTHAGOREAN THEOREM**

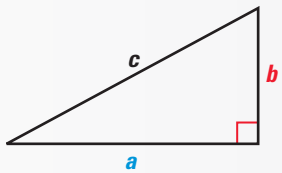
DISTANCE FORMULA

$$(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$



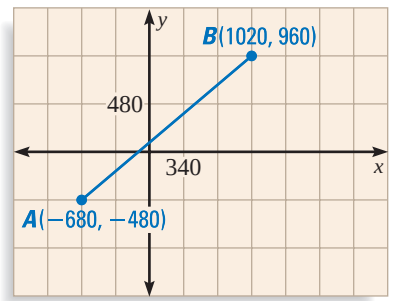
PYTHAGOREAN THEOREM

$$c^2 = a^2 + b^2$$



EXAMPLE 4 *Finding Distances on a City Map*

MAP READING On the map, the city blocks are 340 feet apart east-west and 480 feet apart north-south.

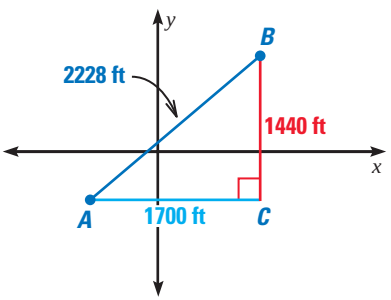


a. Find the walking distance between A and B.

b. What would the distance be if a diagonal street existed between the two points?

SOLUTION

a. To walk from A to B, you would have to walk five blocks east and three blocks north.



$$5 \text{ blocks} \cdot 340 \frac{\text{feet}}{\text{block}} = 1700 \text{ feet}$$

$$3 \text{ blocks} \cdot 480 \frac{\text{feet}}{\text{block}} = 1440 \text{ feet}$$

▶ So, the walking distance is **1700** + **1440**, which is a total of **3140** feet.

b. To find the diagonal distance between A and B, use the Distance Formula.

$$AB = \sqrt{[1020 - (-680)]^2 + [960 - (-480)]^2}$$

$$= \sqrt{1700^2 + 1440^2}$$

$$= \sqrt{4{,}963{,}600} \approx \mathbf{2228} \text{ feet}$$

▶ So, the diagonal distance would be about 2228 feet, which is 912 feet less than the walking distance.

STUDENT HELP

Study Tip
If you use a calculator to compute distances, use the parenthesis keys to group what needs to be squared.

GUIDED PRACTICE

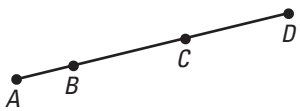
Vocabulary Check ✓

1. What is a *postulate*?

Concept Check ✓

2. Draw a sketch of three collinear points. Label them. Then write the Segment Addition Postulate for the points.

3. Use the diagram. How can you determine BD if you know BC and CD? if you know AB and AD?



Skill Check ✓

Find the distance between the two points.

4. $C(0, 0)$, $D(5, 2)$
5. $G(3, 0)$, $H(8, 10)$
6. $M(1, -3)$, $N(3, 5)$
7. $P(-8, -6)$, $Q(-3, 0)$
8. $S(7, 3)$, $T(1, -5)$
9. $V(-2, -6)$, $W(1, -2)$

Use the Distance Formula to decide whether $\overline{JK} \cong \overline{KL}$.

10. $J(3, -5)$, $K(-1, 2)$, $L(-5, -5)$
11. $J(0, -8)$, $K(4, 3)$, $L(-2, -7)$
12. $J(10, 2)$, $K(7, -3)$, $L(4, -8)$

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 803.

MEASUREMENT **Measure the length of the segment to the nearest millimeter.**

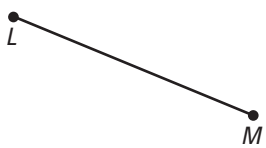
13. 

14. 

15. 

16. 

17. 

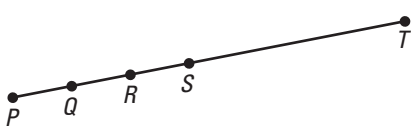
18. 

BETWEENNESS **Draw a sketch of the three collinear points. Then write the Segment Addition Postulate for the points.**

19. E is between D and F.
20. H is between G and J.
21. M is between N and P.
22. R is between Q and S.

LOGICAL REASONING **In the diagram of the collinear points, $PT = 20$, $QS = 6$, and $PQ = QR = RS$. Find each length.**

23. QR
24. RS
25. PQ
26. ST
27. RP
28. RT
29. SP
30. QT



STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 13–18
Example 2: Exs. 19–33
Example 3: Exs. 34–43
Example 4: Exs. 44–54

USING ALGEBRA **Suppose M is between L and N. Use the Segment Addition Postulate to solve for the variable. Then find the lengths of $\overline{LM}$, $\overline{MN}$, and $\overline{LN}$.**

31. $LM = 3x + 8$
$MN = 2x - 5$
$LN = 23$

32. $LM = 7y + 9$
$MN = 3y + 4$
$LN = 143$

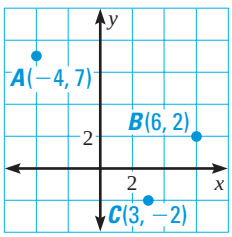
33. $LM = \frac{1}{2}z + 2$
$MN = 3z + \frac{3}{2}$
$LN = 5z + 2$

STUDENT HELP

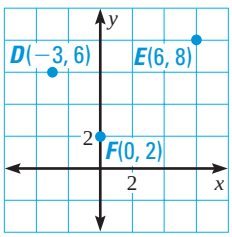
INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 34–36.

DISTANCE FORMULA **Find the distance between each pair of points.**

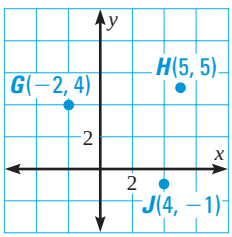
34.



35.

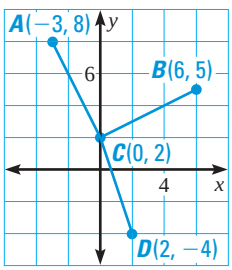


36.

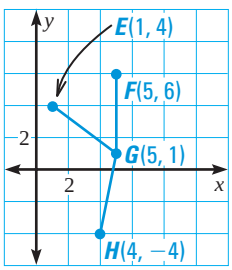


DISTANCE FORMULA **Find the lengths of the segments. Tell whether any of the segments have the same length.**

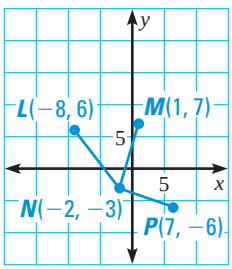
37.



38.



39.



CONGRUENCE **Use the Distance Formula to decide whether $\overline{PQ} \cong \overline{QR}$.**

40. $P(4, -4)$
$Q(1, -6)$
$R(-1, -3)$

41. $P(-1, -6)$
$Q(-8, 5)$
$R(3, -2)$

42. $P(5, 1)$
$Q(-5, -7)$
$R(-3, 6)$

43. $P(-2, 0)$
$Q(10, -14)$
$R(-4, -2)$

CAMBRIA INCLINE **In Exercises 44 and 45, use the information about the incline railway given below.**

In the days before automobiles were available, railways called "inclines" brought people up and down hills in many cities. In Johnstown, Pennsylvania, the Cambria Incline was reputedly the steepest in the world when it was completed in 1893. It rises about 514 feet vertically as it moves 734 feet horizontally.

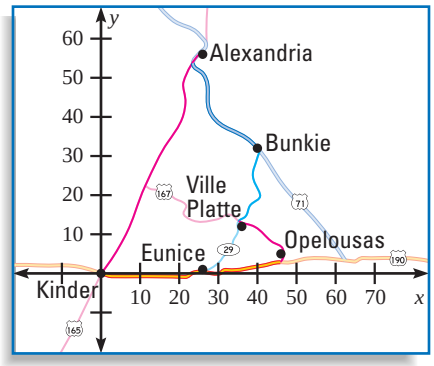
44. On graph paper, draw a coordinate plane and mark the axes using a scale that allows you to plot (0, 0) and (734, 514). Plot the points and connect them with a segment to represent the incline track.

45. Use the Distance Formula to estimate the length of the track.

Workers constructing the Cambria Incline

DRIVING DISTANCES **In Exercises 46 and 47, use the map of cities in Louisiana shown below. Coordinates on the map are given in miles.**

The coordinates of Alexandria, Kinder, Eunice, Opelousas, Ville Platte, and Bunkie are $A(26, 56)$, $K(0, 0)$, $E(26, 1)$, $O(46, 5)$, $V(36, 12)$, and $B(40, 32)$.



46. What is the shortest flying distance between Eunice and Alexandria?

47. Using only roads shown on the map, what is the approximate shortest driving distance between Eunice and Alexandria?

LONG-DISTANCE RATES **In Exercises 48–52, find the distance between the two cities using the information given in the table, which is from a coordinate system used for calculating long-distance telephone rates.**

Buffalo, NY	(5075, 2326)	**Omaha, NE**	(6687, 4595)
Chicago, IL	(5986, 3426)	**Providence, RI**	(4550, 1219)
Dallas, TX	(8436, 4034)	**San Diego, CA**	(9468, 7629)
Miami, FL	(8351, 527)	**Seattle, WA**	(6336, 8896)

48. Buffalo and Dallas

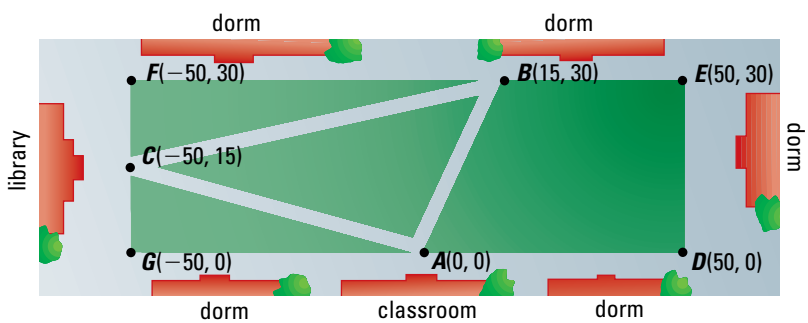
49. Chicago and Seattle

50. Miami and Omaha

51. Providence and San Diego

52. The long-distance coordinate system is measured in units of $\sqrt{0.1}$ mile. Convert the distances you found in Exs. 48–51 to miles.

CAMPUS PATHWAYS **In Exercises 53 and 54, use the campus map below.** Sidewalks around the edge of a campus quadrangle connect the buildings. Students sometimes take shortcuts by walking across the grass along the pathways shown. The coordinate system shown is measured in yards.



53. Find the distances from A to B, from B to C, and from C to A if you have to walk around the quadrangle along the sidewalks.

54. Find the distances from A to B, from B to C, and from C to A if you are able to walk across the grass along the pathways.

Test Preparation

55. MULTIPLE CHOICE Points K and L are on $\overline{AB}$. If $AK > BL$, then which statement must be true?

Ⓐ $AK < KB$ Ⓑ $AL < LB$ Ⓒ $AL > BK$

Ⓓ $KL < LB$ Ⓔ $AL + BK > AB$

56. MULTIPLE CHOICE Suppose point M lies on $\overline{CD}$, $CM = 2 \cdot MD$, and $CD = 18$. What is the length of MD?

Ⓐ 3 Ⓑ 6 Ⓒ 9 Ⓓ 12 Ⓔ 36

★ Challenge

THREE-DIMENSIONAL DISTANCE In Exercises 57–59, use the following information to find the distance between the pair of points.
In a three-dimensional coordinate system, the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

EXTRA CHALLENGE
www.mcdougallittell.com

57. $P(0, 20, -32)$
$Q(2, -10, -20)$

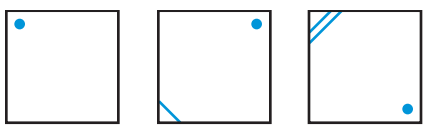
58. $A(-8, 15, -4)$
$B(10, 1, -6)$

59. $F(4, -42, 60)$
$G(-7, -11, 38)$

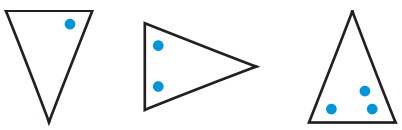
MIXED REVIEW

SKETCHING VISUAL PATTERNS Sketch the next figure in the pattern. (Review 1.1)

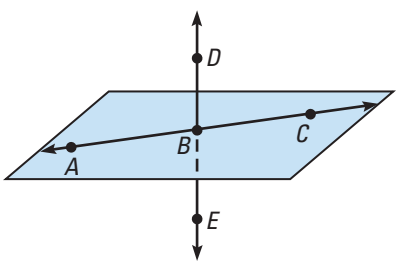
60.



61.



EVALUATING STATEMENTS Determine if the statement is *true* or *false*. (Review 1.2)



62. E lies on $\overleftrightarrow{BD}$.

63. E lies on $\overrightarrow{BD}$.

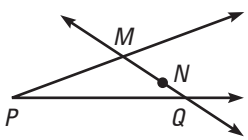
64. A, B, and D are collinear.

65. $\overrightarrow{BD}$ and $\overrightarrow{BE}$ are opposite rays.

66. B lies in plane ADC.

67. The intersection of $\overleftrightarrow{DE}$ and $\overleftrightarrow{AC}$ is B.

NAMING RAYS Name the ray described. (Review 1.2 for 1.4)



68. Name a ray that contains M.

69. Name a ray that has N as an endpoint.

70. Name two rays that intersect at P.

71. Name a pair of opposite rays.

QUIZ 1

Self-Test for Lessons 1.1–1.3

Write the next number in the sequence. (Lesson 1.1)

1. 10, 9.5, 9, 8.5, . . .

2. 0, 2, −2, 4, −4, . . .

Sketch the figure described. (Lesson 1.2)

3. Two segments that do not intersect.

4. Two lines that do not intersect, and a third line that intersects each of them.

5. Two lines that intersect a plane at the same point.

6. Three planes that do not intersect.

7. **MINIATURE GOLF** At a miniature golf course, a water hazard blocks the direct shot from the tee at $T(0, 0)$ to the cup at $C(-1, 7)$. If you hit the ball so it bounces off an angled wall at $B(3, 4)$, it will go into the cup. The coordinate system is measured in feet. Draw a diagram of the situation. Find TB and BC. (Lesson 1.3)

MATH & History — Geometric Constructions

INTERNET **APPLICATION LINK** www.mcdougallittell.com

THEN

MORE THAN 2000 YEARS AGO, the Greek mathematician Euclid published a 13 volume work called *The Elements*. In his systematic approach, figures are *constructed* using only a *compass* and a *straightedge* (a ruler without measuring marks).

An early printed edition of *The Elements*

NOW

TODAY, geometry software may be used to construct geometric figures. Programs allow you to perform constructions as if you have only a compass and straightedge. They also let you make measurements of lengths, angles, and areas.

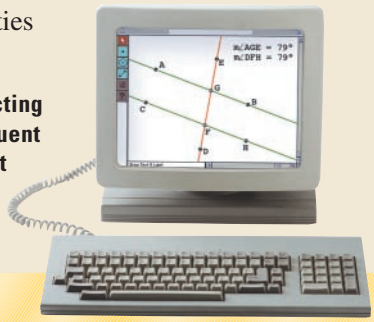
1. Draw two points and use a *straightedge* to construct the line that passes through them.
2. With the points as centers, use a *compass* to draw two circles of different sizes so that the circles intersect in two points. Mark the two points of intersection and construct the line through them.
3. Connect the four points you constructed. What are the properties of the shape formed?

c. 300 B.C.

Euclid develops The Elements.

1796

Gauss proves constructing a shape with 17 congruent sides and 17 congruent angles is possible.



1990s

Geometry software duplicates the tools for construction on screen.

1.4 Angles and Their Measures

What you should learn

GOAL 1 Use angle postulates.

GOAL 2 Classify angles as acute, right, obtuse, or straight.

Why you should learn it

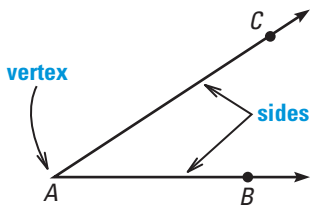
▼ To solve **real-life** problems about angles, such as the field of vision of a horse wearing blinkers in **Example 2**.

REAL LIFE

GOAL 1 USING ANGLE POSTULATES

An **angle** consists of two different rays that have the same initial point. The rays are the **sides** of the angle. The initial point is the **vertex** of the angle.

The angle that has sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is denoted by $\angle BAC$, $\angle CAB$, or $\angle A$. The point A is the vertex of the angle.

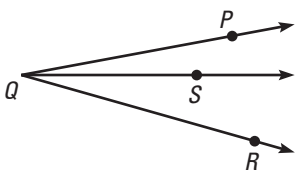


EXAMPLE 1 *Naming Angles*

Name the angles in the figure.

SOLUTION

There are three different angles.

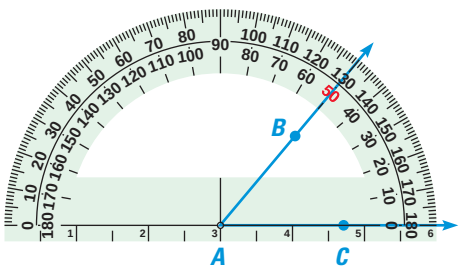


- $\angle PQS$ or $\angle SQP$
- $\angle SQR$ or $\angle RQS$
- $\angle PQR$ or $\angle RQP$

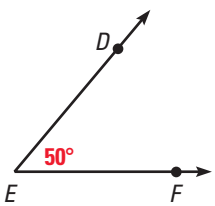
You should not name any of these angles as $\angle Q$ because all three angles have Q as their vertex. The name $\angle Q$ would not distinguish one angle from the others.

The *measure* of $\angle A$ is denoted by $m\angle A$. The measure of an angle can be approximated with a protractor, using units called *degrees* (°). For instance, $\angle BAC$ has a measure of 50°, which can be written as

$$m\angle BAC = 50°.$$



Angles that have the same measure are called **congruent angles**. For instance, $\angle BAC$ and $\angle DEF$ each have a measure of 50°, so they are congruent.



MEASURES ARE EQUAL.	ANGLES ARE CONGRUENT.
$m\angle BAC = m\angle DEF$	$\angle BAC \cong \angle DEF$
"is equal to"	"is congruent to"

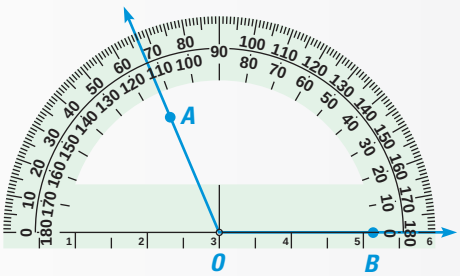
Logical Reasoning

POSTULATE

POSTULATE 3 ***Protractor Postulate***

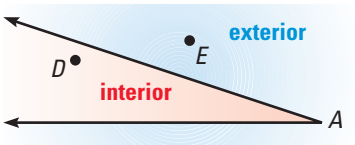
Consider a point A on one side of $\overleftrightarrow{OB}$. The rays of the form $\overrightarrow{OA}$ can be matched one to one with the real numbers from 0 to 180.

The **measure** of $\angle AOB$ is equal to the absolute value of the difference between the real numbers for $\overrightarrow{OA}$ and $\overrightarrow{OB}$.



A point is in the **interior** of an angle if it is between points that lie on each side of the angle.

A point is in the **exterior** of an angle if it is not on the angle or in its interior.

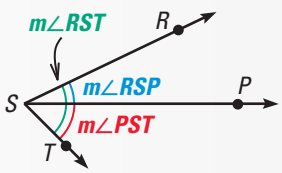


POSTULATE

POSTULATE 4 ***Angle Addition Postulate***

If P is in the interior of $\angle RST$, then

$$m\angle RSP + m\angle PST = m\angle RST.$$



EXAMPLE 2 *Calculating Angle Measures*

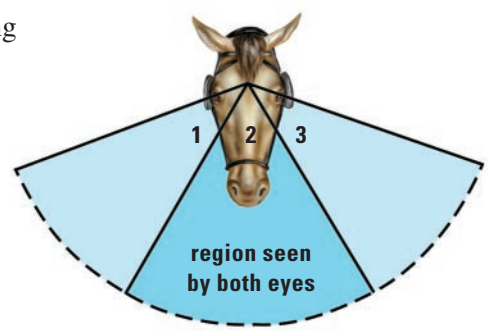
STUDENT HELP

Study Tip

As shown in Example 2, it is sometimes easier to label angles with numbers instead of letters.

REAL LIFE **VISION** Each eye of a horse wearing blinkers has an angle of vision that measures 100°. The angle of vision that is seen by both eyes measures 60°.

Find the angle of vision seen by the left eye alone.



SOLUTION

You can use the Angle Addition Postulate.

$m\angle 2 + m\angle 3 = 100°$	**Total vision for left eye is 100°.**
$m\angle 3 = 100° - m\angle 2$	**Subtract $m\angle 2$ from each side.**
$m\angle 3 = 100° - 60°$	**Substitute 60° for $m\angle 2$.**
$m\angle 3 = 40°$	**Subtract.**

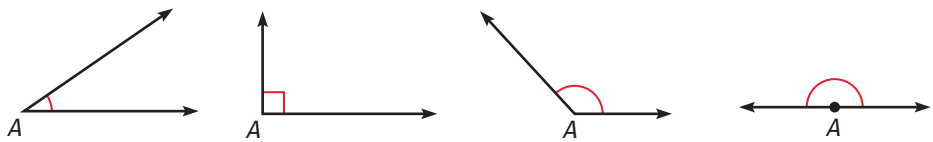
▶ So, the vision for the left eye alone measures 40°.

GOAL 2 CLASSIFYING ANGLES

STUDENT HELP

Study Tip
The mark used to indicate a right angle resembles the corner of a square, which has four right angles.

Angles are classified as **acute**, **right**, **obtuse**, and **straight**, according to their measures. Angles have measures greater than 0° and less than or equal to 180°.



Acute angle	Right angle	Obtuse angle	Straight angle
$0° < m\angle A < 90°$	$m\angle A = 90°$	$90° < m\angle A < 180°$	$m\angle A = 180°$

EXAMPLE 3 *Classifying Angles in a Coordinate Plane*

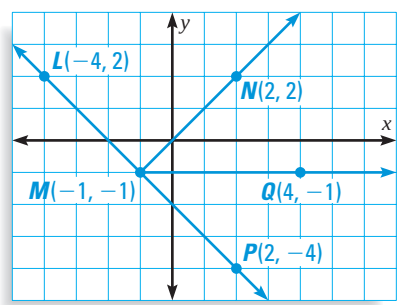
Plot the points $L(-4, 2)$, $M(-1, -1)$, $N(2, 2)$, $Q(4, -1)$, and $P(2, -4)$. Then measure and classify the following angles as acute, right, obtuse, or straight.

a. $\angle LMN$ **b.** $\angle LMP$ **c.** $\angle NMQ$ **d.** $\angle LMQ$

SOLUTION

Begin by plotting the points. Then use a protractor to measure each angle.

	MEASURE	CLASSIFICATION
a.	$m\angle LMN = 90°$	right angle
b.	$m\angle LMP = 180°$	straight angle
c.	$m\angle NMQ = 45°$	acute angle
d.	$m\angle LMQ = 135°$	obtuse angle



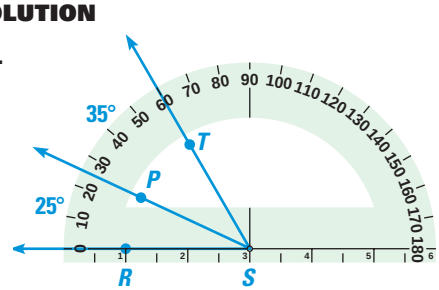
Two angles are **adjacent angles** if they share a common vertex and side, but have no common interior points.

EXAMPLE 4 *Drawing Adjacent Angles*

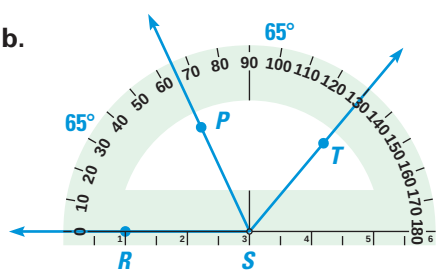
Use a protractor to draw two adjacent acute angles $\angle RSP$ and $\angle PST$ so that $\angle RST$ is (**a**) acute and (**b**) obtuse.

SOLUTION

a.



b.



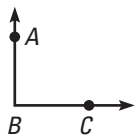
STUDENT HELP

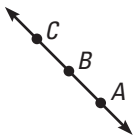
HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

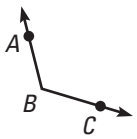
GUIDED PRACTICE

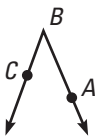
Vocabulary Check ✓ **Match the angle with its classification.**

A. acute **B.** obtuse **C.** right **D.** straight

1.


2.


3.


4.


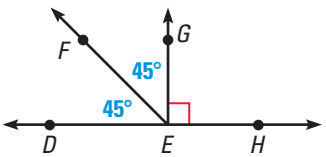
Concept Check ✓ **Use the diagram at the right to answer the questions. Explain your answers.**

5. Is $\angle DEF \cong \angle FEG$?

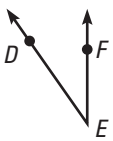
6. Is $\angle DEG \cong \angle HEG$?

7. Are $\angle DEF$ and $\angle FEH$ adjacent?

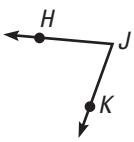
8. Are $\angle GED$ and $\angle DEF$ adjacent?

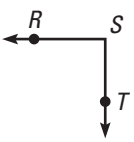


Skill Check ✓ **Name the vertex and sides of the angle. Then estimate its measure.**

9.


10.


11.


12.


Classify the angle as *acute, obtuse, right,* or *straight*.

13. $m\angle A = 180°$ **14.** $m\angle B = 90°$

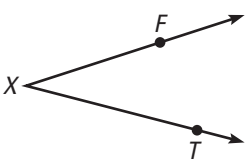
15. $m\angle C = 100°$ **16.** $m\angle D = 45°$

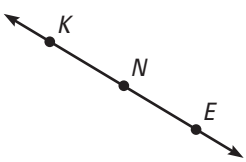
PRACTICE AND APPLICATIONS

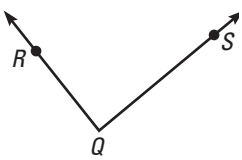
STUDENT HELP

Extra Practice to help you master skills is on pp. 803 and 804.

NAMING PARTS **Name the vertex and sides of the angle.**

17.


18.


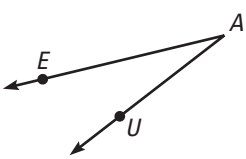
19.


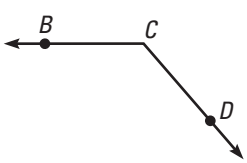
STUDENT HELP

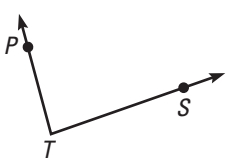
HOMEWORK HELP

Example 1: Exs. 17–22
Example 2: Exs. 23–34
Example 3: Exs. 35–43
Example 4: Exs. 38, 39

NAMING ANGLES **Write two names for the angle.**

20.


21.


22.


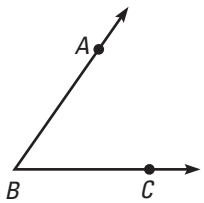
FOCUS ON CAREERS

SURVEYOR Surveyors use a tool called a theodolite, which can measure angles to the nearest 1/3600 of a degree.

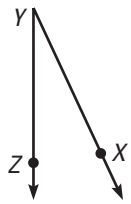
CAREER LINK www.mcdougallittell.com

MEASURING ANGLES **Copy the angle, extend its sides, and use a protractor to measure it to the nearest degree.**

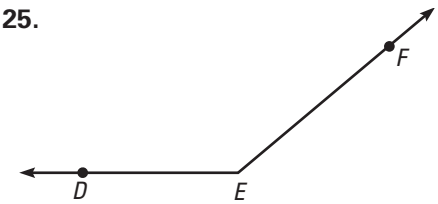
23.



24.

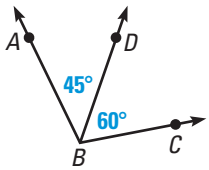


25.

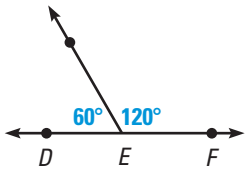


ANGLE ADDITION **Use the Angle Addition Postulate to find the measure of the unknown angle.**

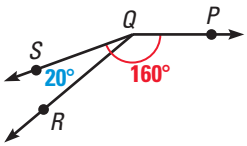
26. $m\angle ABC = \underline{?}$



27. $m\angle DEF = \underline{?}$



28. $m\angle PQR = \underline{?}$



LOGICAL REASONING **Draw a sketch that uses all of the following information.**

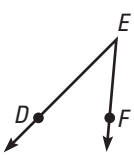
D is in the interior of $\angle BAE$. $m\angle BAC = 130°$
E is in the interior of $\angle DAF$. $m\angle EAC = 100°$
F is in the interior of $\angle EAC$. $m\angle BAD = m\angle EAF = m\angle FAC$

29. Find $m\angle FAC$. **30.** Find $m\angle BAD$. **31.** Find $m\angle FAB$.

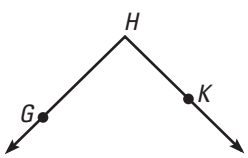
32. Find $m\angle DAE$. **33.** Find $m\angle FAD$. **34.** Find $m\angle BAE$.

CLASSIFYING ANGLES **State whether the angle appears to be *acute, right, obtuse,* or *straight*. Then estimate its measure.**

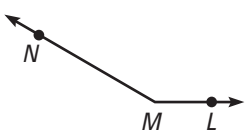
35.



36.



37.



LOGICAL REASONING **Draw five points, *A, B, C, D,* and *E* so that all three statements are true.**

38. $\angle DBE$ is a straight angle.
$\angle DBA$ is a right angle.
$\angle ABC$ is a straight angle.

39. C is in the interior of $\angle ADE$.
$m\angle ADC + m\angle CDE = 120°$.
$\angle CDB$ is a straight angle.

USING ALGEBRA **In a coordinate plane, plot the points and sketch $\angle ABC$. Classify the angle. Write the coordinates of a point that lies in the interior of the angle and the coordinates of a point that lies in the exterior of the angle.**

40. $A(3, -2)$
$B(5, -1)$
$C(4, -4)$

41. $A(5, -1)$
$B(3, -2)$
$C(4, -4)$

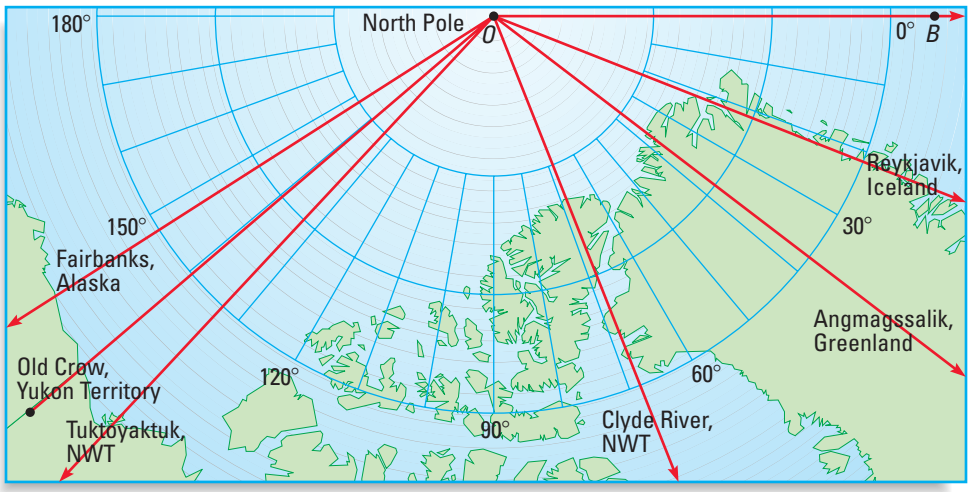
42. $A(5, -1)$
$B(3, -2)$
$C(0, -1)$

43. $A(-3, 1)$
$B(-2, 2)$
$C(-1, 4)$

GEOGRAPHY **For each city on the polar map, estimate the measure of $\angle BOA$, where B is on the Prime Meridian (0° longitude), O is the North Pole, and A is the city.**

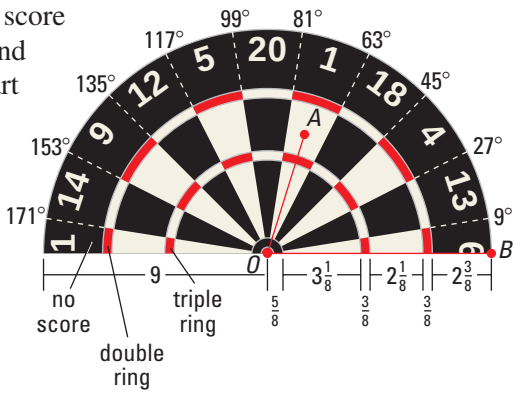
44. Clyde River, Canada **45.** Fairbanks, Alaska **46.** Angmagssalik, Greenland

47. Old Crow, Canada **48.** Reykjavik, Iceland **49.** Tuktoyaktuk, Canada



PLAYING DARTS **In Exercises 50–53, use the following information to find the score for the indicated dart toss landing at point A.**

A dartboard is 18 inches across. It is divided into twenty wedges of equal size. The score of a toss is indicated by numbers around the board. The score is doubled if a dart lands in the *double ring* and tripled if it lands in the *triple ring*. Only the top half of the dart board is shown.



50. $m\angle BOA = 160°$; $AO = 3$ in.

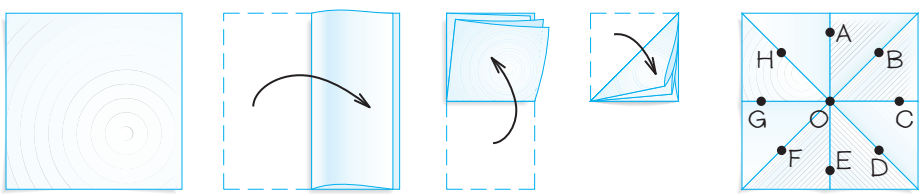
51. $m\angle BOA = 35°$; $AO = 4$ in.

52. $m\angle BOA = 60°$; $AO = 5$ in.

53. $m\angle BOA = 90°$; $AO = 6.5$ in.

Test Preparation

54. **MULTI-STEP PROBLEM** Use a piece of paper folded in half three times and labeled as shown.



a. Name eight congruent acute angles.

b. Name eight right angles.

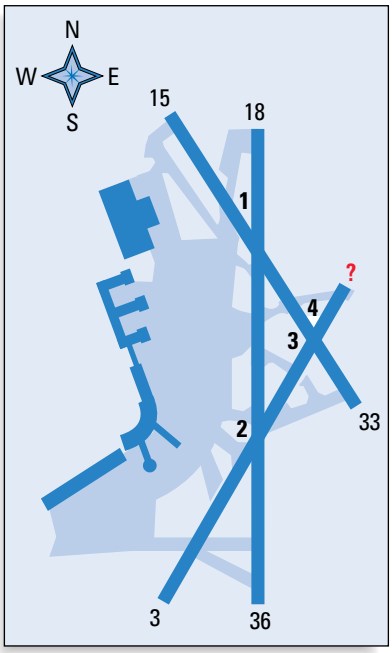
c. Name eight congruent obtuse angles.

d. Name two adjacent angles that combine to form a straight angle.

★ Challenge

AIRPORT RUNWAYS **In Exercises 55–60, use the diagram of Ronald Reagan Washington National Airport and the information about runway numbering on page 1.**

An airport runway is named by dividing its *bearing* (the angle measured clockwise from due north) by 10. Because a full circle contains 360°, runway numbers range from 1 to 36.



55. Find the measure of $\angle 1$.

56. Find the measure of $\angle 2$.

57. Find the measure of $\angle 3$.

58. Find the measure of $\angle 4$.

59. What is the number of the unlabeled runway in the diagram?

60. *Writing* Explain why the difference between the numbers at the opposite ends of a runway is always 18.

STUDENT HELP

HOMEWORK HELP

Bearings are measured around a circle, so they can have values larger than 180°. You can think of bearings between 180° and 360° as angles that are "bigger" than a straight angle.

EXTRA CHALLENGE

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MIXED REVIEW

STUDENT HELP

Skills Review

For help solving equations, see p.790.

USING ALGEBRA **Solve for *x*.** (Skills Review, p. 790, for 1.5)

61. $\frac{x+3}{2} = 3$

62. $\frac{5+x}{2} = 5$

63. $\frac{x+4}{2} = -4$

64. $\frac{-8+x}{2} = 12$

65. $\frac{x+7}{2} = -10$

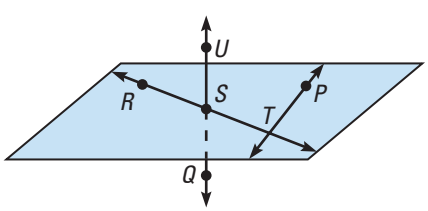
66. $\frac{-9+x}{2} = -7$

67. $\frac{x+(-1)}{2} = 7$

68. $\frac{8+x}{2} = -1$

69. $\frac{x+(-3)}{2} = -4$

EVALUATING STATEMENTS **Decide whether the statement is *true* or *false*.** (Review 1.2)



70. U, S, and Q are collinear.

71. T, Q, S, and P are coplanar.

72. $\overleftrightarrow{UQ}$ and $\overleftrightarrow{PT}$ intersect.

73. $\overrightarrow{SR}$ and $\overrightarrow{TS}$ are opposite rays.

DISTANCE FORMULA **Find the distance between the two points.** (Review 1.3 for 1.5)

74. $A(3, 10)$, $B(-2, -2)$

75. $C(0, 8)$, $D(-8, 3)$

76. $E(-3, 11)$, $F(4, 4)$

77. $G(10, -2)$, $H(0, 9)$

78. $J(5, 7)$, $K(7, 5)$

79. $L(0, -3)$, $M(-3, 0)$

1.5 Segment and Angle Bisectors

What you should learn

GOAL 1 Bisect a segment.

GOAL 2 Bisect an angle, as applied in **Exs. 50–55**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the angle measures of a kite in **Example 4**.

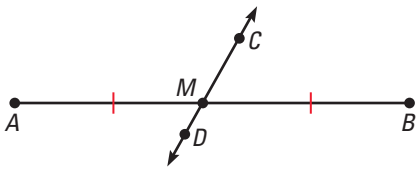
REAL LIFE

GOAL 1 BISECTING A SEGMENT

The **midpoint** of a segment is the point that divides, or **bisects**, the segment into two congruent segments. In this book, matching red *congruence marks* identify congruent segments in diagrams.

A **segment bisector** is a segment, ray, line, or plane that intersects a segment at its midpoint.

***M* is the midpoint of $\overline{AB}$ if *M* is on $\overline{AB}$ and *AM* = *MB*.**



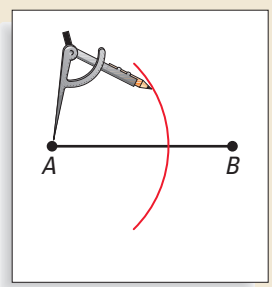
$\overleftrightarrow{CD}$ is a bisector of $\overline{AB}$.

You can use a **compass** and a **straightedge** (a ruler without marks) to **construct** a segment bisector and midpoint of $\overline{AB}$. A **construction** is a geometric drawing that uses a limited set of tools, usually a compass and a straightedge.

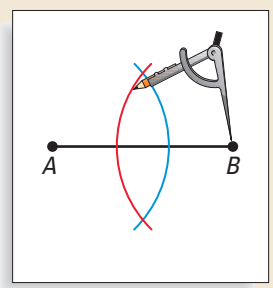
ACTIVITY
Construction

Segment Bisector and Midpoint

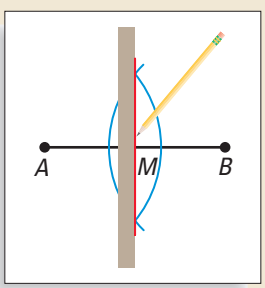
Use the following steps to construct a bisector of $\overline{AB}$ and find the midpoint M of $\overline{AB}$.



1 Place the compass point at A. Use a compass setting greater than half the length of $\overline{AB}$. Draw an arc.



2 Keep the same compass setting. Place the compass point at B. Draw an arc. It should intersect the other arc in two places.



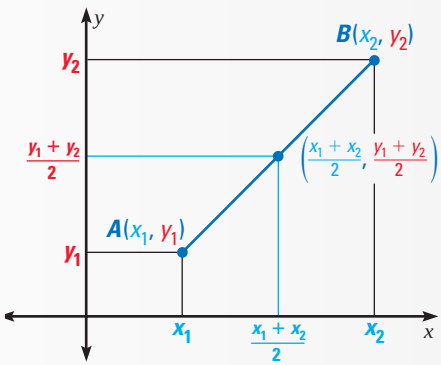
3 Use a straightedge to draw a segment through the points of intersection. This segment bisects $\overline{AB}$ at M, the midpoint of $\overline{AB}$.

If you know the coordinates of the endpoints of a segment, you can calculate the coordinates of the midpoint. You simply take the mean, or average, of the x-coordinates and of the y-coordinates. This method is summarized as the **Midpoint Formula**.

THE MIDPOINT FORMULA

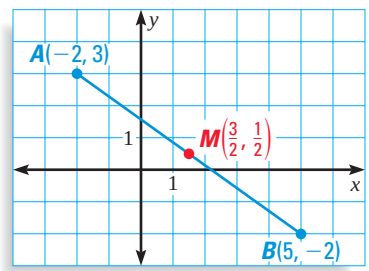
If $A(x_1, y_1)$ and $B(x_2, y_2)$ are points in a coordinate plane, then the midpoint of $\overline{AB}$ has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



EXAMPLE 1 *Finding the Coordinates of the Midpoint of a Segment*

Find the coordinates of the midpoint of $\overline{AB}$ with endpoints $A(-2, 3)$ and $B(5, -2)$.



SOLUTION

Use the Midpoint Formula as follows.

$$M = \left(\frac{-2 + 5}{2}, \frac{3 + (-2)}{2}\right)$$

$$= \left(\frac{3}{2}, \frac{1}{2}\right)$$

EXAMPLE 2 *Finding the Coordinates of an Endpoint of a Segment*

Using Algebra

The midpoint of $\overline{RP}$ is $M(2, 4)$. One endpoint is $R(-1, 7)$. Find the coordinates of the other endpoint.

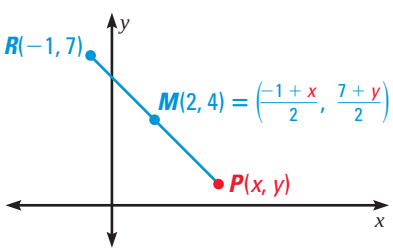
SOLUTION

Let (x, y) be the coordinates of P. Use the Midpoint Formula to write equations involving x and y.

$$\frac{-1 + x}{2} = 2 \qquad \frac{7 + y}{2} = 4$$

$$-1 + x = 4 \qquad 7 + y = 8$$

$$x = 5 \qquad y = 1$$



So, the other endpoint of the segment is $P(5, 1)$.

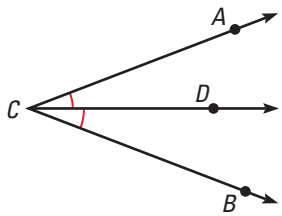
STUDENT HELP

Study Tip
Sketching the points in a coordinate plane helps you check your work. You should sketch a drawing of a problem even if the directions don't ask for a sketch.

GOAL 2 BISECTING AN ANGLE

An **angle bisector** is a ray that divides an angle into two adjacent angles that are congruent. In the diagram at the right, the ray $\overrightarrow{CD}$ bisects $\angle ABC$ because it divides the angle into two congruent angles, $\angle ACD$ and $\angle BCD$.

In this book, matching *congruence arcs* identify congruent angles in diagrams.

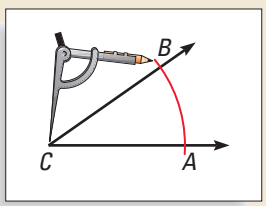


$m\angle ACD = m\angle BCD$

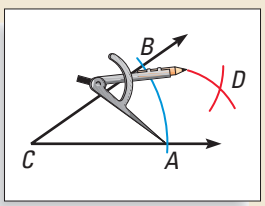
ACTIVITY
Construction

Angle Bisector

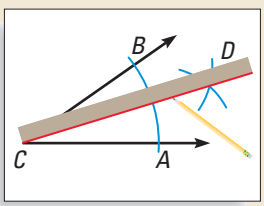
Use the following steps to construct an angle bisector of $\angle C$.



1 Place the compass point at C. Draw an arc that intersects both sides of the angle. Label the intersections A and B.



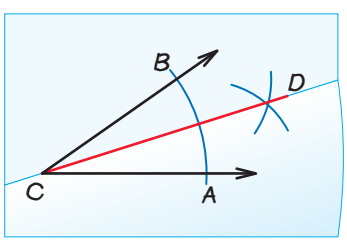
2 Place the compass point at A. Draw an arc. Then place the compass point at B. Using the same compass setting, draw another arc.



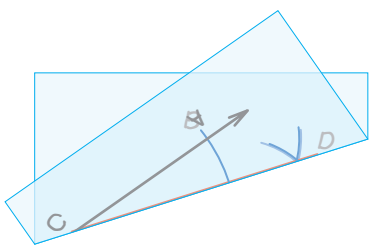
3 Label the intersection D. Use a straightedge to draw a ray through C and D. This is the angle bisector.

After you have constructed an angle bisector, you should check that it divides the original angle into two congruent angles. One way to do this is to use a protractor to check that the angles have the same measure.

Another way is to fold the piece of paper along the angle bisector. When you hold the paper up to a light, you should be able to see that the sides of the two angles line up, which implies that the angles are congruent.



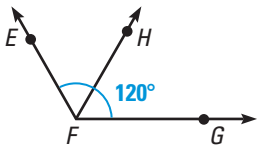
Fold on $\overrightarrow{CD}$.



The sides of angles $\angle BCD$ and $\angle ACD$ line up.

EXAMPLE 3 *Dividing an Angle Measure in Half*

The ray $\overrightarrow{FH}$ bisects the angle $\angle EFG$. Given that $m\angle EFG = 120°$, what are the measures of $\angle EFH$ and $\angle HFG$?



SOLUTION

An angle bisector divides an angle into two congruent angles, each of which has half the measure of the original angle. So,

$$m\angle EFH = m\angle HFG = \frac{120°}{2} = 60°.$$

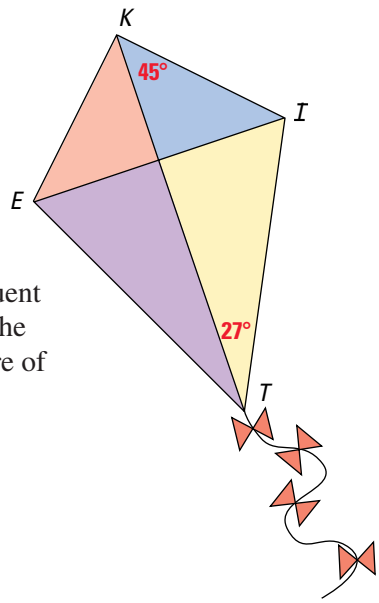
EXAMPLE 4 *Doubling an Angle Measure*

KITE DESIGN In the kite, two angles are bisected.

$\angle EKI$ is bisected by $\overrightarrow{KT}$.

$\angle ITE$ is bisected by $\overrightarrow{TK}$.

Find the measures of the two angles.



SOLUTION

You are given the measure of one of the two congruent angles that make up the larger angle. You can find the measure of the larger angle by doubling the measure of the smaller angle.

$$m\angle EKI = 2m\angle TKI = 2(45°) = 90°$$

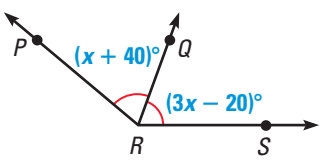
$$m\angle ITE = 2m\angle KTI = 2(27°) = 54°$$

FOCUS ON PEOPLE

JOSÉ SAÍNZ, a San Diego kite designer, uses colorful patterns in his kites. The struts of his kites often bisect the angles they support.

EXAMPLE 5 *Finding the Measure of an Angle*

In the diagram, $\overrightarrow{RQ}$ bisects $\angle PRS$. The measures of the two congruent angles are $(x + 40)°$ and $(3x - 20)°$. Solve for x.



Using Algebra

SOLUTION

$m\angle PRQ = m\angle QRS$	**Congruent angles have equal measures.**
$(x + 40)° = (3x - 20)°$	**Substitute given measures.**
$x + 60 = 3x$	**Add 20° to each side.**
$60 = 2x$	**Subtract x from each side.**
$30 = x$	**Divide each side by 2.**

So, $x = 30$. You can check by substituting to see that each of the congruent angles has a measure of 70°.

GUIDED PRACTICE

Vocabulary Check ✓

1. What kind of geometric figure is an *angle bisector*?

Concept Check ✓

2. How do you indicate congruent segments in a diagram? How do you indicate congruent angles in a diagram?

3. What is the simplified form of the Midpoint Formula if one of the endpoints of a segment is (0, 0) and the other is (x, y)?

Skill Check ✓

Find the coordinates of the midpoint of a segment with the given endpoints.

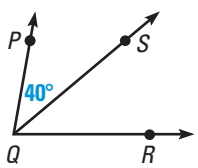
4. $A(5, 4)$, $B(-3, 2)$
5. $A(-1, -9)$, $B(11, -5)$
6. $A(6, -4)$, $B(1, 8)$

Find the coordinates of the other endpoint of a segment with the given endpoint and midpoint *M*.

7. $C(3, 0)$, $M(3, 4)$
8. $D(5, 2)$, $M(7, 6)$
9. $E(-4, 2)$, $M(-3, -2)$

10. Suppose $m\angle JKL$ is 90°. If the ray $\overrightarrow{KM}$ bisects $\angle JKL$, what are the measures of $\angle JKM$ and $\angle LKM$?

$\overrightarrow{QS}$ is the angle bisector of $\angle PQR$. Find the two angle measures not given in the diagram.

11.


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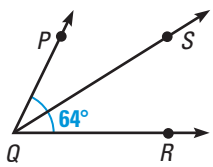

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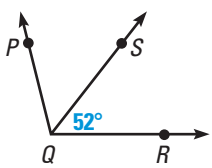
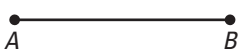


PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 804.

CONSTRUCTION **Use a ruler to measure and redraw the line segment on a piece of paper. Then use construction tools to construct a segment bisector.**

14.


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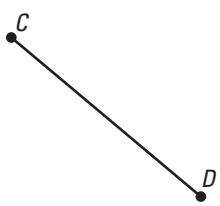


16.


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 17–24
Example 2: Exs. 25–30
Example 3: Exs. 37–42
Example 4: Exs. 37–42
Example 5: Exs. 44–49

FINDING THE MIDPOINT **Find the coordinates of the midpoint of a segment with the given endpoints.**

17. $A(0, 0)$, $B(-8, 6)$
18. $J(-1, 7)$, $K(3, -3)$
19. $C(10, 8)$, $D(-2, 5)$
20. $P(-12, -9)$, $Q(2, 10)$
21. $S(0, -8)$, $T(-6, 14)$
22. $E(4, 4)$, $F(4, -18)$
23. $V(-1.5, 8)$, $W(0.25, -1)$
24. $G(-5.5, -6.1)$, $H(-0.5, 9.1)$

USING ALGEBRA **Find the coordinates of the other endpoint of a segment with the given endpoint and midpoint *M*.**

25. $R(2, 6)$
$M(-1, 1)$

26. $T(-8, -1)$
$M(0, 3)$

27. $W(3, -12)$
$M(2, -1)$

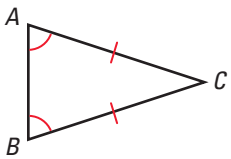
28. $Q(-5, 9)$
$M(-8, -2)$

29. $A(6, 7)$
$M(10, -7)$

30. $D(-3.5, -6)$
$M(1.5, 4.5)$

RECOGNIZING CONGRUENCE **Use the marks on the diagram to name the congruent segments and congruent angles.**

31.



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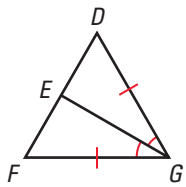
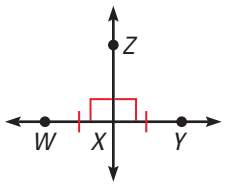
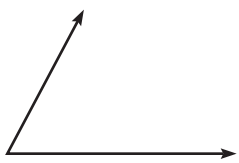


33.



CONSTRUCTION **Use a protractor to measure and redraw the angle on a piece of paper. Then use construction tools to find the angle bisector.**

34.



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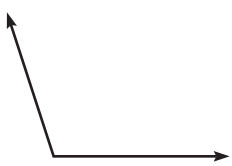

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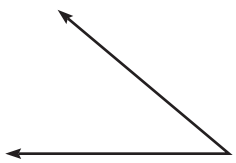
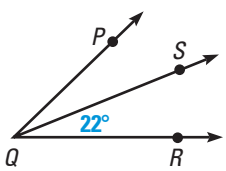
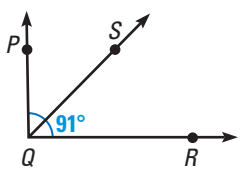


ANALYZING ANGLE BISECTORS **$\overrightarrow{QS}$ is the angle bisector of $\angle PQR$. Find the two angle measures not given in the diagram.**

37.



38.



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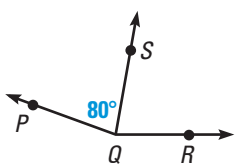

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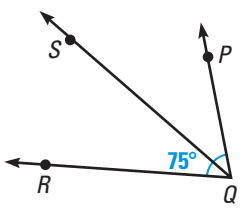
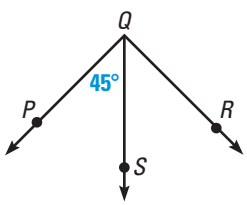
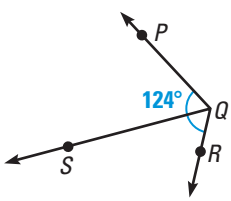


41.



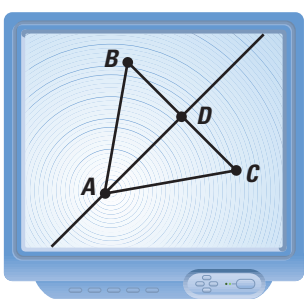
42.



STUDENT HELP

SOFTWARE HELP
Visit our Web site www.mcdougallittell.com to see instructions for several software applications.

43. **TECHNOLOGY** Use geometry software to draw a triangle. Construct the angle bisector of one angle. Then find the midpoint of the opposite side of the triangle. Change your triangle and observe what happens.



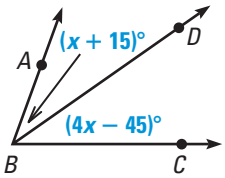
Does the angle bisector *always* pass through the midpoint of the opposite side? Does it *ever* pass through the midpoint?

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Ex. 44–49.

USING ALGEBRA **$\overrightarrow{BD}$ bisects $\angle ABC$. Find the value of x.**

44.



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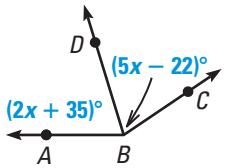

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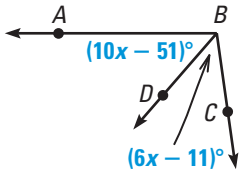

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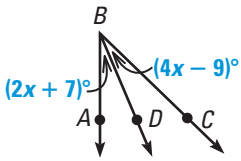

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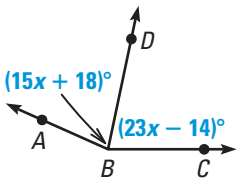

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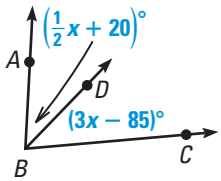
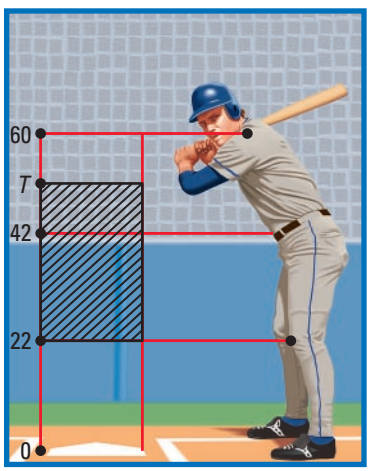


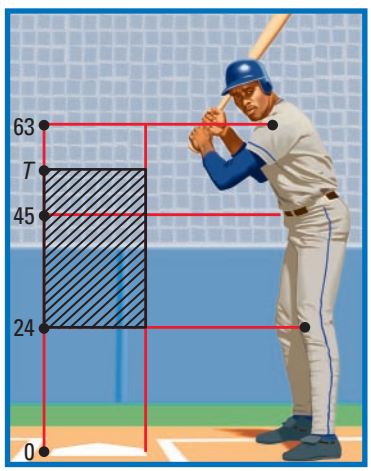
STRIKE ZONE **In Exercises 50 and 51, use the information below. For each player, find the coordinate of *T*, a point on the top of the strike zone.**
In baseball, the "strike zone" is the region a baseball needs to pass through in order for an umpire to declare it a strike if it is not hit. The *top of the strike zone* is a horizontal plane passing through the midpoint between the top of the hitter's shoulders and the top of the uniform pants when the player is in a batting stance.

▶ Source: Major League Baseball

50.

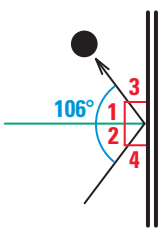


51.



AIR HOCKEY **When an air hockey puck is hit into the sideboards, it bounces off so that $\angle 1$ and $\angle 2$ are congruent. Find $m\angle 1$, $m\angle 2$, $m\angle 3$, and $m\angle 4$.**

52.



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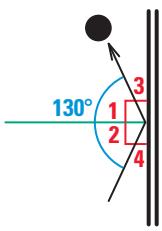

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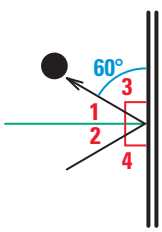
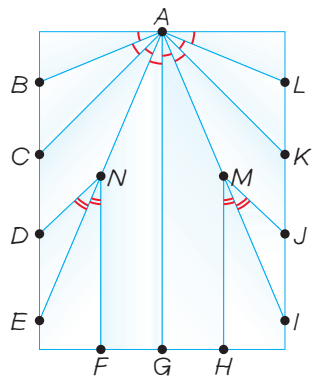


55. **PAPER AIRPLANES** The diagram represents an unfolded piece of paper used to make a paper airplane. The segments represent where the paper was folded to make the airplane.

Using the diagram, name as many pairs of congruent segments and as many congruent angles as you can.



56. *Writing* Explain, in your own words, how you would divide a line segment into four congruent segments using a compass and straightedge. Then explain how you could do it using the Midpoint Formula.

57. **MIDPOINT FORMULA REVISITED** Another version of the Midpoint Formula, for $A(x_1, y_1)$ and $B(x_2, y_2)$, is

$$M\left[x_1 + \frac{1}{2}(x_2 - x_1), y_1 + \frac{1}{2}(y_2 - y_1)\right].$$

Redo Exercises 17–24 using this version of the Midpoint Formula. Do you get the same answers as before? Use algebra to explain why the formula above is equivalent to the one in the lesson.

Test Preparation

58. **MULTI-STEP PROBLEM** Sketch a triangle with three sides of different lengths.

a. Using construction tools, find the midpoints of all three sides and the angle bisectors of all three angles of your triangle.

b. Determine whether or not the angle bisectors pass through the midpoints.

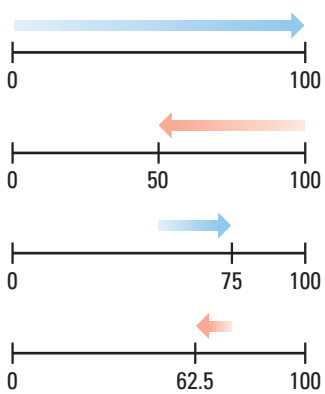
c. *Writing* Write a brief paragraph explaining your results. Determine if your results would be different if you used a different kind of triangle.

★ Challenge

INFINITE SERIES A football team practices running back and forth on the field in a special way. First they run from one end of the 100 yd field to the other. Then they turn around and run half the previous distance. Then they turn around again and run half the previous distance, and so on.

59. Suppose the athletes continue the running drill with smaller and smaller distances. What is the coordinate of the point that they approach?

60. What is the total distance that the athletes cover?

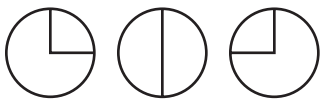


EXTRA CHALLENGE
www.mcdougallittell.com

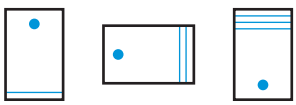
MIXED REVIEW

SKETCHING VISUAL PATTERNS **Sketch the next figure in the pattern.** (Review 1.1)

61.



62.



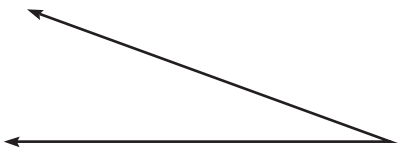
DISTANCE FORMULA **Find the distance between the two points.** (Review 1.3)

63. $A(3, 12)$, $B(-5, -1)$ **64.** $C(-6, 9)$, $D(-2, -7)$ **65.** $E(8, -8)$, $F(2, 14)$

66. $G(3, -8)$, $H(0, -2)$ **67.** $J(-4, -5)$, $K(5, -1)$ **68.** $L(-10, 1)$, $M(-4, 9)$

MEASURING ANGLES **Use a protractor to find the measure of the angle.** (Review 1.4 for 1.6)

69.



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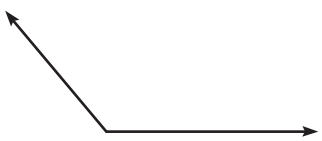
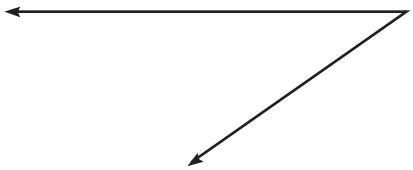


71.

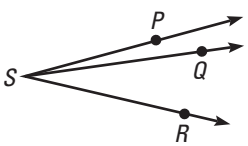
72.



QUIZ 2

Self-Test for Lessons 1.4 and 1.5

1. State the Angle Addition Postulate for the three angles shown at the right. (Lesson 1.4)



In a coordinate plane, plot the points and sketch $\angle DEF$. Classify the angle. Write the coordinates of a point that lies in the interior of the angle and the coordinates of a point that lies in the exterior of the angle. (Lesson 1.4)

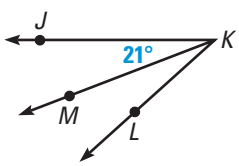
2. $D(-2, 3)$
$E(4, -3)$
$F(2, 6)$

3. $D(-6, -3)$
$E(0, -5)$
$F(8, -5)$

4. $D(-1, 8)$
$E(-4, 0)$
$F(4, 0)$

5. $D(1, 10)$
$E(1, 1)$
$F(8, 1)$

6. In the diagram, $\overrightarrow{KM}$ is the angle bisector of $\angle JKL$. Find $m\angle MKL$ and $m\angle JKL$. (Lesson 1.5)



1.6 Angle Pair Relationships

What you should learn

GOAL 1 Identify vertical angles and linear pairs.

GOAL 2 Identify complementary and supplementary angles.

Why you should learn it

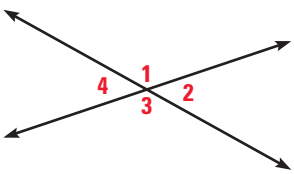
▼ To solve **real-life** problems, such as finding the measures of angles formed by the cables of a bridge in **Ex. 53**.

REAL LIFE

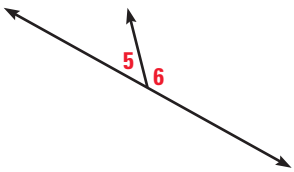
GOAL 1 VERTICAL ANGLES AND LINEAR PAIRS

In Lesson 1.4, you learned that two angles are *adjacent* if they share a common vertex and side but have no common interior points. In this lesson, you will study other relationships between pairs of angles.

Two angles are **vertical angles** if their sides form two pairs of opposite rays. Two adjacent angles are a **linear pair** if their noncommon sides are opposite rays.



∠1 and ∠3 are vertical angles.
∠2 and ∠4 are vertical angles.



∠5 and ∠6 are a linear pair.

In this book, you can assume from a diagram that two adjacent angles form a linear pair if the noncommon sides appear to lie on the same line.

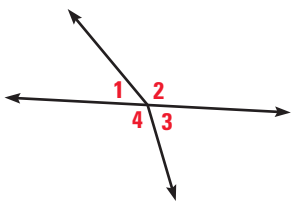
EXAMPLE 1 *Identifying Vertical Angles and Linear Pairs*

a. Are ∠2 and ∠3 a linear pair?

b. Are ∠3 and ∠4 a linear pair?

c. Are ∠1 and ∠3 vertical angles?

d. Are ∠2 and ∠4 vertical angles?



SOLUTION

a. No. The angles are adjacent but their noncommon sides are not opposite rays.

b. Yes. The angles are adjacent and their noncommon sides are opposite rays.

c. No. The sides of the angles do not form two pairs of opposite rays.

d. No. The sides of the angles do not form two pairs of opposite rays.

In Activity 1.6 on page 43, you may have discovered two results:

- *Vertical angles are congruent.*
- *The sum of the measures of angles that form a linear pair is 180°.*

Both of these results will be stated formally in Chapter 2.

Logical Reasoning

EXAMPLE 2 *Finding Angle Measures*

In the stair railing shown at the right, $\angle 6$ has a measure of $130°$. Find the measures of the other three angles.

SOLUTION

$\angle 6$ and $\angle 7$ are a linear pair. So, the sum of their measures is $180°$.

$$m\angle 6 + m\angle 7 = 180°$$

$$130° + m\angle 7 = 180°$$

$$m\angle 7 = 50°$$

$\angle 6$ and $\angle 5$ are also a linear pair. So, it follows that $m\angle 5 = 50°$.

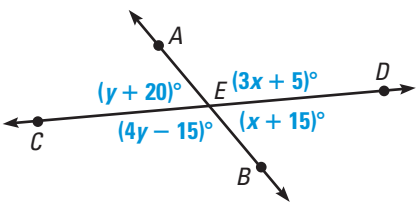
$\angle 6$ and $\angle 8$ are vertical angles. So, they are congruent and have the same measure.

$$m\angle 8 = m\angle 6 = 130°$$

Using Algebra

EXAMPLE 3 *Finding Angle Measures*

Solve for x and y.
Then find the angle measures.



SOLUTION

Use the fact that the sum of the measures of angles that form a linear pair is $180°$.

$$m\angle AED + m\angle DEB = 180° \qquad m\angle AEC + m\angle CEB = 180°$$

$$(3x + 5)° + (x + 15)° = 180° \qquad (y + 20)° + (4y - 15)° = 180°$$

$$4x + 20 = 180 \qquad 5y + 5 = 180$$

$$4x = 160 \qquad 5y = 175$$

$$x = 40 \qquad y = 35$$

Use substitution to find the angle measures.

$$m\angle AED = (3x + 5)° = (3 \cdot 40 + 5)° = 125°$$

$$m\angle DEB = (x + 15)° = (40 + 15)° = 55°$$

$$m\angle AEC = (y + 20)° = (35 + 20)° = 55°$$

$$m\angle CEB = (4y - 15)° = (4 \cdot 35 - 15)° = 125°$$

▶ So, the angle measures are $125°$, $55°$, $55°$, and $125°$. Because the vertical angles are congruent, the result is reasonable.

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

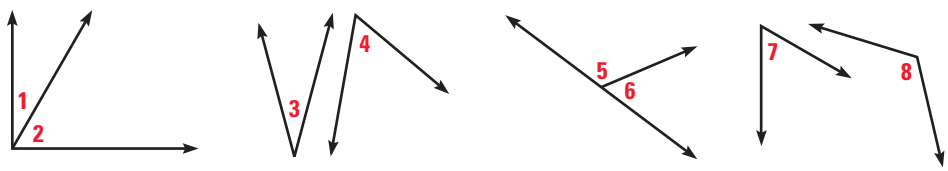
GOAL 2 COMPLEMENTARY AND SUPPLEMENTARY ANGLES

STUDENT HELP

Study Tip
In mathematics, the word *complement* is related to the phrase *to complete.* When you draw the complement of an angle, you are "completing" a right angle. (The word *compliment* is different. It means something said in praise.)

Two angles are **complementary angles** if the sum of their measures is 90°. Each angle is the **complement** of the other. Complementary angles can be adjacent or nonadjacent.

Two angles are **supplementary angles** if the sum of their measures is 180°. Each angle is the **supplement** of the other. Supplementary angles can be adjacent or nonadjacent.



complementary adjacent | **complementary nonadjacent** | **supplementary adjacent** | **supplementary nonadjacent**

EXAMPLE 4 *Identifying Angles*

State whether the two angles are complementary, supplementary, or neither.

SOLUTION

The angle showing 4:00 has a measure of 120° and the angle showing 10:00 has a measure of 60°. Because the sum of these two measures is 180°, the angles are supplementary.

EXAMPLE 5 *Finding Measures of Complements and Supplements*

a. Given that $\angle A$ is a complement of $\angle C$ and $m\angle A = 47°$, find $m\angle C$.

b. Given that $\angle P$ is a supplement of $\angle R$ and $m\angle R = 36°$, find $m\angle P$.

SOLUTION

a. $m\angle C = 90° - m\angle A = 90° - 47° = 43°$

b. $m\angle P = 180° - m\angle R = 180° - 36° = 144°$

EXAMPLE 6 *Finding the Measure of a Complement*

Using Algebra

$\angle W$ and $\angle Z$ are complementary. The measure of $\angle Z$ is five times the measure of $\angle W$. Find $m\angle W$.

SOLUTION

Because the angles are complementary, $m\angle W + m\angle Z = 90°$. But $m\angle Z = 5(m\angle W)$, so $m\angle W + 5(m\angle W) = 90°$. Because $6(m\angle W) = 90°$, you know that $m\angle W = 15°$.

GUIDED PRACTICE

Vocabulary Check ✓

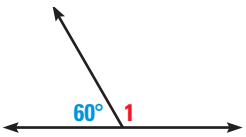
1. Explain the difference between *complementary angles* and *supplementary angles*.

Concept Check ✓

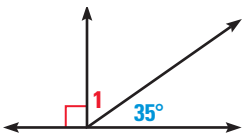
2. Sketch examples of acute vertical angles and obtuse vertical angles.
3. Sketch examples of adjacent congruent complementary angles and adjacent congruent supplementary angles.

Skill Check ✓

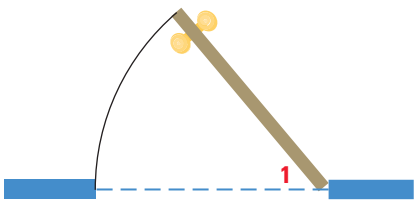
FINDING ANGLE MEASURES **Find the measure of ∠1.**

4. 

5. 

6. 

7. **OPENING A DOOR** The figure shows a doorway viewed from above. If you open the door so that the measure of ∠1 is 50°, how many more degrees would you have to open the door so that the angle between the wall and the door is 90°?



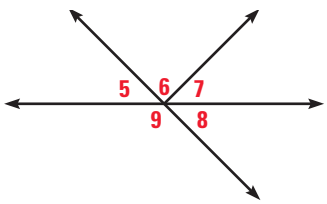
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 804.

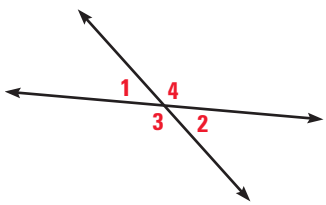
IDENTIFYING ANGLE PAIRS **Use the figure at the right.**

8. Are ∠5 and ∠6 a linear pair?
9. Are ∠5 and ∠9 a linear pair?
10. Are ∠5 and ∠8 a linear pair?
11. Are ∠5 and ∠8 vertical angles?
12. Are ∠5 and ∠7 vertical angles?
13. Are ∠9 and ∠6 vertical angles?



EVALUATING STATEMENTS **Decide whether the statement is *always, sometimes,* or *never* true.**

14. If $m\angle 1 = 40°$, then $m\angle 2 = 140°$.
15. If $m\angle 4 = 130°$, then $m\angle 2 = 50°$.
16. ∠1 and ∠4 are congruent.
17. $m\angle 2 + m\angle 3 = m\angle 1 + m\angle 4$
18. $\angle 2 \cong \angle 1$
19. $m\angle 2 = 90° - m\angle 3$

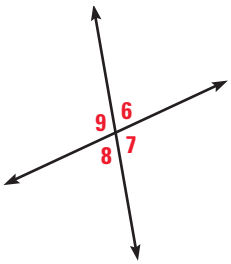


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 8–13
Example 2: Exs. 14–27
Example 3: Exs. 28–36
Example 4: Exs. 37–40
Example 5: Exs. 41, 42
Example 6: Exs. 43, 44

FINDING ANGLE MEASURES **Use the figure at the right.**



20. If $m\angle 6 = 72°$, then $m\angle 7 =$ __?__.

21. If $m\angle 8 = 80°$, then $m\angle 6 =$ __?__.

22. If $m\angle 9 = 110°$, then $m\angle 8 =$ __?__.

23. If $m\angle 9 = 123°$, then $m\angle 7 =$ __?__.

24. If $m\angle 7 = 142°$, then $m\angle 8 =$ __?__.

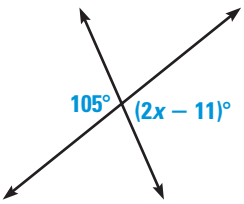
25. If $m\angle 6 = 13°$, then $m\angle 9 =$ __?__.

26. If $m\angle 9 = 170°$, then $m\angle 6 =$ __?__.

27. If $m\angle 8 = 26°$, then $m\angle 7 =$ __?__.

USING ALGEBRA **Find the value(s) of the variable(s).**

28.



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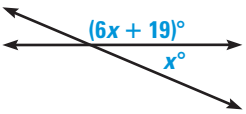

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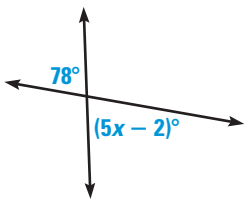

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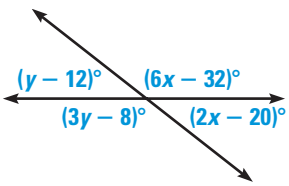

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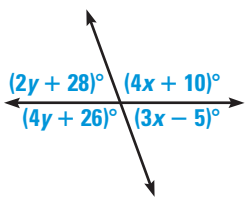

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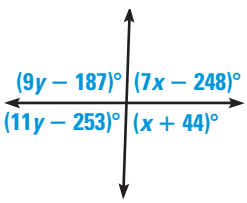

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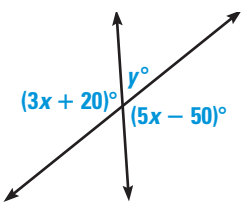

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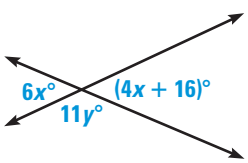
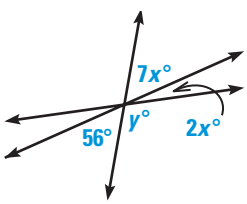


36.



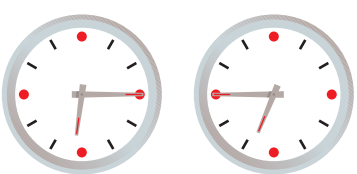
IDENTIFYING ANGLES **State whether the two angles shown are *complementary, supplementary,* or *neither*.**

37.

38.

39.

40.



41. FINDING COMPLEMENTS In the table, assume that $\angle 1$ and $\angle 2$ are complementary. Copy and complete the table.

$m\angle 1$	2°	10°	25°	33°	40°	49°	55°	62°	76°	86°
$m\angle 2$	?	?	?	?	?	?	?	?	?	?

42. FINDING SUPPLEMENTS In the table, assume that $\angle 1$ and $\angle 2$ are supplementary. Copy and complete the table.

$m\angle 1$	4°	16°	48°	72°	90°	99°	120°	152°	169°	178°
$m\angle 2$	?	?	?	?	?	?	?	?	?	?

43. USING ALGEBRA $\angle A$ and $\angle B$ are complementary. The measure of $\angle B$ is three times the measure of $\angle A$. Find $m\angle A$ and $m\angle B$.

44. USING ALGEBRA $\angle C$ and $\angle D$ are supplementary. The measure of $\angle D$ is eight times the measure of $\angle C$. Find $m\angle C$ and $m\angle D$.

FINDING ANGLES **$\angle A$ and $\angle B$ are complementary. Find $m\angle A$ and $m\angle B$.**

45. $m\angle A = 5x + 8$
$m\angle B = x + 4$

46. $m\angle A = 3x - 7$
$m\angle B = 11x - 1$

47. $m\angle A = 8x - 7$
$m\angle B = x - 11$

48. $m\angle A = \frac{3}{4}x - 13$
$m\angle B = 3x - 17$

FINDING ANGLES **$\angle A$ and $\angle B$ are supplementary. Find $m\angle A$ and $m\angle B$.**

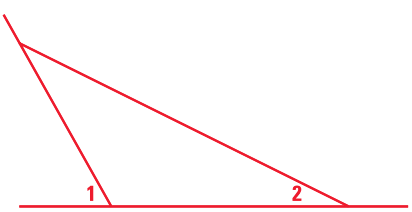
49. $m\angle A = 3x$
$m\angle B = x + 8$

50. $m\angle A = 6x - 1$
$m\angle B = 5x - 17$

51. $m\angle A = 12x + 1$
$m\angle B = x + 10$

52. $m\angle A = \frac{3}{8}x + 50$
$m\angle B = x + 31$

53. BRIDGES The Alamillo Bridge in Seville, Spain, was designed by Santiago Calatrava. In the bridge, $m\angle 1 = 58°$ and $m\angle 2 = 24°$. Find the supplements of both $\angle 1$ and $\angle 2$.



54. BASEBALL The foul lines of a baseball field intersect at home plate to form a right angle. Suppose you hit a baseball whose path forms an angle of 34° with the third base foul line. What is the angle between the first base foul line and the path of the baseball?

FOCUS ON PEOPLE

SANTIAGO CALATRAVA, a Spanish born architect, has developed designs for bridges, train stations, stadiums, and art museums.

APPLICATION LINK
www.mcdougallittell.com

55. PLANTING TREES To support a young tree, you attach wires from the trunk to the ground. The obtuse angle the wire makes with the ground is supplementary to the acute angle the wire makes, and it is three times as large. Find the measures of the angles.

56. *Writing* Give an example of an angle that *does not* have a complement. In general, what is true about an angle that has a complement?

Test Preparation

57. MULTIPLE CHOICE In the diagram shown at the right, what are the values of x and y?

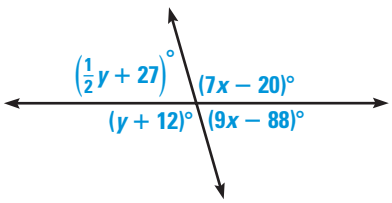
Ⓐ $x = 74, y = 106$

Ⓑ $x = 16, y = 88$

Ⓒ $x = 74, y = 16$

Ⓓ $x = 18, y = 118$

Ⓔ $x = 18, y = 94$

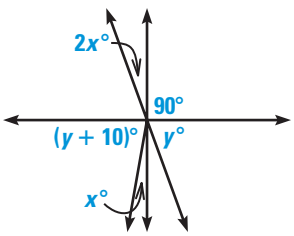


58. MULTIPLE CHOICE $\angle F$ and $\angle G$ are supplementary. The measure of $\angle G$ is six and one half times the measure of $\angle F$. What is $m\angle F$?

Ⓐ 20° Ⓑ 24° Ⓒ 24.5° Ⓓ 26.5° Ⓔ 156°

★ Challenge

59. USING ALGEBRA Find the values of x and y in the diagram shown at the right.



MIXED REVIEW

SOLVING EQUATIONS **Solve the equation.** (Skills Review, p. 802, for 1.7)

60. $3x = 96$ **61.** $\frac{1}{2} \cdot 5 \cdot h = 20$ **62.** $\frac{1}{2} \cdot b \cdot 6 = 15$

63. $s^2 = 200$ **64.** $2 \cdot 3.14 \cdot r = 40$ **65.** $3.14 \cdot r^2 = 314$

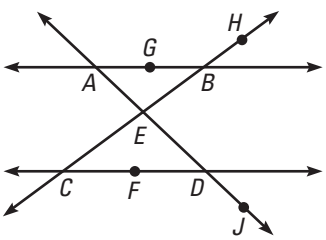
FINDING COLLINEAR POINTS **Use the diagram to find a third point that is collinear with the given points.** (Review 1.2)

66. A and J

67. D and F

68. H and E

69. B and G



FINDING THE MIDPOINT **Find the coordinates of the midpoint of a segment with the given endpoints.** (Review 1.5)

70. $A(0, 0), B(-6, -4)$ **71.** $F(2, 5), G(-10, 7)$ **72.** $K(8, -6), L(-2, -2)$

73. $M(-14, -9), N(0, 11)$ **74.** $P(-1.5, 4), Q(5, -9)$ **75.** $S(-2.4, 5), T(7.6, 9)$

1.7 Introduction to Perimeter, Circumference, and Area

What you should learn

GOAL 1 Find the perimeter and area of common plane figures.

GOAL 2 Use a general problem-solving plan.

Why you should learn it

▼ To solve **real-life** problems about perimeter and area, such as finding the number of bags of seed you need for a field in **Example 4**.

GOAL 1 REVIEWING PERIMETER, CIRCUMFERENCE, AND AREA

In this lesson, you will review some common formulas for perimeter, circumference, and area. You will learn more about area in Chapters 6, 11, and 12.

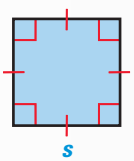
PERIMETER, CIRCUMFERENCE, AND AREA FORMULAS

Formulas for the perimeter P, area A, and circumference C of some common plane figures are given below.

SQUARE
side length s

$P = 4s$

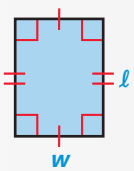
$A = s^2$



RECTANGLE
length ℓ and width w

$P = 2\ell + 2w$

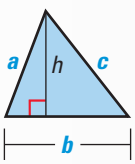
$A = \ell w$



TRIANGLE
side lengths a, b, and c, base b, and height h

$P = a + b + c$

$A = \frac{1}{2}bh$



CIRCLE
radius r

$C = 2\pi r$

$A = \pi r^2$

Pi (π) is the ratio of the circle's circumference to its diameter.

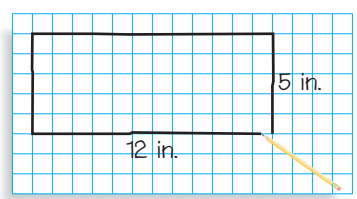
The measurements of perimeter and circumference use units such as centimeters, meters, kilometers, inches, feet, yards, and miles. The measurements of area use units such as square centimeters (cm^2), square meters (m^2), and so on.

EXAMPLE 1 *Finding the Perimeter and Area of a Rectangle*

Find the perimeter and area of a rectangle of length 12 inches and width 5 inches.

SOLUTION

Begin by drawing a diagram and labeling the length and width. Then, use the formulas for perimeter and area of a rectangle.



$$P = 2\ell + 2w \qquad A = \ell w$$
$$= 2(12) + 2(5) \qquad = (12)(5)$$
$$= 34 \qquad = 60$$

▶ So, the perimeter is 34 inches and the area is 60 square inches.

EXAMPLE 2 *Finding the Area and Circumference of a Circle*

Find the diameter, radius, circumference, and area of the circle shown at the right. Use 3.14 as an approximation for π.

STUDENT HELP

Study Tip
Some approximations for $\pi = 3.141592654\ldots$ are 3.14 and $\frac{22}{7}$.

SOLUTION

From the diagram, you can see that the diameter of the circle is

$$d = 13 - 5 = 8 \text{ cm}.$$

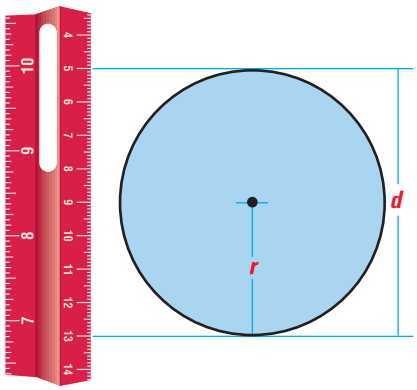
The radius is one half the diameter.

$$r = \frac{1}{2}(8) = 4 \text{ cm}$$

Using the formulas for circumference and area, you have

$$C = 2\pi r \approx 2(3.14)(4) \approx 25.1 \text{ cm}$$

$$A = \pi r^2 \approx 3.14(4^2) \approx 50.2 \text{ cm}^2.$$

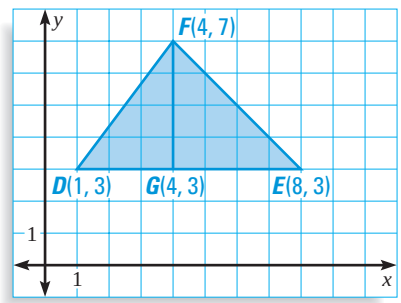


EXAMPLE 3 *Finding Measurements of a Triangle in a Coordinate Plane*

Find the area and perimeter of the triangle defined by $D(1, 3)$, $E(8, 3)$, and $F(4, 7)$.

SOLUTION

Plot the points in a coordinate plane. Draw the height from F to side $\overline{DE}$. Label the point where the height meets $\overline{DE}$ as G. Point G has coordinates (4, 3).



base: $DE = 8 - 1 = 7$

height: $FG = 7 - 3 = 4$

$$A = \frac{1}{2}(\text{base})(\text{height})$$

$$= \frac{1}{2}(7)(4)$$

$$= 14 \text{ square units}$$

To find the perimeter, use the Distance Formula.

STUDENT HELP

Skills Review
For help with simplifying radicals, see page 799.

$$EF = \sqrt{(4-8)^2 + (7-3)^2} = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2} \text{ units}$$

$$DF = \sqrt{(4-1)^2 + (7-3)^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \text{ units}$$

▶ So, the perimeter is $DE + EF + DF = (7 + 4\sqrt{2} + 5)$, or $12 + 4\sqrt{2}$, units.

GOAL 2 USING A PROBLEM-SOLVING PLAN

A problem-solving plan can help you organize solutions to geometry problems.

A PROBLEM-SOLVING PLAN

1. Ask yourself what you need to solve the problem. Write a **verbal model** or **draw a sketch** that will help you find what you need to know.
2. **Label known and unknown facts** on or near your sketch.
3. Use labels and facts to **choose related definitions**, **theorems**, **formulas**, or other results you may need.
4. **Reason logically** to link the facts, using a proof or other written argument.
5. Write a **conclusion** that answers the original problem. **Check** that your reasoning is correct.

EXAMPLE 4 *Using the Area of a Rectangle*

REAL LIFE **SOCCER FIELD** You have a part-time job at a school. You need to buy enough grass seed to cover the school's soccer field. The field is 50 yards wide and 100 yards long. The instructions on the seed bags say that one bag will cover 5000 square feet. How many bags do you need?

SOLUTION

Begin by rewriting the dimensions of the field in feet. Multiplying each of the dimensions by 3, you find that the field is 150 feet wide and 300 feet long.

PROBLEM SOLVING STRATEGY

VERBAL MODEL

$$\boxed{\text{Area of field}} = \boxed{\text{Bags of seed}} \cdot \boxed{\text{Coverage per bag}}$$

LABELS

Area of field = **150 · 300** (square feet)

Bags of seed = n (bags)

Coverage per bag = **5000** (square feet per bag)

REASONING

$150 \cdot 300 = n \cdot 5000$ **Write model for area of field.**

$\frac{150 \cdot 300}{5000} = n$ **Divide each side by 5000.**

$9 = n$ **Simplify.**

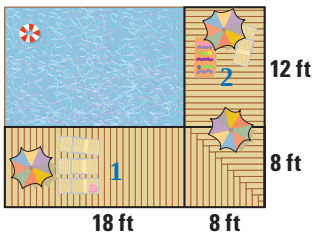
▶ You need 9 bags of seed.

✓**UNIT ANALYSIS** You can use *unit analysis* to verify the units of measure.

$$\text{ft}^2 = \text{bags} \cdot \frac{\text{ft}^2}{\text{bag}}$$

EXAMPLE 5 *Using the Area of a Square*

SWIMMING POOL You are planning a deck along two sides of a pool. The pool measures 18 feet by 12 feet. The deck is to be 8 feet wide. What is the area of the deck?



SOLUTION

PROBLEM SOLVING STRATEGY

DRAW A SKETCH From your diagram, you can see that the area of the deck can be represented as the sum of the areas of two rectangles and a square.

VERBAL MODEL

$$\boxed{\text{Area of deck}} = \boxed{\text{Area of rectangle 1}} + \boxed{\text{Area of rectangle 2}} + \boxed{\text{Area of square}}$$

LABELS

Area of deck $= A$	(square feet)
Area of rectangle 1 $= 8 \cdot 18$	(square feet)
Area of rectangle 2 $= 8 \cdot 12$	(square feet)
Area of square $= 8 \cdot 8$	(square feet)

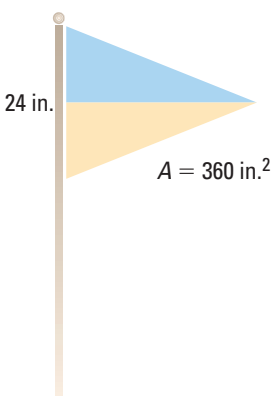
REASONING

$A = 8 \cdot 18 + 8 \cdot 12 + 8 \cdot 8$	Write model for deck area.
$= 304$	Simplify.

▶ The area of the deck is 304 square feet.

EXAMPLE 6 *Using the Area of a Triangle*

FLAG DESIGN You are making a triangular flag with a base of 24 inches and an area of 360 square inches. How long should it be?



SOLUTION

PROBLEM SOLVING STRATEGY

VERBAL MODEL

$$\boxed{\text{Area of flag}} = \frac{1}{2} \cdot \boxed{\text{Base of flag}} \cdot \boxed{\text{Length of flag}}$$

LABELS

Area of flag $= 360$	(square inches)
Base of flag $= 24$	(inches)
Length of flag $= L$	(inches)

REASONING

$360 = \frac{1}{2}(24)L$	Write model for flag area.
$360 = 12L$	Simplify.
$30 = L$	Divide each side by 12.

▶ The flag should be 30 inches long.

GUIDED PRACTICE

Vocabulary Check ✓

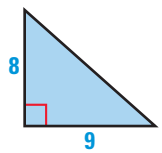
1. The perimeter of a circle is called its ___?___.

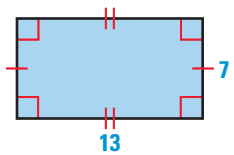
Concept Check ✓

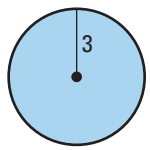
2. Explain how to find the perimeter of a rectangle.

Skill Check ✓

In Exercises 3–5, find the area of the figure. (Where necessary, use $\pi \approx 3.14$.)

3. 

4. 

5. 

6. The perimeter of a square is 12 meters. What is the length of a side of the square?

7. The radius of a circle is 4 inches. What is the circumference of the circle? (Use $\pi \approx 3.14$.)

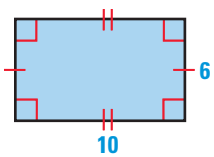
8. **FENCING** You are putting a fence around a rectangular garden with length 15 feet and width 8 feet. What is the length of the fence that you will need?

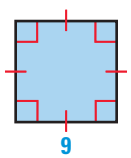
PRACTICE AND APPLICATIONS

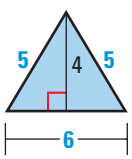
STUDENT HELP

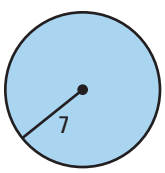
Extra Practice to help you master skills is on p. 804.

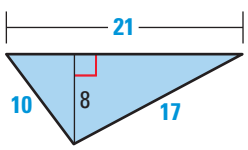
FINDING PERIMETER, CIRCUMFERENCE, AND AREA **Find the perimeter (or circumference) and area of the figure. (Where necessary, use $\pi \approx 3.14$.)**

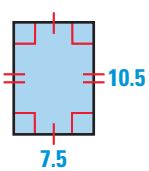
9. 

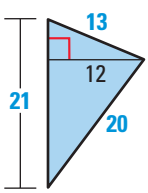
10. 

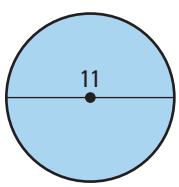
11. 

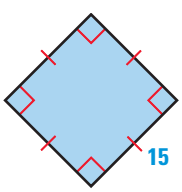
12. 

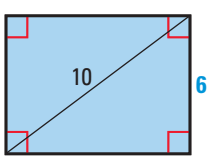
13. 

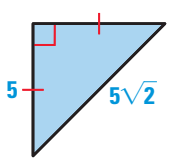
14. 

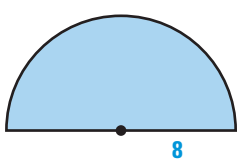
15. 

16. 

17. 

18. 

19. 

20. 

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 9–26
Example 2: Exs. 9–26
Example 3: Exs. 27–33
Example 4: Exs. 34–40
Example 5: Exs. 34–40
Example 6: Exs. 41–48

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 21–26.

FINDING AREA **Find the area of the figure described.**

21. Triangle with height 6 cm and base 5 cm

22. Rectangle with length 12 yd and width 9 yd

23. Square with side length 8 ft

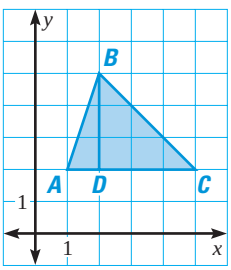
24. Circle with radius 10 m (Use $\pi \approx 3.14$.)

25. Square with perimeter 24 m

26. Circle with diameter 100 ft (Use $\pi \approx 3.14$.)

FINDING AREA **Find the area of the figure.**

27.



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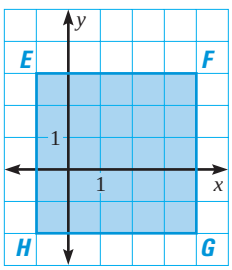

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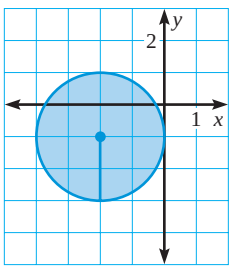


FINDING AREA **Draw the figure in a coordinate plane and find its area.**

30. Triangle defined by $A(3, 4)$, $B(7, 4)$, and $C(5, 7)$

31. Triangle defined by $R(-2, -3)$, $S(6, -3)$, and $T(5, 4)$

32. Rectangle defined by $L(-2, -4)$, $M(-2, 1)$, $N(7, 1)$, and $P(7, -4)$

33. Square defined by $W(5, 0)$, $X(0, 5)$, $Y(-5, 0)$, and $Z(0, -5)$

34. **CARPETING** How many square yards of carpet are needed to carpet a room that is 15 feet by 25 feet?

35. **WINDOWS** A rectangular pane of glass measuring 12 inches by 18 inches is surrounded by a wooden frame that is 2 inches wide. What is the area of the window, including the frame?

36. **MILLENNIUM DOME** The largest fabric dome in the world, the Millennium Dome covers a circular plot of land with a diameter of 320 meters. What is the circumference of the covered land? What is its area? (Use $\pi \approx 3.14$.)

37. **SPREADSHEET** Use a spreadsheet to show many different possible values of length and width for a rectangle with an area of 100 m^2. For each possible rectangle, calculate the perimeter. What are the dimensions of the rectangle with the smallest perimeter?

FOCUS ON APPLICATIONS

MILLENNIUM DOME
Built for the year 2000, this dome in Greenwich, England, is over 50 m tall and is covered by more than 100,000 square meters of fabric.

APPLICATION LINK
www.mcdougallittell.com

Perimeter of Rectangle

	A	B	C	D	E	F	G	H
1	Length	1.00	2.00	3.00	4.00	5.00	6.00	...
2	Width	100.00	50.00	33.33	25.00	20.00	16.67	...
3	Area	100.00	100.00	100.00	100.00	100.00	100.00	...
4	Perimeter	202.00	104.00	72.67	58.00	50.00	45.33	...
5								

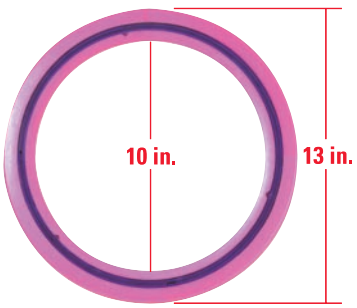
FOCUS ON APPLICATIONS

CRANBERRIES Cranberries were once called "bounceberries" because they bounce when they are ripe.

38. CRANBERRY HARVEST To harvest cranberries, the field is flooded so that the berries float. The berries are gathered with an inflatable boom. What area of cranberries can be gathered into a circular region with a radius of 5.5 meters? (Use $\pi \approx 3.14$.)

39. BICYCLES How many times does a bicycle tire that has a radius of 21 inches rotate when it travels 420 inches? (Use $\pi \approx 3.14$.)

40. FLYING DISC A plastic flying disc is circular and has a circular hole in the middle. If the diameter of the outer edge of the ring is 13 inches and the diameter of the inner edge of the ring is 10 inches, what is the area of plastic in the ring? (Use $\pi \approx 3.14$.)



LOGICAL REASONING Use the given measurements to find the unknown measurement. (Where necessary, use $\pi \approx 3.14$.)

41. A rectangle has an area of 36 in.^{2} and a length of 9 in. Find its perimeter.

42. A square has an area of 10,000 m^2. Find its perimeter.

43. A triangle has an area of 48 ft^2 and a base of 16 ft. Find its height.

44. A triangle has an area of 52 yd^2 and a height of 13 yd. Find its base.

45. A circle has an area of 200π cm^2. Find its radius.

46. A circle has an area of 1 m^2. Find its diameter.

47. A circle has a circumference of 100 yd. Find its area.

48. A right triangle has sides of length 4.5 cm, 6 cm, and 7.5 cm. Find its area.

Test Preparation

49. MULTI-STEP PROBLEM Use the following information.
Earth has a radius of about 3960 miles at the equator. Because there are 5280 feet in one mile, the radius of Earth is about 20,908,800 feet.

a. Suppose you could wrap a cable around Earth to form a circle that is snug against the ground. Find the length of the cable in feet by finding the circumference of Earth. (Assume that Earth is perfectly round. Use $\pi \approx 3.14$.)

b. Suppose you add 6 feet to the cable length in part (a). Use this length as the circumference of a new circle. Find the radius of the larger circle.

c. Use your results from parts (a) and (b) to find how high off of the ground the longer cable would be if it was evenly spaced around Earth.

d. Would the answer to part (c) be different on a planet with a different radius? Explain.

★ Challenge

50. DOUBLING A RECTANGLE'S SIDES The length and width of a rectangle are doubled. How do the perimeter and area of the new rectangle compare with the perimeter and area of the original rectangle? Illustrate your answer.

MIXED REVIEW

SKETCHING FIGURES **Sketch the points, lines, segments, and rays.** **(Review 1.2 for 2.1)**

51. Draw opposite rays using the points A, B, and C, with B as the initial point for both rays.

52. Draw four noncollinear points, W, X, Y, and Z, no three of which are collinear. Then sketch $\overleftrightarrow{XY}$, $\overrightarrow{YW}$, $\overline{XZ}$ and $\overleftrightarrow{ZY}$.

USING ALGEBRA **Plot the points in a coordinate plane and sketch $\angle DEF$. Classify the angle. Write the coordinates of one point in the interior of the angle and one point in the exterior of the angle.** **(Review 1.4)**

53. $D(2, -2)$, $E(4, -3)$, $F(6, -2)$

54. $D(0, 0)$, $E(-3, 0)$, $F(0, -2)$

55. $D(0, 1)$, $E(2, 3)$, $F(4, 1)$

56. $D(-3, -2)$, $E(3, -4)$, $F(1, 3)$

FINDING THE MIDPOINT **Find the coordinates of the midpoint of a segment with the given endpoints.** **(Review 1.5)**

57. $A(0, 0)$, $B(5, 3)$

58. $C(2, -3)$, $D(4, 4)$

59. $E(-3, 4)$, $F(-2, -1)$

60. $G(-2, 0)$, $H(-7, -6)$

61. $J(0, 5)$, $K(14, 1)$

62. $M(-44, 9)$, $N(6, -7)$

QUIZ 3

Self-Test for Lessons 1.6 and 1.7

In Exercises 1–4, find the measure of the angle. **(Lesson 1.6)**

1. Complement of $\angle A$; $m\angle A = 41°$

2. Supplement of $\angle B$; $m\angle B = 127°$

3. Supplement of $\angle C$; $m\angle C = 22°$

4. Complement of $\angle D$; $m\angle D = 35°$

5. $\angle A$ and $\angle B$ are complementary. The measure of $\angle A$ is five times the measure of $\angle B$. Find $m\angle A$ and $m\angle B$. **(Lesson 1.6)**

In Exercises 6–9, use the given information to find the unknown measurement. **(Lesson 1.7)**

6. Find the area and circumference of a circle with a radius of 18 meters. (Use $\pi \approx 3.14$.)

7. Find the area of a triangle with a base of 13 inches and a height of 11 inches.

8. Find the area and perimeter of a rectangle with a length of 10 centimeters and a width of 4.6 centimeters.

9. Find the area of a triangle defined by $P(-3, 4)$, $Q(7, 4)$, and $R(-1, 12)$.

10. **WALLPAPER** You are buying rolls of wallpaper to paper the walls of a rectangular room. The room measures 12 feet by 24 feet and the walls are 8 feet high. A roll of wallpaper contains 28 ft^2. About how many rolls of wallpaper will you need? **(Lesson 1.7)**

CHAPTER 1

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Find and describe patterns. **(1.1)**	Use a pattern to predict a figure or number in a sequence. **(p. 3)**
Use inductive reasoning. **(1.1)**	Make and verify conjectures such as a conjecture about the frequency of full moons. **(p. 5)**
Use defined and undefined terms. **(1.2)**	Understand the basic elements of geometry.
Sketch intersections of lines and planes. **(1.2)**	Visualize the basic elements of geometry and the ways they can intersect.
Use segment postulates and the Distance Formula. **(1.3)**	Solve real-life problems, such as finding the distance between two points on a map. **(p. 20)**
Use angle postulates and classify angles. **(1.4)**	Solve problems in geometry and in real life, such as finding the measure of the angle of vision for a horse wearing blinkers. **(p. 27)**
Bisect a segment and bisect an angle. **(1.5)**	Solve problems in geometry and in real life, such as finding an angle measure of a kite. **(p. 37)**
Identify vertical angles, linear pairs, complementary angles, and supplementary angles. **(1.6)**	Find the angle measures of geometric figures and real-life structures, such as intersecting metal supports of a stair railing. **(p. 45)**
Find the perimeter, circumference, and area of common plane figures. **(1.7)**	To solve problems related to measurement, such as finding the area of a deck for a pool. **(p. 54)**
Use a general problem-solving plan. **(1.7)**	To solve problems related to mathematics and real life, such as finding the number of bags of grass seed you need for a soccer field. **(p. 53)**

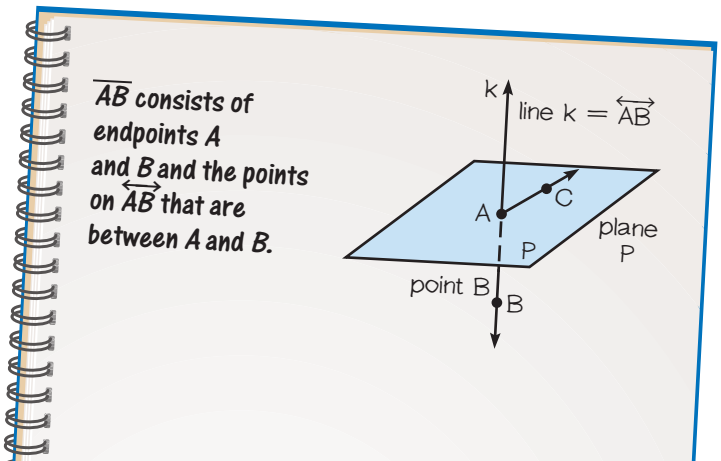
How does Chapter 1 fit into the BIGGER PICTURE of geometry?

In this chapter, you learned a basic reasoning skill—inductive reasoning. You also learned many fundamental terms—*point*, *line*, *plane*, *segment*, and *angle*, to name a few. Added to this were four basic postulates. These building blocks will be used throughout the remainder of this book to develop new terms, postulates, and theorems to explain the geometry of the world around you.

STUDY STRATEGY

How did you use your vocabulary pages?

The definitions of vocabulary terms you made, using the **Study Strategy** on page 2, may resemble this one.



CHAPTER 1

Chapter Review

VOCABULARY

- conjecture, p. 4
- inductive reasoning, p. 4
- counterexample, p. 4
- definition, undefined, p. 10
- point, line, plane, p. 10
- collinear, coplanar, p. 10
- line segment, p. 11
- endpoints, p. 11
- ray, p. 11
- initial point, p. 11
- opposite rays, p. 11
- intersect, intersection, p. 12
- postulates, or axioms, p. 17
- coordinate, p. 17
- distance, length, p. 17
- between, p. 18
- Distance Formula, p. 19
- congruent segments, p. 19
- angle, p. 26
- sides, vertex of an angle, p. 26
- congruent angles, p. 26
- measure of an angle, p. 27
- interior of an angle, p. 27
- exterior of an angle, p. 27
- acute, obtuse angles, p. 28
- right, straight angles, p. 28
- adjacent angles, p. 28
- midpoint, p. 34
- bisect, p. 34
- segment bisector, p. 34
- compass, straightedge, p. 34
- construct, construction, p. 34
- Midpoint Formula, p. 35
- angle bisector, p. 36
- vertical angles, p. 44
- linear pair, p. 44
- complementary angles, p. 46
- complement of an angle, p. 46
- supplementary angles, p. 46
- supplement of an angle, p. 46

1.1 PATTERNS AND INDUCTIVE REASONING

Examples on pp. 3–5

EXAMPLE Make a conjecture based on the results shown.

Conjecture: Given a 3-digit number, form a 6-digit number by repeating the digits. Divide the number by 7, then 11, then 13. The result is the original number.

$456{,}456 \div 7 \div 11 \div 13 = 456$
$562{,}562 \div 7 \div 11 \div 13 = 562$
$109{,}109 \div 7 \div 11 \div 13 = 109$

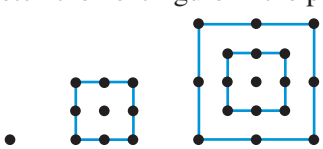
In Exercises 1–3, describe a pattern in the sequence of numbers.

1. 5, 12, 19, 26, 33, . . .

2. 0, 2, 6, 14, 30, . . .

3. 4, 12, 36, 108, 324, . . .

4. Sketch the next figure in the pattern.



5. Make a conjecture based on the results.

$4 \cdot 5 \cdot 6 \cdot 7 + 1 = 29 \cdot 29$
$5 \cdot 6 \cdot 7 \cdot 8 + 1 = 41 \cdot 41$
$6 \cdot 7 \cdot 8 \cdot 9 + 1 = 55 \cdot 55$

6. Show the conjecture is false by finding a counterexample:

Conjecture: *The cube of a number is always greater than the number.*

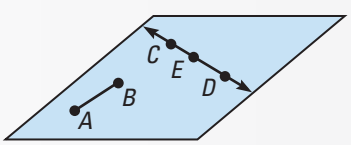
1.2 POINTS, LINES, AND PLANES

Examples on pp. 10–12

EXAMPLE

C, E, and D are collinear.
$\overleftrightarrow{CD}$ is a line. $\overline{AB}$ is a segment.

A, B, C, D, and E are coplanar.
$\overrightarrow{EC}$ and $\overrightarrow{ED}$ are opposite rays.



7. Draw five coplanar points, A, B, C, D, and E so that $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays, and $\overline{DE}$ intersects $\overleftrightarrow{AC}$ at B.
8. Sketch three planes that do not intersect.
9. Sketch two lines that are not coplanar and do not intersect.

1.3 SEGMENTS AND THEIR MEASURES

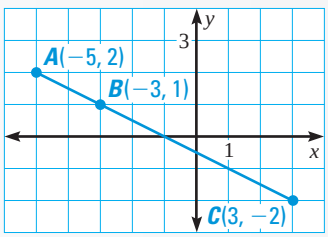
Examples on pp. 17–20

EXAMPLE B is between A and C, so $AB + BC = AC$.
Use the Distance Formula to find AB and BC.

$$AB = \sqrt{[-3-(-5)]^2 + (1-2)^2} = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$BC = \sqrt{[3-(-3)]^2 + (-2-1)^2} = \sqrt{6^2 + (-3)^2} = \sqrt{45}$$

Because $AB \neq BC$, $\overline{AB}$ and $\overline{BC}$ are *not* congruent segments.



10. Q is between P and S. R is between Q and S. S is between Q and T. $PT = 30$, $QS = 16$, and $PQ = QR = RS$. Find PQ, ST, and RP.

Use the Distance Formula to decide whether $\overline{PQ} \cong \overline{QR}$.

11. $P(-4, 3)$, $Q(-2, 1)$, $R(0, -1)$
12. $P(-3, 5)$, $Q(1, 3)$, $R(4, 1)$
13. $P(-2, -2)$, $Q(0, 1)$, $R(1, 4)$

1.4 ANGLES AND THEIR MEASURES

Examples on pp. 26–28

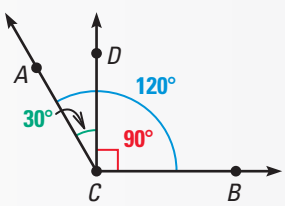
EXAMPLE

$m\angle ACD + m\angle DCB = m\angle ACB$

$\angle ACD$ is an acute angle: $m\angle ACD < 90°$.

$\angle DCB$ is a right angle: $m\angle DCB = 90°$.

$\angle ACB$ is an obtuse angle: $m\angle ACB > 90°$.

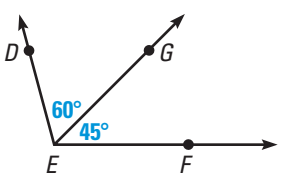


Classify the angle as *acute, right, obtuse,* or *straight*. Sketch the angle. Then use a protractor to check your results.

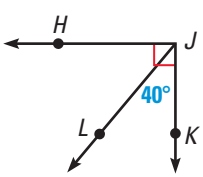
14. $m\angle KLM = 180°$
15. $m\angle A = 150°$
16. $m\angle Y = 45°$

Use the Angle Addition Postulate to find the measure of the unknown angle.

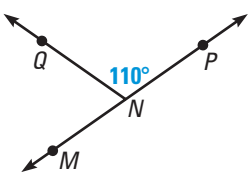
17. $m\angle DEF$



18. $m\angle HJL$



19. $m\angle QNM$

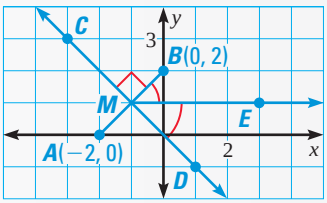


1.5 SEGMENT AND ANGLE BISECTORS

Examples on pp. 35–37

EXAMPLE If $\overleftrightarrow{CD}$ is a bisector of $\overline{AB}$, then $\overleftrightarrow{CD}$ intersects $\overline{AB}$ at its midpoint M: $M = \left(\frac{-2+0}{2}, \frac{0+2}{2}\right) = (-1, 1)$.

$\overrightarrow{ME}$ bisects $\angle BMD$, so $m\angle BME = m\angle EMD = 45°$.

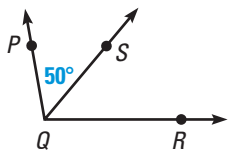


Find the coordinates of the midpoint of a segment with the given endpoints.

20. $A(0, 0)$, $B(-8, 6)$ **21.** $J(-1, 7)$, $K(3, -3)$ **22.** $P(-12, -9)$, $Q(2, 10)$

$\overrightarrow{QS}$ is the bisector of $\angle PQR$. Find any angle measures not given in the diagram.

23.



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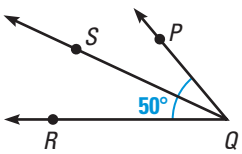

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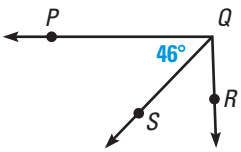
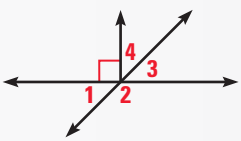


1.6 ANGLE PAIR RELATIONSHIPS

Examples on pp. 44–46

EXAMPLE $\angle 1$ and $\angle 3$ are vertical angles.
$\angle 1$ and $\angle 2$ are a linear pair and are supplementary angles.
$\angle 3$ and $\angle 4$ are complementary angles.



Use the diagram above to decide whether the statement is *always*, *sometimes*, or *never* true.

26. If $m\angle 2 = 115°$, then $m\angle 3 = 65°$.

27. $\angle 3$ and $\angle 4$ are congruent.

28. If $m\angle 1 = 40°$, then $m\angle 3 = 50°$.

29. $\angle 1$ and $\angle 4$ are complements.

1.7 INTRODUCTION TO PERIMETER, CIRCUMFERENCE, AND AREA

Examples on pp. 51–54

EXAMPLES A circle has diameter 24 ft.
Its circumference is $C = 2\pi r \approx 2(3.14)(12) = 75.36$ feet.
Its area is $A = \pi r^2 \approx 3.14(12^2) = 452.16$ square feet.

Find the perimeter (or circumference) and area of the figure described.

30. Rectangle with length 10 cm and width 4.5 cm

31. Circle with radius 9 in. (Use $\pi \approx 3.14$.)

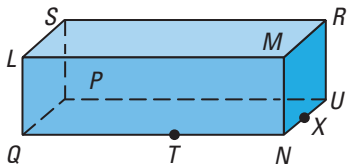
32. Triangle defined by $A(-6, 0)$, $B(2, 0)$, and $C(-2, -3)$

33. A square garden has sides of length 14 ft. What is its perimeter?

CHAPTER 1

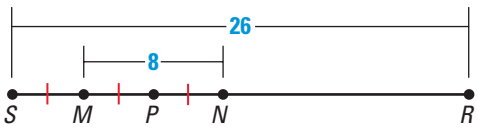
Chapter Test

Use the diagram to name the figures.



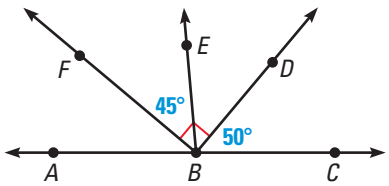
1. Three collinear points
2. Four noncoplanar points
3. Two opposite rays
4. Two intersecting lines
5. The intersection of plane *LMN* and plane *QLS*

Find the length of the segment.

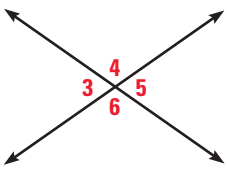


6. $\overline{MP}$
7. $\overline{SM}$
8. $\overline{NR}$
9. $\overline{MR}$

Find the measure of the angle.

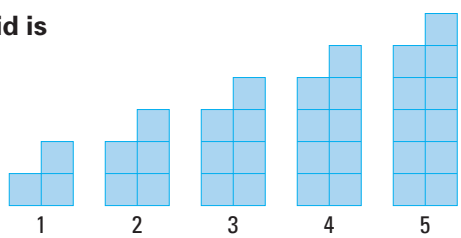


10. $\angle DBE$
11. $\angle FBC$
12. $\angle ABF$
13. $\angle DBA$



14. Refer to the diagram for Exercises 10–13. Name an obtuse angle, an acute angle, a right angle, and two complementary angles.
15. Q is between P and R. $PQ = 2w - 3$, $QR = 4 + w$, and $PR = 34$. Find the value of w. Then find the lengths of $\overline{PQ}$ and $\overline{QR}$.
16. $\overline{RT}$ has endpoints $R(-3, 8)$ and $T(3, 6)$. Find the coordinates of the midpoint, S, of $\overline{RT}$. Then use the Distance Formula to verify that $RS = ST$.
17. Use the diagram. If $m\angle 3 = 68°$, find the measures of $\angle 5$ and $\angle 4$.
18. Suppose $m\angle PQR = 130°$. If $\overrightarrow{QT}$ bisects $\angle PQR$, what is the measure of $\angle PQT$?

The first five figures in a pattern are shown. Each square in the grid is 1 unit × 1 unit.



19. Make a table that shows the distance around each figure at each stage.
20. Describe the pattern of the distances and use it to predict the distance around the figure at stage 20.

A center pivot irrigation system uses a fixed water supply to water a circular region of a field. The radius of the watering system is 560 feet long. (Use $\pi \approx 3.14$.)

21. If some workers walked around the circumference of the watered region, how far would they have to walk? Round to the nearest foot.
22. Find the area of the region watered. Round to the nearest square foot.

2.1 Conditional Statements

What you should learn

GOAL 1 Recognize and analyze a conditional statement.

GOAL 2 Write postulates about points, lines, and planes using conditional statements.

Why you should learn it

▼ Point, line, and plane postulates help you analyze **real-life** objects, such as the research buggy below and in **Ex. 54**.

REAL LIFE

Coastal Research Amphibious Buggy

GOAL 1 RECOGNIZING CONDITIONAL STATEMENTS

In this lesson you will study a type of logical statement called a conditional statement. A **conditional statement** has two parts, a *hypothesis* and a *conclusion*. When the statement is written in **if-then form**, the "if" part contains the **hypothesis** and the "then" part contains the **conclusion**. Here is an example:

$$\text{If } \underbrace{\textbf{it is noon in Georgia}}_{\text{Hypothesis}}, \text{ then } \underbrace{\textbf{it is 9 A.M. in California}}_{\text{Conclusion}}.$$

EXAMPLE 1 *Rewriting in If-Then Form*

Rewrite the conditional statement in *if-then form.*

a. Two points are collinear if they lie on the same line.

b. All sharks have a boneless skeleton.

c. A number divisible by 9 is also divisible by 3.

SOLUTION

a. If two points lie on the same line, then they are collinear.

b. If a fish is a shark, then it has a boneless skeleton.

c. If a number is divisible by 9, then it is divisible by 3.

Conditional statements can be either true or false. To show that a conditional statement is true, you must present an argument that the conclusion follows for *all* cases that fulfill the hypothesis. To show that a conditional statement is false, describe a single counterexample that shows the statement is not always true.

EXAMPLE 2 *Writing a Counterexample*

Write a counterexample to show that the following conditional statement is false.

$$\text{If } x^2 = 16, \text{ then } x = 4.$$

SOLUTION

As a counterexample, let $x = -4$. The hypothesis is true, because $(-4)^2 = 16$. However, the conclusion is false. This implies that the given conditional statement is false.

The **converse** of a conditional statement is formed by switching the hypothesis and conclusion. Here is an example.

Statement: If you see lightning, then you hear thunder.

Converse: If you hear thunder, then you see lightning.

EXAMPLE 3 *Writing the Converse of a Conditional Statement*

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

Write the converse of the following conditional statement.

Statement: If two segments are congruent, then they have the same length.

SOLUTION

Converse: If two segments have the same length, then they are congruent.

. . . . . .

A statement can be altered by **negation**, that is, by writing the negative of the statement. Here are some examples.

STATEMENT	NEGATION
$m\angle A = 30°$	$m\angle A \neq 30°$
$\angle A$ is acute.	$\angle A$ is not acute.

When you negate the hypothesis and conclusion of a conditional statement, you form the **inverse**. When you negate the hypothesis and conclusion of the converse of a conditional statement, you form the **contrapositive**.

Original	If $m\angle A = 30°$, then $\angle A$ is acute.
Inverse	If $m\angle A \neq 30°$, then $\angle A$ is not acute.
Converse	If $\angle A$ is acute, then $m\angle A = 30°$.
Contrapositive	If $\angle A$ is not acute, then $m\angle A \neq 30°$.

Both false **Both true**

When two statements are both true or both false, they are called **equivalent statements**. A conditional statement is equivalent to its contrapositive. Similarly, the inverse and converse of any conditional statement are equivalent. This is shown in the table above.

FOCUS ON APPLICATIONS

REAL LIFE **CROCUS** There are some exceptions to the statement in Example 4. For instance, crocuses can bloom when snow is on the ground.

EXAMPLE 4 *Writing an Inverse, Converse, and Contrapositive*

Write the (**a**) inverse, (**b**) converse, and (**c**) contrapositive of the statement.

If there is snow on the ground, then flowers are not in bloom.

SOLUTION

a. Inverse: If there is no snow on the ground, then flowers are in bloom.

b. Converse: If flowers are not in bloom, then there is snow on the ground.

c. Contrapositive: If flowers are in bloom, then there is no snow on the ground.

GOAL 2 USING POINT, LINE, AND PLANE POSTULATES

In Chapter 1, you studied four postulates.

Ruler Postulate	(Lesson 1.3, page 17)
Segment Addition Postulate	(Lesson 1.3, page 18)
Protractor Postulate	(Lesson 1.4, page 27)
Angle Addition Postulate	(Lesson 1.4, page 27)

Remember that postulates are assumed to be true—they form the foundation on which other statements (called *theorems*) are built.

STUDENT HELP

Study Tip
There is a list of all the postulates in this course at the end of the book beginning on page 827.

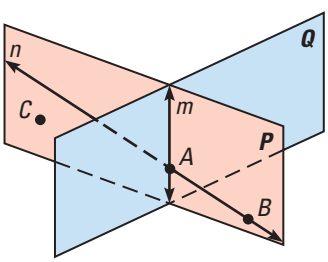
POINT, LINE, AND PLANE POSTULATES

POSTULATE 5	Through any two points there exists exactly one line.
POSTULATE 6	A line contains at least two points.
POSTULATE 7	If two lines intersect, then their intersection is exactly one point.
POSTULATE 8	Through any three noncollinear points there exists exactly one plane.
POSTULATE 9	A plane contains at least three noncollinear points.
POSTULATE 10	If two points lie in a plane, then the line containing them lies in the plane.
POSTULATE 11	If two planes intersect, then their intersection is a line.

EXAMPLE 5 *Identifying Postulates*

Logical Reasoning

Use the diagram at the right to give examples of Postulates 5 through 11.



SOLUTION

a. Postulate 5: There is exactly one line (line n) that passes through the points A and B.

b. Postulate 6: Line n contains at least two points. For instance, line n contains the points A and B.

c. Postulate 7: Lines m and n intersect at point A.

d. Postulate 8: Plane P passes through the noncollinear points A, B, and C.

e. Postulate 9: Plane P contains at least three noncollinear points, A, B, and C.

f. Postulate 10: Points A and B lie in plane P. So, line n, which contains points A and B, also lies in plane P.

g. Postulate 11: Planes P and Q intersect. So, they intersect in a line, labeled in the diagram as line m.

EXAMPLE 6 *Rewriting a Postulate*

a. Rewrite Postulate 5 in if-then form.

b. Write the inverse, converse, and contrapositive of Postulate 5.

SOLUTION

a. Postulate 5 can be rewritten in if-then form as follows:

> If two points are distinct, then there is exactly one line that passes through them.

b. **Inverse:** If two points are not distinct, then it is not true that there is exactly one line that passes through them.

Converse: If exactly one line passes through two points, then the two points are distinct.

Contrapositive: If it is not true that exactly one line passes through two points, then the two points are not distinct.

EXAMPLE 7 *Using Postulates and Counterexamples*

Logical Reasoning

Decide whether the statement is *true* or *false*. If it is false, give a counterexample.

a. A line can be in more than one plane.

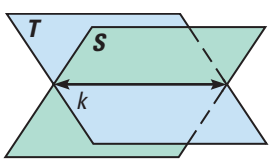
b. Four noncollinear points are always coplanar.

c. Two nonintersecting lines can be noncoplanar.

SOLUTION

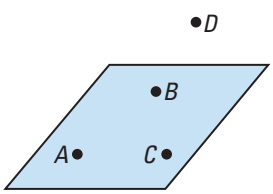
a. In the diagram at the right, line k is in plane S and line k is in plane T.

So, it is *true* that a line can be in more than one plane.



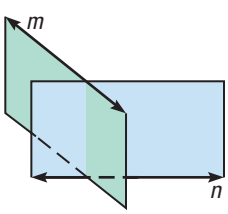
b. Consider the points A, B, C, and D at the right. The points A, B, and C lie in a plane, but there is no plane that contains all four points.

So, as shown in the counterexample at the right, it is *false* that four noncollinear points are always coplanar.



c. In the diagram at the right, line m and line n are nonintersecting and are also noncoplanar.

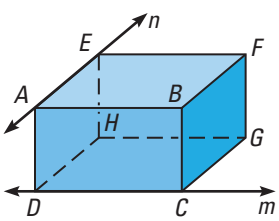
So, it is *true* that two nonintersecting lines can be noncoplanar.



STUDENT HELP

Study Tip

A box can be used to help visualize points and lines in space. For instance, the diagram shows that $\overleftrightarrow{AE}$ and $\overleftrightarrow{DC}$ are noncoplanar.



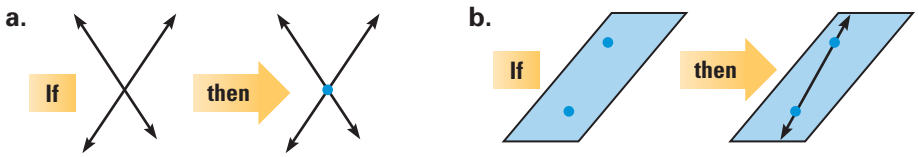
GUIDED PRACTICE

Vocabulary Check ✓

1. The ___?___ of a conditional statement is found by switching the hypothesis and conclusion.

Concept Check ✓

2. State the postulate described in each diagram.



Skill Check ✓

3. Write the hypothesis and conclusion of the statement, "If the dew point equals the air temperature, then it will rain."

In Exercises 4 and 5, write the statement in if-then form.

4. When threatened, the African ball python protects itself by coiling into a ball with its head in the middle.
5. The measure of a right angle is 90°.
6. Write the inverse, converse, and contrapositive of the conditional statement, "If a cactus is of the *cereus* variety, then its flowers open at night."

Decide whether the statement is *true* or *false*. Make a sketch to help you decide.

7. Through three noncollinear points there exists exactly one line.
8. If a line and a plane intersect, and the line does not lie in the plane, then their intersection is a point.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 805.

REWRITING STATEMENTS **Rewrite the conditional statement in if-then form.**

9. An object weighs one ton if it weighs 2000 pounds.
10. An object weighs 16 ounces if it weighs one pound.
11. Three points are collinear if they lie on the same line.
12. Blue trunkfish live in the waters of a coral reef.
13. Hagfish live in salt water.

STUDENT HELP

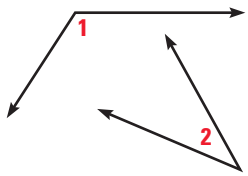
HOMEWORK HELP

Example 1: Exs. 9–13
Example 2: Exs. 14–17
Example 3: Exs. 18–21
Example 4: Exs. 46–52
Example 5: Exs. 25–34
Example 6: Exs. 22–24
Example 7: Exs. 35–38

ANALYZING STATEMENTS **Decide whether the statement is *true* or *false*. If false, provide a counterexample.**

14. A point may lie in more than one plane.
15. If x^4 equals 81, then x must equal 3.
16. If it is snowing, then the temperature is below freezing.
17. If four points are collinear, then they are coplanar.

WRITING CONVERSES **Write the converse of the statement.**



18. If $\angle 1$ measures 123°, then $\angle 1$ is obtuse.

19. If $\angle 2$ measures 38°, then $\angle 2$ is acute.

20. I will go to the mall if it is not raining.

21. I will go to the movies if it is raining.

REWRITING POSTULATES **Rewrite the postulate in if-then form. Then write the inverse, converse, and contrapositive of the conditional statement.**

22. A line contains at least two points.

23. Through any three noncollinear points there exists exactly one plane.

24. A plane contains at least three noncollinear points.

ILLUSTRATING POSTULATES **Fill in the blank. Then draw a sketch that helps illustrate your answer.**

25. If two lines intersect, then their intersection is ___?___ point(s).

26. Through any ___?___ points there exists exactly one line.

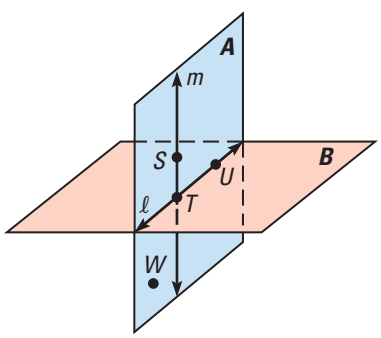
27. If two points lie in a plane, then the ___?___ containing them lies in the plane.

28. If two planes intersect, then their intersection is ___?___.

STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 29–34.

LINKING POSTULATES **Use the diagram to state the postulate(s) that verifies the truth of the statement.**



29. The points U and T lie on line ℓ.

30. Line ℓ contains points U and T.

31. The points W, S, and T lie in plane A.

32. The points S and T lie in plane A. Therefore, line m lies in plane A.

33. The planes A and B intersect in line ℓ.

34. Lines m and ℓ intersect at point T.

USING POSTULATES **In Exercises 35–38, state the postulate that shows that the statement is false.**

35. A line contains only one point.

36. Two planes intersect in exactly one point.

37. Three points, A, B, and C, are noncollinear, and two planes, M and N, each contain points A, B, and C.

38. Two points, P and Q, are collinear and two different lines, $\overleftrightarrow{RS}$ and $\overleftrightarrow{XY}$, each pass through points P and Q.

39. *Writing* Give an example of a true conditional statement with a true converse.

POINTS AND LINES IN SPACE **Think of the intersection of the ceiling and the front wall of your classroom as line *k*. Think of the center of the floor as point *A* and the center of the ceiling as point *B*.**

40. Is there more than one line that contains both points A and B?

41. Is there more than one plane that contains both points A and B?

42. Is there a plane that contains line k and point A?

43. Is there a plane that contains points A, B, and a point on the front wall?

USING ALGEBRA **Find the inverse, converse, and contrapositive of the statement.**

44. If $x = y$, then $5x = 5y$.

45. $6x - 6 = x + 14$ if $x = 4$.

QUOTES OF WISDOM **Rewrite the statement in if-then form. Then (a) determine the hypothesis and conclusion, and (b) find the inverse of the conditional statement.**

46. "If you tell the truth, you don't have to remember anything." — Mark Twain

47. "One can never consent to creep when one feels the impulse to soar." — Helen Keller

48. "Freedom is not worth having if it does not include the freedom to make mistakes." — Mahatma Ghandi

49. "Early to bed and early to rise, makes a man healthy, wealthy, and wise." — Benjamin Franklin

ADVERTISING **In Exercises 50–52, use the following advertising slogan: "You want a great selection of used cars? Come and see Bargain Bob's Used Cars!"**

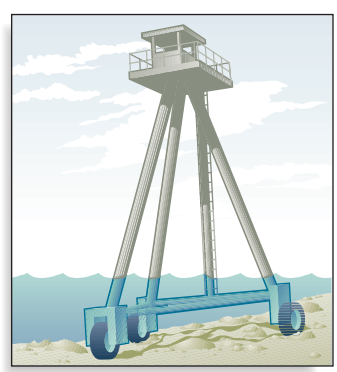
50. Write the slogan in if-then form. What are the hypothesis and conclusion of the conditional statement?

51. Write the inverse, converse, and contrapositive of the conditional statement.

52. *Writing* Find a real-life advertisement or slogan similar to the one given. Then repeat Exercises 50 and 51 using the advertisement or slogan.

53. **TECHNOLOGY** Use geometry software to draw a segment with endpoints A and C. Draw a third point B not on $\overline{AC}$. Measure $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. Move B closer to $\overline{AC}$ and observe the measures of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.

54. **RESEARCH BUGGY** The diagram at the right shows the 35 foot tall Coastal Research Amphibious Buggy, also known as CRAB. This vehicle moves along the ocean floor collecting data that are used to make an accurate map of the ocean floor. Using the postulates you have learned, make a conjecture about why the CRAB was built with three legs instead of four.



FOCUS ON CAREERS

REAL LIFE ADVERTISING COPYWRITER

Advertising copywriters write the advertisements you see and hear everyday. These ads appear in many forms including Internet home pages.

INTERNET CAREER LINK

www.mcdougallittell.com

Test Preparation

55. MULTIPLE CHOICE Use the conditional statement "If the measure of an angle is 44°, then the angle is acute" to decide which of the following are true.

I. The statement is true.
II. The converse of the statement is true.
III. The contrapositive of the statement is true.

Ⓐ I only Ⓑ II only Ⓒ I and II Ⓓ I and III Ⓔ I, II, and III

56. MULTIPLE CHOICE Which one of the following statements is *not* true?

Ⓐ If $x = 2$, then $x^2 = 4$.

Ⓑ If $x = -2$, then $x^2 = 4$.

Ⓒ If $x^3 = -8$, then $x = -2$.

Ⓓ If $x^2 = 4$, then $x = 2$.

Ⓔ If $x = -2$, then $x^3 = -8$.

★ Challenge

MAKING A CONJECTURE Sketch a line *k* and a point *P* not on line *k*. Make a conjecture about how many planes can be drawn through line *k* and point *P*, and then answer the following questions.

57. Which postulate allows you to state that there are two points, R and S, on line k?

58. Which postulate allows you to conclude that exactly one plane X can be drawn to contain points P, R, and S?

59. Which postulate guarantees that line k is contained in plane X?

60. Was your conjecture correct?

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

DRAWING ANGLES Plot the points in a coordinate plane. Then classify $\angle ABC$. (Review 1.4 for 2.2)

61. $A(0, 7)$, $B(2, 2)$, $C(6, -1)$

62. $A(-1, 0)$, $B(-6, 4)$, $C(-6, -1)$

63. $A(1, 3)$, $B(1, -5)$, $C(-5, -5)$

64. $A(-3, -1)$, $B(2, 5)$, $C(3, -2)$

FINDING THE MIDPOINT Find the coordinates of the midpoint of the segment joining the two points. (Review 1.5)

65. $A(-2, 8)$, $B(4, -12)$

66. $A(8, 8)$, $B(-6, 1)$

67. $A(-7, -4)$, $B(4, 7)$

68. $A(0, -9)$, $B(-8, 5)$

69. $A(1, 4)$, $B(11, -6)$

70. $A(-10, -10)$, $B(2, 12)$

FINDING PERIMETER AND AREA Find the area and perimeter (or circumference) of the figure described. (Use $\pi \approx 3.14$ when necessary.) (Review 1.7 for 2.2)

71. circle, radius = 6 m

72. square, side = 11 cm

73. square, side = 38.75 mm

74. circle, diameter = 23 ft

2.2 Definitions and Biconditional Statements

What you should learn

GOAL 1 Recognize and use definitions.

GOAL 2 Recognize and use biconditional statements.

Why you should learn it

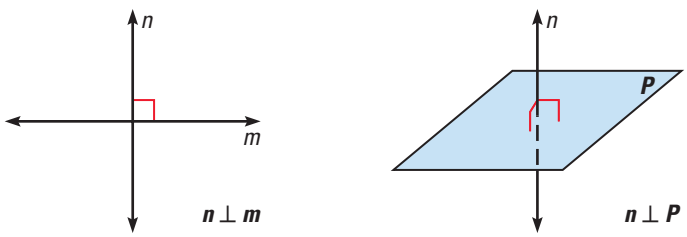
▼ You can use biconditional statements to help analyze geographic relations, such as whether three cities in Florida lie on the same line, as in **Ex. 50**.

REAL LIFE

GOAL 1 RECOGNIZING AND USING DEFINITIONS

In Lesson 1.2 you learned that a *definition* uses known words to describe a new word. Here are two examples.

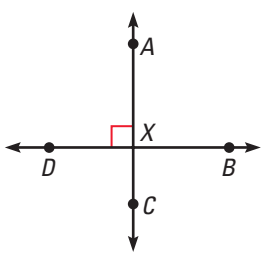
Two lines are called **perpendicular lines** if they intersect to form a right angle. A **line perpendicular to a plane** is a line that intersects the plane in a point and is perpendicular to every line in the plane that intersects it. The symbol $\perp$ is read as "is perpendicular to."



All definitions can be interpreted "forward" and "backward." For instance, the definition of perpendicular lines means (1) if two lines are perpendicular, then they intersect to form a right angle, *and* (2) if two lines intersect to form a right angle, then they are perpendicular.

EXAMPLE 1 *Using Definitions*

Decide whether each statement about the diagram is true. Explain your answer using the definitions you have learned.



a. Points D, X, and B are collinear.

b. $\overleftrightarrow{AC}$ is perpendicular to $\overleftrightarrow{DB}$.

c. $\angle AXB$ is adjacent to $\angle CXD$.

SOLUTION

a. This statement is true. Two or more points are *collinear* if they lie on the same line. The points D, X, and B all lie on line $\overleftrightarrow{DB}$ so they are collinear.

b. This statement is true. The right angle symbol in the diagram indicates that the lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{DB}$ intersect to form a right angle. So, the lines are perpendicular.

c. This statement is false. By definition, adjacent angles must share a common side. Because $\angle AXB$ and $\angle CXD$ do not share a common side, they are not adjacent.

STUDENT HELP

Study Tip
When a conditional statement contains the word "if," the hypothesis does not always follow the "if." This is shown in the "only-if" statement at the right.

GOAL 2 USING BICONDITIONAL STATEMENTS

Conditional statements are not always written in if-then form. Another common form of a conditional statement is *only-if* form. Here is an example.

$$\underbrace{\textbf{It is Saturday}}_{\textbf{Hypothesis}}\text{, only if }\underbrace{\textbf{I am working at the restaurant}}_{\textbf{Conclusion}}.$$

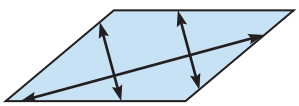
You can rewrite this conditional statement in if-then form as follows:

If **it is Saturday**, then **I am working at the restaurant**.

A **biconditional statement** is a statement that contains the phrase "if and only if." Writing a biconditional statement is equivalent to writing a conditional statement *and* its converse.

EXAMPLE 2 *Rewriting a Biconditional Statement*

The biconditional statement below can be rewritten as a conditional statement and its converse.



Three lines are coplanar if and only if they lie in the same plane.

Conditional statement: If three lines are coplanar, then they lie in the same plane.

Converse: If three lines lie in the same plane, then they are coplanar.

A biconditional statement can be either true or false. To be true, *both* the conditional statement and its converse must be true. This means that a true biconditional statement is true both "forward" and "backward." All definitions can be written as true biconditional statements.

EXAMPLE 3 *Analyzing a Biconditional Statement*

Using Algebra

Consider the following statement: $x = 3$ if and only if $x^2 = 9$.

a. Is this a biconditional statement?

b. Is the statement true?

SOLUTION

a. The statement is biconditional because it contains "if and only if."

b. The statement can be rewritten as the following statement and its converse.

Conditional statement: If $x = 3$, then $x^2 = 9$.

Converse: If $x^2 = 9$, then $x = 3$.

▶ The first of these statements is true, but the second is false. So, the biconditional statement is false.

Logical Reasoning

EXAMPLE 4 *Writing a Biconditional Statement*

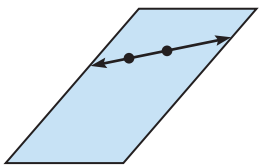
Each of the following statements is true. Write the converse of each statement and decide whether the converse is *true* or *false*. If the converse is true, combine it with the original statement to form a true biconditional statement. If the converse is false, state a counterexample.

a. If two points lie in a plane, then the line containing them lies in the plane.

b. If a number ends in 0, then the number is divisible by 5.

SOLUTION

a. **Converse:** If a line containing two points lies in a plane, then the points lie in the plane.



The converse is true, as shown in the diagram. So, it can be combined with the original statement to form the true biconditional statement written below.

Biconditional statement: Two points lie in a plane if and only if the line containing them lies in the plane.

b. **Converse:** If a number is divisible by 5, then the number ends in 0.

The converse is false. As a counterexample, consider the number 15. It is divisible by 5, but it does not end in 0, as shown at the right.

$10 \div 5 = 2$
▶ $\mathbf{15 \div 5 = 3}$
$20 \div 5 = 4$

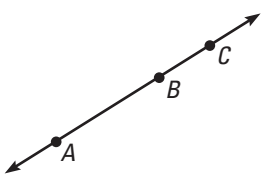
Knowing how to use true biconditional statements is an important tool for reasoning in geometry. For instance, if you can write a true biconditional statement, then you can use the conditional statement or the converse to justify an argument.

EXAMPLE 5 *Writing a Postulate as a Biconditional*

The second part of the Segment Addition Postulate is the converse of the first part. Combine the statements to form a true biconditional statement.

SOLUTION

The first part of the Segment Addition Postulate can be written as follows:



If B lies between points A and C, then $AB + BC = AC$.

The converse of this is as follows:

If $AB + BC = AC$, then B lies between A and C.

Combining these statements produces the following true biconditional statement:

Point B lies between points A and C if and only if $AB + BC = AC$.

STUDENT HELP

Study Tip

Unlike definitions, not all postulates can be written as true biconditional statements.

GUIDED PRACTICE

Vocabulary Check ✓ 1. Describe in your own words what a *true biconditional statement* is.

Concept Check ✓ 2. **ERROR ANALYSIS** What is wrong with Jared's argument below?

> ~~The statements "I eat cereal only if it is morning" and "If I eat cereal, then it is morning" are not equivalent.~~

Skill Check ✓ **Tell whether the statement is a biconditional.**

3. I will work after school only if I have the time.
4. An angle is called a right angle if and only if it measures 90°.
5. Two segments are congruent if and only if they have the same length.

Rewrite the biconditional statement as a conditional statement and its converse.

6. The ceiling fan runs if and only if the light switch is on.
7. You scored a touchdown if and only if the football crossed the goal line.
8. The expression $3x + 4$ is equal to 10 if and only if x is 2.

WINDOWS Decide whether the statement about the window shown is true. Explain your answer using the definitions you have learned.

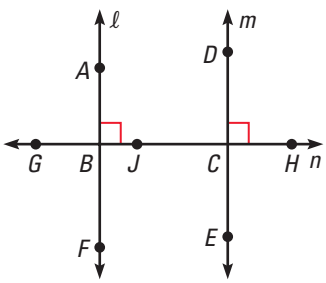
9. The points D, E, and F are collinear.
10. $m\angle CBA = 90°$
11. $\angle DBA$ and $\angle EBC$ are not complementary.
12. $\overline{DE} \perp \overline{AC}$

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 805.

PERPENDICULAR LINES Use the diagram to determine whether the statement is *true* or *false*.



13. Points A, F, and G are collinear.
14. $\angle DCJ$ and $\angle DCH$ are supplementary.
15. $\overline{DC}$ is perpendicular to line ℓ.
16. $\overline{FB}$ is perpendicular to line n.
17. $\angle FBJ$ and $\angle JBA$ are complementary.
18. Line m bisects $\angle JCH$.
19. $\angle ABJ$ and $\angle DCH$ are supplementary.

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 13–19
Example 2: Exs. 20–23
Example 3: Exs. 28–31
Example 4: Exs. 32–37
Example 5: Exs. 44–46

BICONDITIONAL STATEMENTS Rewrite the biconditional statement as a conditional statement and its converse.

20. Two angles are congruent if and only if they have the same measure.

21. A ray bisects an angle if and only if it divides the angle into two congruent angles.

22. Two lines are perpendicular if and only if they intersect to form right angles.

23. A point is a midpoint of a segment if and only if it divides the segment into two congruent segments.

FINDING COUNTEREXAMPLES Give a counterexample that demonstrates that the converse of the statement is false.

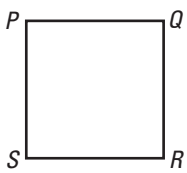
24. If an angle measures 94°, then it is obtuse.

25. If two angles measure 42° and 48°, then they are complementary.

26. If Terry lives in Tampa, then she lives in Florida.

27. If a polygon is a square, then it has four sides.

ANALYZING BICONDITIONAL STATEMENTS Determine whether the biconditional statement about the diagram is *true* or *false*. If false, provide a counterexample.



28. $\overline{SR}$ is perpendicular to $\overline{QR}$ if and only if $\angle SRQ$ measures 90°.

29. PQ and PS are equal if and only if PQ and PS are both 8 centimeters.

30. $\angle PQR$ and $\angle QRS$ are supplementary if and only if $m\angle PQR = m\angle QRS = 90°$.

31. $\angle PSR$ measures 90° if and only if $\angle PSR$ is a right angle.

FOCUS ON APPLICATIONS

REAL LIFE SNOW LEOPARDS The pale coat of the snow leopard, as mentioned in Ex. 37, allows the animal to blend in with the snow 3960 meters (13,000 feet) high in the mountains of Central Asia.

REWRITING STATEMENTS Rewrite the true statement in if-then form and write the converse. If the converse is true, combine it with the if-then statement to form a true biconditional statement. If the converse is false, provide a counterexample.

32. Adjacent angles share a common side.

33. Two circles have the same circumference if they have the same diameter.

34. The perimeter of a triangle is the sum of the lengths of its sides.

35. All leopards have spots.

36. Panthers live in the forest.

37. A leopard is a snow leopard if the leopard has pale gray fur.

USING ALGEBRA Determine whether the statement can be combined with its converse to form a true biconditional.

38. If $3u + 2 = u + 12$, then $u = 5$.

39. If $v = 1$, then $9v - 4v = 2v + 3v$.

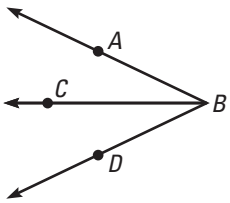
40. If $w^2 - 10 = w + 2$, then $w = 4$.

41. If $x^3 - 27 = 0$, then $x = 3$.

42. If $y = -3$, then $y^2 = 9$.

43. If $z = 3$, then $7 + 18z = 5z + 7 + 13z$.

44. REWRITING A POSTULATE Write the converse of the Angle Addition Postulate and decide whether the converse is *true* or *false*. If true, write the postulate as a true biconditional. If false, provide a counterexample.



Angle Addition Postulate: If C is in the interior of $\angle ABD$, then $m\angle ABC + m\angle CBD = m\angle ABD$.

45. *Writing* Give an example of a true biconditional statement.

46. MUSICAL GROUPS The table shows four different groups, along with the number of instrumentalists in each group. Write your own definitions of the musical groups and verify that they are true biconditional statements by writing each definition "forward" and "backward." The first one is started for you.

Sample: A musical group is a *piano trio* if and only if it contains exactly one pianist, one violinist, and one cellist.

Musical group	Pianist	Violinist	Cellist	Violist
Piano trio	1	1	1	—
String quartet	—	2	1	1
String quintet	—	2	1	2
Piano quintet	1	2	1	1

TECHNOLOGY In Exercises 47–49, use geometry software to complete the statement.

47. If the sides of a square are doubled, then the area is ___?___.

48. If the sides of a square are doubled, then the perimeter is ___?___.

49. Decide whether the statements in Exercises 47 and 48 can be written as true biconditionals. If not, provide a counterexample.

50. AIR DISTANCES The air distance between Jacksonville, Florida, and Merritt Island, Florida, is 148 miles and the air distance between Merritt Island and Fort Pierce, Florida, is 70 miles. Given that the air distance between Jacksonville and Fort Pierce is 218 miles, does Merritt Island fall on the line connecting Jacksonville and Fort Pierce?

WINDS AT SEA Use the portion of the Beaufort wind scale table shown to determine whether the biconditional statement is *true* or *false*. If false, provide a counterexample.

51. A storm is a hurricane if and only if the winds of the storm measure 64 knots or greater.

52. Winds at sea are classified as a strong gale if and only if the winds measure 34–40 knots.

53. Winds are classified as 10 on the Beaufort scale if and only if the winds measure 41–55 knots.

Beaufort Wind Scale for Open Sea		
Number	Knots	Description
8	34–40	gale winds
9	41–47	strong gale
10	48–55	storm
11	56–63	violent storm
12	64+	hurricane

FOCUS ON APPLICATIONS

REAL LIFE **WINDS AT SEA** Along with wind speed, sailors need to know the direction of the wind. Flags, also known as telltales, help sailors determine wind direction.

INTERNET APPLICATION LINK
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Test Preparation

54. MULTIPLE CHOICE Which one of the following statements cannot be written as a true biconditional statement?

(A) Any angle that measures between 90° and 180° is obtuse.

(B) $2x - 5 = x + 1$ only if $x = 6$.

(C) Any angle that measures between 0° and 90° is acute.

(D) If two angles measure 110° and 70°, then they are supplementary.

(E) If the sum of the measures of two angles equals 180°, then they are supplementary.

55. MULTIPLE CHOICE Which of the following statements about the conditional statement "If two lines intersect to form a right angle, then they are perpendicular" is true?

I. The converse is true.
II. The statement can be written as a true biconditional.
III. The statement is false.

(A) I only (B) I and II only (C) II and III only
(D) III only (E) I, II, and III

★ Challenge

WRITING STATEMENTS In Exercises 56 and 57, determine (a) whether the contrapositive of the true statement is *true* or *false* and (b) whether the true statement can be written as a true biconditional.

56. If I am in Des Moines, then I am in the capital of Iowa.

57. If two angles measure 10° and 80°, then they are complementary.

58. LOGICAL REASONING You are given that the contrapositive of a statement is true. Will that help you determine whether the statement can be written as a true biconditional? Explain. (*Hint:* Use your results from Exercises 56 and 57.)

EXTRA CHALLENGE
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MIXED REVIEW

STUDYING ANGLES Find the measures of a complement and a supplement of the angle. (Review 1.6 for 2.3)

59. 87° **60.** 73° **61.** 14° **62.** 29°

FINDING PERIMETER AND AREA Find the area and perimeter, or circumference of the figure described. (Use $\pi \approx 3.14$ when necessary.) (Review 1.7 for 2.3)

63. rectangle: $w = 3$ ft, $l = 12$ ft

64. rectangle: $w = 7$ cm, $l = 10$ cm

65. circle: $r = 8$ in.

66. square: $s = 6$ m

CONDITIONAL STATEMENTS Write the converse of the statement. (Review 2.1 for 2.3)

67. If the sides of a rectangle are all congruent, then the rectangle is a square.

68. If $8x + 1 = 3x + 16$, then $x = 3$.

2.3 Deductive Reasoning

What you should learn

GOAL 1 Use symbolic notation to represent logical statements.

GOAL 2 Form conclusions by applying the laws of logic to true statements, such as statements about a trip to Alabama in **Example 6**.

Why you should learn it

▼ The laws of logic help you with classification. For instance, the Law of Syllogism is used to determine true statements about birds in **Example 5**.

REAL LIFE

GOAL 1 USING SYMBOLIC NOTATION

In Lesson 2.1 you learned that a conditional statement has a hypothesis and a conclusion. Conditional statements can be written using symbolic notation, where p represents the hypothesis, q represents the conclusion, and $\rightarrow$ is read as "implies." Here are some examples.

$$\text{If } \underbrace{\textbf{the sun is out}}_{p}, \text{ then } \underbrace{\textbf{the weather is good}}_{q}.$$

This conditional statement can be written symbolically as follows:

$$\text{If } p, \text{ then } q \qquad \text{or} \qquad p \rightarrow q.$$

To form the converse of an "If p, then q" statement, simply switch p and q.

$$\text{If } \underbrace{\textbf{the weather is good}}_{q}, \text{ then } \underbrace{\textbf{the sun is out}}_{p}.$$

The converse can be written symbolically as follows:

$$\text{If } q, \text{ then } p \qquad \text{or} \qquad q \rightarrow p.$$

A biconditional statement can be written using symbolic notation as follows:

$$\text{If } p, \text{ then } q \text{ and if } q, \text{ then } p \qquad \text{or} \qquad p \leftrightarrow q.$$

Most often a biconditional statement is written in this form:

$$p \text{ if and only if } q.$$

EXAMPLE 1 *Using Symbolic Notation*

Let p be "the value of x is -5" and let q be "the absolute value of x is 5."

a. Write $p \rightarrow q$ in words.

b. Write $q \rightarrow p$ in words.

c. Decide whether the biconditional statement $p \leftrightarrow q$ is true.

SOLUTION

a. If the value of x is -5, then the absolute value of x is 5.

b. If the absolute value of x is 5, then the value of x is -5.

c. The conditional statement in part (a) is true, but its converse in part (b) is false. So, the biconditional statement $p \leftrightarrow q$ is false.

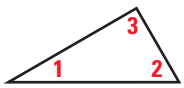
STUDENT HELP

Study Tip
The negation of a negative statement is a positive statement. This is shown at the right with the statement, "$\angle 3$ is not acute."

To write the inverse and contrapositive in symbolic notation, you need to be able to write the negation of a statement symbolically. The symbol for negation ($\sim$) is written before the letter. Here are some examples.

STATEMENT	SYMBOL	NEGATION	SYMBOL
$\angle 3$ measures 90°.	p	$\angle 3$ does not measure 90°.	$\sim p$
$\angle 3$ is not acute.	q	$\angle 3$ is acute.	$\sim q$

The inverse and contrapositive of $p \rightarrow q$ are as follows:



Inverse: $\sim p \rightarrow \sim q$

If $\angle 3$ does not measure 90°, then $\angle 3$ is acute.

Contrapositive: $\sim q \rightarrow \sim p$

If $\angle 3$ is acute, then $\angle 3$ does not measure 90°.

Notice that the inverse is false, but the contrapositive is true.

EXAMPLE 2 *Writing an Inverse and a Contrapositive*

STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

Let p be "it is raining" and let q be "the soccer game is canceled."

a. Write the contrapositive of $p \rightarrow q$.

b. Write the inverse of $p \rightarrow q$.

SOLUTION

a. Contrapositive: $\sim q \rightarrow \sim p$

If the soccer game is not canceled, then it is not raining.

b. Inverse: $\sim p \rightarrow \sim q$

If it is not raining, then the soccer game is not canceled.

. . . . . .

Recall from Lesson 2.1 that a conditional statement is equivalent to its contrapositive and that the converse and inverse are equivalent.

Equivalent Statements	Equivalent Statements
Conditional Statement $p \rightarrow q$ If the car will start, then the battery is charged.	**Converse** $q \rightarrow p$ If the battery is charged, then the car will start.
Contrapositive $\sim q \rightarrow \sim p$ If the battery is not charged, then the car will not start.	**Inverse** $\sim p \rightarrow \sim q$ If the car will not start, then the battery is not charged.

In the table above the conditional statement and its contrapositive are true. The converse and inverse are false. (Just because a car won't start does not imply that its battery is dead.)

GOAL 2 USING THE LAWS OF LOGIC

Deductive reasoning uses facts, definitions, and accepted properties in a logical order to write a **logical argument**. This differs from *inductive reasoning*, in which previous examples and patterns are used to form a conjecture.

EXAMPLE 3 *Using Inductive and Deductive Reasoning*

The following examples show how inductive and deductive reasoning differ.

a. Andrea knows that Robin is a sophomore and Todd is a junior. All the other juniors that Andrea knows are older than Robin. Therefore, Andrea reasons *inductively* that Todd is older than Robin based on past observations.

b. Andrea knows that Todd is older than Chan. She also knows that Chan is older than Robin. Andrea reasons *deductively* that Todd is older than Robin based on accepted statements.

There are two *laws of deductive reasoning*. The first is the **Law of Detachment**, shown below. The **Law of Syllogism** follows on the next page.

LAW OF DETACHMENT

If $p \rightarrow q$ is a true conditional statement and p is true, then q is true.

EXAMPLE 4 *Using the Law of Detachment*

State whether the argument is valid.

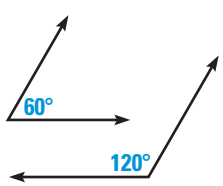
a. Jamal knows that if he misses the practice the day before a game, then he will not be a starting player in the game. Jamal misses practice on Tuesday so he concludes that he will not be able to start in the game on Wednesday.

b. If two angles form a linear pair, then they are supplementary; $\angle A$ and $\angle B$ are supplementary. So, $\angle A$ and $\angle B$ form a linear pair.

SOLUTION

a. This logical argument is a valid use of the Law of Detachment. It is given that both a statement ($p \rightarrow q$) and its hypothesis (p) are true. So, it is valid for Jamal to conclude that the conclusion (q) is true.

b. This logical argument is not a valid use of the Law of Detachment. Given that a statement ($p \rightarrow q$) and its conclusion (q) are true does not mean the hypothesis (p) is true. The argument implies that all supplementary angles form a linear pair.



The diagram shows that this is not a valid conclusion.

FOCUS ON CAREERS

ZOOLOGY Zoologists use facts about a bird's appearance to classify it. Ornithology is the branch of zoology that studies birds.

CAREER LINK
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LAW OF SYLLOGISM

If $p \rightarrow q$ and $q \rightarrow r$ are true conditional statements, then $p \rightarrow r$ is true.

EXAMPLE 5 *Using the Law of Syllogism*

ZOOLOGY Write some conditional statements that can be made from the following true statements using the Law of Syllogism.

1. If a bird is the fastest bird on land, then it is the largest of all birds.
2. If a bird is the largest of all birds, then it is an ostrich.
3. If a bird is a bee hummingbird, then it is the smallest of all birds.
4. If a bird is the largest of all birds, then it is flightless.
5. If a bird is the smallest bird, then it has a nest the size of a walnut half-shell.

SOLUTION

Here are the conditional statements that use the Law of Syllogism.

a. If a bird is the fastest bird on land, then it is an ostrich. (Use 1 and 2.)

b. If a bird is a bee hummingbird, then it has a nest the size of a walnut half-shell. (Use 3 and 5.)

c. If a bird is the fastest bird on land, then it is flightless. (Use 1 and 4.)

Logical Reasoning

EXAMPLE 6 *Using the Laws of Deductive Reasoning*

Over the summer, Mike visited Alabama. Given the following true statements, can you conclude that Mike visited the Civil Rights Memorial?

If Mike visits Alabama, then he will spend a day in Montgomery.

If Mike spends a day in Montgomery, then he will visit the Civil Rights Memorial.

Civil Rights Memorial in Montgomery, Alabama

SOLUTION

Let p, q, and r represent the following.

p: Mike visits Alabama.

q: Mike spends a day in Montgomery.

r: Mike visits the Civil Rights Memorial.

Because $p \rightarrow q$ is true and $q \rightarrow r$ is true, you can apply the Law of Syllogism to conclude that $p \rightarrow r$ is true.

If Mike visits Alabama, then he will visit the Civil Rights Memorial.

▶ You are told that Mike visited Alabama, which means p is true. Using the Law of Detachment, you can conclude that he visited the Civil Rights Memorial.

GUIDED PRACTICE

Vocabulary Check ✓

1. If the statements $p \rightarrow q$ and $q \rightarrow r$ are true, then the statement $p \rightarrow r$ is true by the Law of ___?___. If the statement $p \rightarrow q$ is true and p is true, then q is true by the Law of ___?___.

Concept Check ✓

2. State whether the following argument uses inductive or deductive reasoning: "If it is Friday, then Kendra's family has pizza for dinner. Today is Friday, therefore, Kendra's family will have pizza for dinner."

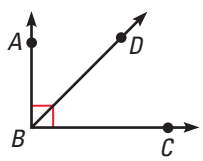
Skill Check ✓

3. Given the notation for a conditional statement is $p \rightarrow q$, what statement is represented by $q \rightarrow p$?

4. A conditional statement is defined in symbolic notation as $p \rightarrow q$. Use symbolic notation to write the inverse of $p \rightarrow q$.

5. Write the contrapositive of the following statement: "If you don't enjoy scary movies, then you wouldn't have liked this one."

6. If a ray bisects a right angle, then the congruent angles formed are complementary. In the diagram, $\angle ABC$ is a right angle. Are $\angle ABD$ and $\angle CBD$ complementary? Explain your reasoning.



7. If $f \rightarrow g$ and $g \rightarrow h$ are true statements, and f is true, does it follow that h is true? Explain.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 805.

WRITING STATEMENTS **Using *p* and *q* below, write the symbolic statement in words.**

p: Points X, Y, and Z are collinear.

q: Points X, Y, and Z lie on the same line.

8. $q \rightarrow p$
9. $\sim q$
10. $\sim p$
11. $\sim p \rightarrow \sim q$
12. $p \longleftrightarrow q$
13. $\sim q \rightarrow \sim p$

WRITING INVERSE AND CONTRAPOSITIVE **Given that the statement is of the form $p \rightarrow q$, write p and q. Then write the inverse and the contrapositive of $p \rightarrow q$ both symbolically and in words.**

14. If Jed gets a C on the exam, then he will get an A for the quarter.
15. If Alberto finds a summer job, then he will buy a car.
16. If the fuse has blown, then the light will not go on.
17. If the car is running, then the key is in the ignition.
18. If you dial 911, then there is an emergency.
19. If Gina walks to the store, then she will buy a newspaper.
20. If it is not raining, then Petra will ride her bike to school.

STUDENT HELP

HOMEWORK HELP

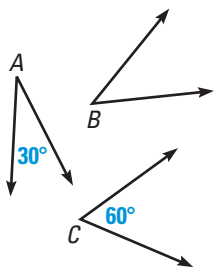
Example 1: Exs. 8–13
Example 2: Exs. 14–20
Example 3: Exs. 21, 22
Example 4: Exs. 23–25
Example 5: Exs. 30–48
Example 6: Exs. 30–48

LOGICAL REASONING **Decide whether *inductive* or *deductive* reasoning is used to reach the conclusion. Explain your reasoning.**

21. For the past three Wednesdays the cafeteria has served macaroni and cheese for lunch. Dana concludes that the cafeteria will serve macaroni and cheese for lunch this Wednesday.

22. If you live in Nevada and are between the ages of 16 and 18, then you must take driver's education to get your license. Marcus lives in Nevada, is 16 years old, and has his driver's license. Therefore, Marcus took driver's education.

USING THE LAW OF DETACHMENT **State whether the argument is valid. Explain your reasoning.**



23. If the sum of the measures of two angles is 90°, then the two angles are complementary. Because $m\angle A + m\angle C = 90°$, $\angle A$ and $\angle C$ are complementary.

24. If two adjacent angles form a right angle, then the two angles are complementary. Because $\angle A$ and $\angle C$ are complementary, $\angle A$ and $\angle C$ are adjacent.

25. If $\angle A$ and $\angle C$ are acute angles, then any angle whose measure is between the measures of $\angle A$ and $\angle C$ is also acute. In the diagram above it is shown that $m\angle A \leq m\angle B \leq m\angle C$, so $\angle B$ must be acute.

USING ALGEBRA **State whether any conclusions can be made using the true statement, given that $x = 3$.**

26. If $x > 2x - 10$, then $x = y$.

27. If $2x + 3 < 4x < 5x$, then $y \leq x$.

28. If $4x \geq 12$, then $y = 6x$.

29. If $x + 3 = 10$, then $y = x$.

MAKING CONCLUSIONS **Use the Law of Syllogism to write the statement that follows from the pair of true statements.**

30. If the sun is shining, then it is a beautiful day.

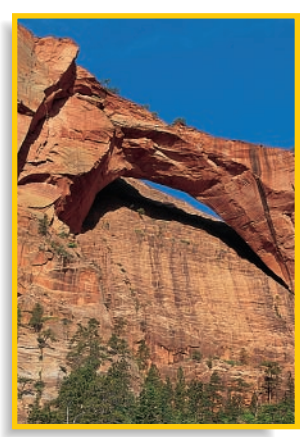
If it is a beautiful day, then we will have a picnic.

31. If the stereo is on, then the volume is loud.

If the volume is loud, then the neighbors will complain.

32. If Ginger goes to the movies, then Marta will go to the movies.

If Yumi goes to the movies, then Ginger will go to the movies.



The Kolob Arch mentioned in Ex. 34, spans 310 feet.

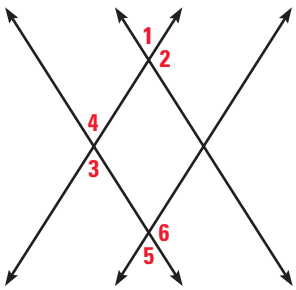
USING DEDUCTIVE REASONING **Select the word that makes the concluding statement true.**

33. The Oak Terrace apartment building does not allow dogs. Serena lives at Oak Terrace. So, Serena (must, may, may not) keep a dog.

34. The Kolob Arch is the world's widest natural arch. The world's widest arch is in Zion National Park. So, the Kolob Arch (is, may be, is not) in Zion.

35. Zion National Park is in Utah. Jeremy spent a week in Utah. So, Jeremy (must have, may have, never) visited Zion National Park.

USING THE LAWS OF LOGIC **In Exercises 36–42, use the diagram to give a reason for each true statement. In the diagram, $m\angle 2 = 115°$, $\angle 1 \cong \angle 4$, $\angle 3 \cong \angle 5$.**



36. p_1: $m\angle 2 = 115°$

37. $p_1 \rightarrow p_2$: If $m\angle 2 = 115°$, then $m\angle 1 = 65°$.

38. $p_2 \rightarrow p_3$: If $m\angle 1 = 65°$, then $m\angle 4 = 65°$.

39. $p_3 \rightarrow p_4$: If $m\angle 4 = 65°$, then $m\angle 3 = 65°$.

40. $p_4 \rightarrow p_5$: If $m\angle 3 = 65°$, then $m\angle 5 = 65°$.

41. $p_5 \rightarrow p_6$: If $m\angle 5 = 65°$, then $m\angle 6 = 115°$.

42. $p_1 \rightarrow p_6$: If $m\angle 2 = 115°$, then $m\angle 6 = 115°$.

43. *Writing* Describe a time in your life when you use deductive reasoning.

44. **CRITICAL THINKING** Describe an instance where inductive reasoning can lead to an incorrect conclusion.

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 45–48.

LOGICAL REASONING **In Exercises 45–48, use the true statements to determine whether the conclusion is *true* or *false*. Explain your reasoning.**

- If Diego goes shopping, then he will buy a pretzel.
- If the mall is open, then Angela and Diego will go shopping.
- If Angela goes shopping, then she will buy a pizza.
- The mall is open.

45. Diego bought a pretzel.

46. Angela and Diego went shopping.

47. Angela bought a pretzel.

48. Diego had some of Angela's pizza.

49. **ROBOTICS** Because robots can withstand higher temperatures than humans, a fire-fighting robot is under development. Write the following statements about the robot in order. Then use the Law of Syllogism to complete the statement, "If there is a fire, then __?__."

A. If the robot sets off a fire alarm, then it concludes there is a fire.

B. If the robot senses high levels of smoke and heat, then it sets off a fire alarm.

C. If the robot locates the fire, then the robot extinguishes the fire.

D. If there is a fire, then the robot senses high levels of smoke and heat.

E. If the robot concludes there is a fire, then it locates the fire.

50. **DOGS** Use the true statements to form other conditional statements.

A. If a dog is a gazehound, then it hunts by sight.

B. If a hound bays (makes long barks while hunting), then it is a scent hound.

C. If a dog is a foxhound, then it does not hunt primarily by sight.

D. If a dog is a coonhound, then it bays when it hunts.

E. If a dog is a greyhound, then it is a gazehound.

Test Preparation

51. MULTI-STEP PROBLEM Let p be "Jana wins the contest" and q be "Jana gets two free tickets to the concert."

a. Write $p \rightarrow q$ in words.

b. Write the converse of $p \rightarrow q$, both in words and symbols.

c. Write the contrapositive of $p \rightarrow q$, both in words and symbols.

d. Suppose Jana gets two free tickets to the concert but does not win the contest. Is this a counterexample to the converse or to the contrapositive?

e. What do you need to know about the conditional statement from part (a) so the Law of Detachment can be used to conclude that Jana gets two free tickets to the concert?

f. *Writing* Use the statement in part (a) to write a second statement that uses the Law of Syllogism to reach a valid conclusion.

★ Challenge

CONTRAPOSITIVES Use the true statements to answer the questions.

- If a creature is a fly, then it has six legs.
- If a creature has six legs, then it is an insect.

52. Use symbolic notation to describe the statements.

53. Use the statements and the Law of Syllogism to write a conditional statement, both in words and symbols.

54. Write the contrapositive of each statement, both in words and symbols.

55. Using the contrapositives and the Law of Syllogism, write a conditional statement. Is the statement true? Does the Law of Syllogism work for contrapositives?

EXTRA CHALLENGE
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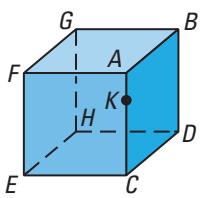
MIXED REVIEW

NAMING POINTS Use the diagram to name a point. (Review 1.2)



56. A third point collinear with A and C

57. A fourth point coplanar with A, C, and E

58. A point coplanar with A and B, but not coplanar with A, B, and C

59. A point coplanar with A and C, but not coplanar with E and F

FINDING ANGLE MEASURES Find $m\angle ABD$ given that $\angle ABC$ and $\angle CBD$ are adjacent angles. (Review 1.4 for 2.4)

60. $m\angle ABC = 20°$, $m\angle CBD = 10°$

61. $m\angle CBD = 13°$, $m\angle ABC = 28°$

62. $m\angle ABC = 3y + 1$, $m\angle CBD = 12 - y$

63. $m\angle CBD = 11 + 2f - g$, $m\angle ABC = 5g - 4 + f$

QUIZ 1

Self-Test for Lessons 2.1–2.3

Write the true statement in if-then form and write its converse. Determine whether the statement and its converse can be combined to form a true biconditional statement. (Lesson 2.1 and Lesson 2.2)

1. If today is June 4, then tomorrow is June 5.
2. A century is a period of 100 years.
3. Two circles are congruent if they have the same diameter.

LOGICAL REASONING **Use the true statements to answer the questions.** (Lesson 2.3)

- If John drives into the fence, then John's father will be angry.
- If John backs the car out, then John will drive into the fence.
- John backs the car out.

4. Does John drive into the fence?
5. Is John's father angry?

MATH & History — History of Recreational Logic Puzzles

APPLICATION LINK
www.mcdougallittell.com

THEN

IN THE 1600S, puzzles involving "formal" logic first became popular in Europe. However, logic has been a part of games such as mancala and chess for thousands of years.

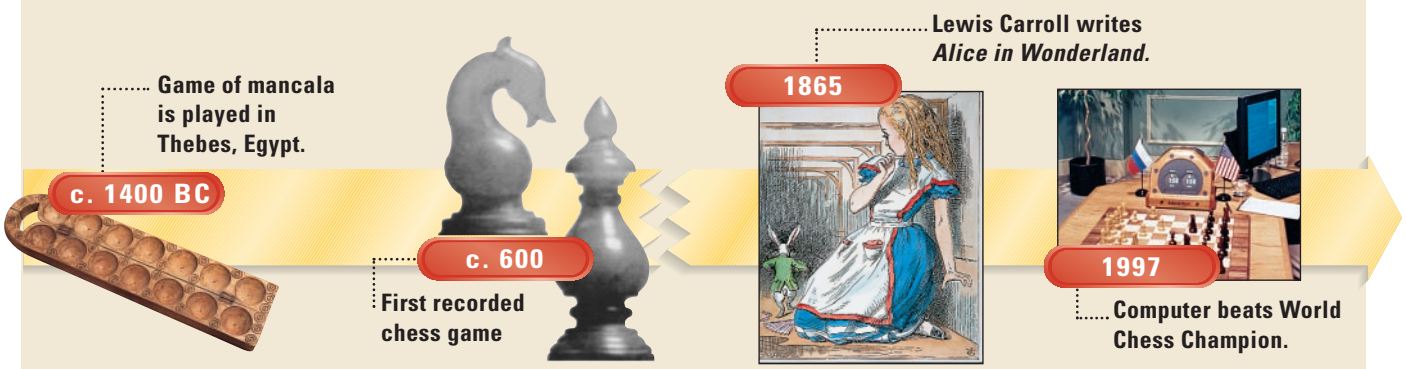
NOW

TODAY, logic games and puzzles are a popular pastime throughout the world. Lewis Carroll, author of *Alice in Wonderland*, was also a mathematician who wrote books on logic. The following problem is based on notes he wrote in his diary in the 1890s.

A says B lies; B says C lies; C says A and B lie.
Who is telling the truth? Who is lying?

Complete the exercises to solve the problem.

1. If A is telling the truth, then B is lying. What can you conclude about C's statement?
2. Assume A is telling the truth. Explain how this leads to a contradiction.
3. Who is telling the truth? Who is lying? How do you know? (*Hint*: For C to be lying, only one other person (A or B) must be telling the truth.)



2.4 Reasoning with Properties from Algebra

What you should learn

GOAL 1 Use properties from algebra.

GOAL 2 Use properties of length and measure to justify segment and angle relationships, such as the angles at the turns of a racetrack, as in **Example 5** and **Ex. 28**.

Why you should learn it

▼ Using algebraic properties helps you when rewriting a formula, such as the formula for an athlete's target heart rate in **Example 3**.

REAL LIFE

GOAL 1 USING PROPERTIES FROM ALGEBRA

Many properties from algebra concern the equality of real numbers. Several of these are summarized in the following list.

ALGEBRAIC PROPERTIES OF EQUALITY

Let a, b, and c be real numbers.

ADDITION PROPERTY	If $a = b$, then $a + c = b + c$.
SUBTRACTION PROPERTY	If $a = b$, then $a - c = b - c$.
MULTIPLICATION PROPERTY	If $a = b$, then $ac = bc$.
DIVISION PROPERTY	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
REFLEXIVE PROPERTY	For any real number a, $a = a$.
SYMMETRIC PROPERTY	If $a = b$, then $b = a$.
TRANSITIVE PROPERTY	If $a = b$ and $b = c$, then $a = c$.
SUBSTITUTION PROPERTY	If $a = b$, then a can be substituted for b in any equation or expression.

Properties of equality along with other properties from algebra, such as the *distributive property*,

$$a(b + c) = ab + ac$$

can be used to solve equations. For instance, you can use the subtraction property of equality to solve the equation $x + 3 = 7$. By subtracting 3 from each side of the equation, you obtain $x = 4$.

EXAMPLE 1 *Writing Reasons*

Using Algebra

Solve $5x - 18 = 3x + 2$ and write a reason for each step.

SOLUTION

$5x - 18 = 3x + 2$	**Given**
$2x - 18 = 2$	**Subtraction property of equality**
$2x = 20$	**Addition property of equality**
$x = 10$	**Division property of equality**

EXAMPLE 2 *Writing Reasons*

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

Solve $55z - 3(9z + 12) = -64$ and write a reason for each step.

SOLUTION

$55z - 3(9z + 12) = -64$	**Given**
$55z - 27z - 36 = -64$	**Distributive property**
$28z - 36 = -64$	**Simplify.**
$28z = -28$	**Addition property of equality**
$z = -1$	**Division property of equality**

EXAMPLE 3 *Using Properties in Real Life*

REAL LIFE **FITNESS** Before exercising, you should find your target heart rate. This is the rate at which you achieve an effective workout while not placing too much strain on your heart. Your target heart rate r (in beats per minute) can be determined from your age a (in years) using the equation $a = 220 - \frac{10}{7}r$.

a. Solve the formula for r and write a reason for each step.

b. Use the result to find the target heart rate for a 16 year old.

c. Find the target heart rate for the following ages: 20, 30, 40, 50, and 60. What happens to the target heart rate as a person gets older?

SOLUTION

a.

$a = 220 - \frac{10}{7}r$	**Given**
$a + \frac{10}{7}r = 220$	**Addition property of equality**
$\frac{10}{7}r = 220 - a$	**Subtraction property of equality**
$r = \frac{7}{10}(220 - a)$	**Multiplication property of equality**

b. Using $a = 16$, the target heart rate is:

$r = \frac{7}{10}(220 - a)$	**Given**
$r = \frac{7}{10}(220 - 16)$	**Substitute 16 for *a*.**
$r = 142.8$	**Simplify.**

▶ The target heart rate for a 16 year old is about 143 beats per minute.

c. From the table, the target heart rate appears to decrease as a person ages.

Age	20	30	40	50	60
Rate	140	133	126	119	112

GOAL 2 USING PROPERTIES OF LENGTH AND MEASURE

The algebraic properties of equality can be used in geometry.

CONCEPT SUMMARY — PROPERTIES OF EQUALITY

	SEGMENT LENGTH	ANGLE MEASURE
REFLEXIVE	For any segment AB, $AB = AB$.	For any angle A, $m\angle A = m\angle A$.
SYMMETRIC	If $AB = CD$, then $CD = AB$.	If $m\angle A = m\angle B$, then $m\angle B = m\angle A$.
TRANSITIVE	If $AB = CD$ and $CD = EF$, then $AB = EF$.	If $m\angle A = m\angle B$ and $m\angle B = m\angle C$, then $m\angle A = m\angle C$.

EXAMPLE 4 *Using Properties of Length*

Logical Reasoning

In the diagram, $AB = CD$. The argument below shows that $AC = BD$.

$AB = CD$	**Given**
$AB + BC = BC + CD$	**Addition property of equality**
$AC = AB + BC$	**Segment Addition Postulate**
$BD = BC + CD$	**Segment Addition Postulate**
$AC = BD$	**Substitution property of equality**

EXAMPLE 5 *Using Properties of Measure*

FOCUS ON APPLICATIONS

AUTO RACING The Talladega Superspeedway racetrack in Alabama has four banked turns, which are described in the diagram at the left. Use the given information about the maximum banking angle of the four turns to find $m\angle 4$.

$m\angle 1 + m\angle 2 = 66°$

$m\angle 1 + m\angle 2 + m\angle 3 = 99°$

$m\angle 3 = m\angle 1$

$m\angle 1 = m\angle 4$

REAL LIFE AUTO RACING Banked turns help the cars travel around the track at high speeds. The angles provide an inward force that helps keep the cars from flying off the track.

SOLUTION

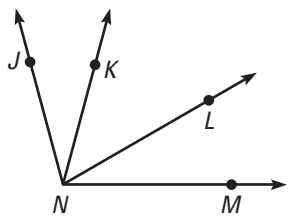
$m\angle 1 + m\angle 2 = 66°$	**Given**
$m\angle 1 + m\angle 2 + m\angle 3 = 99°$	**Given**
$66° + m\angle 3 = 99°$	**Substitution property of equality**
$m\angle 3 = 33°$	**Subtraction property of equality**
$m\angle 3 = m\angle 1, m\angle 1 = m\angle 4$	**Given**
$m\angle 3 = m\angle 4$	**Transitive property of equality**
$m\angle 4 = 33°$	**Substitution property of equality**

GUIDED PRACTICE

Vocabulary Check ✓

1. Name the property that makes the following statement true: "If $m\angle 3 = m\angle 5$, then $m\angle 5 = m\angle 3$."

Concept Check ✓ **Use the diagram at the right.**



2. Explain how the addition property of equality supports this statement: "If $m\angle JNK = m\angle LNM$, then $m\angle JNL = m\angle KNM$."

3. Explain how the subtraction property of equality supports this statement: "If $m\angle JNL = m\angle KNM$, then $m\angle JNK = m\angle LNM$."

Skill Check ✓ **In Exercises 4–8, match the conditional statement with the property of equality.**

4. If $JK = PQ$ and $PQ = ST$, then $JK = ST$.
5. If $m\angle S = 30°$, then $5° + m\angle S = 35°$.
6. If $ST = 2$ and $SU = ST + 3$, then $SU = 5$.
7. If $m\angle K = 45°$, then $3(m\angle K) = 135°$.
8. If $m\angle P = m\angle Q$, then $m\angle Q = m\angle P$.

A. Addition property
B. Substitution property
C. Transitive property
D. Symmetric property
E. Multiplication property

9. **WIND-CHILL FACTOR** If the wind is blowing at 20 miles per hour, you can find the wind-chill temperature W (in degrees Fahrenheit) by using the equation $W = 1.42T - 38.5$, where T is the actual temperature (in degrees Fahrenheit). Solve this equation for T and write a reason for each step. What is the actual temperature if the wind chill temperature is $-24.3°$F and the wind is blowing at 20 miles per hour?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 806.

COMPLETING STATEMENTS **In Exercises 10–14, use the property to complete the statement.**

10. Symmetric property of equality: If $m\angle A = m\angle B$, then __?__.
11. Transitive property of equality: If $BC = CD$ and $CD = EF$, then __?__.
12. Substitution property of equality: If $LK + JM = 12$ and $LK = 2$, then __?__.
13. Subtraction property of equality: If $PQ + ST = RS + ST$, then __?__.
14. Division property of equality: If $3(m\angle A) = 90°$, then $m\angle A =$ __?__.
15. Copy and complete the argument below, giving a reason for each step.

$2(3x + 1) = 5x + 14$	**Given**
$6x + 2 = 5x + 14$	__?__
$x + 2 = 14$	__?__
$x = 12$	__?__

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 10–23
Example 2: Exs. 15–23
Example 3: Exs. 29–31
Example 4: Exs. 24–27
Example 5: Ex. 28

SOLVING EQUATIONS **In Exercises 16–23, solve the equation and state a reason for each step.**

16. $p - 1 = 6$

17. $q + 9 = 13$

18. $2r - 7 = 9$

19. $7s + 20 = 4s - 13$

20. $3(2t + 9) = 30$

21. $-2(-w + 3) = 15$

22. $26u + 4(12u - 5) = 128$

23. $3(4v - 1) - 8v = 17$

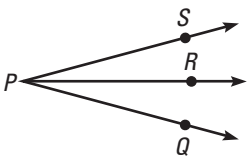
24. **LOGICAL REASONING** In the diagram, $m\angle RPQ = m\angle RPS$. Verify each step in the argument that shows $m\angle SPQ = 2(m\angle RPQ)$.

$m\angle RPQ = m\angle RPS$

$m\angle SPQ = m\angle RPQ + m\angle RPS$

$m\angle SPQ = m\angle RPQ + m\angle RPQ$

$m\angle SPQ = 2(m\angle RPQ)$



25. **LOGICAL REASONING** In the diagram, $m\angle ABF = m\angle BCG$ and $m\angle ABF = 90°$. Verify each step in the argument that shows $\overleftrightarrow{GK} \perp \overleftrightarrow{AD}$.

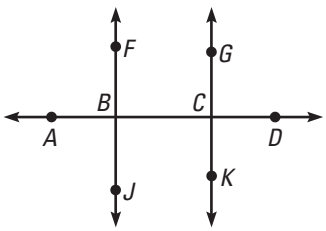
$m\angle ABF = 90°$

$m\angle ABF = m\angle BCG$

$m\angle BCG = 90°$

$\angle BCG$ is a right angle.

$\overleftrightarrow{GK} \perp \overleftrightarrow{AD}$



DEVELOPING ARGUMENTS **In Exercises 26 and 27, give an argument for the statement, including a reason for each step.**

26. If $\angle 1$ and $\angle 2$ are right angles, then they are supplementary.

27. If B lies between A and C and $AB = 3$ and $BC = 8$, then $AC = 11$.

28. **AUTO RACING** Some facts about the maximum banking angles of Daytona International Speedway at corners 1, 2, 3, and 4 are at the right. Find $m\angle 3$. Explain your steps. (Banked corners are described on page 98.)

$m\angle 1 + m\angle 3 + m\angle 4 = 93°$

$m\angle 2 + m\angle 4 = 62°$

$m\angle 2 = m\angle 3$

$m\angle 1 = m\angle 2$

FOCUS ON PEOPLE

REAL LIFE **BILL ELLIOTT** holds the qualifying record at Daytona International Speedway with a speed of 210.364 miles per hour.

PAY RAISES **In Exercises 29–31, suppose you receive a raise at work. You can calculate your percent increase by using the pay raise formula $c(r + 1) = n$, where c is your current wage (in dollars per hour), r is your percent increase (as a decimal), and n is your new wage (in dollars per hour).**

29. Solve the formula for r and write a reason for each step.

30. Use the result from Exercise 29 to find your percent increase if your current wage is \$10.00 and your new wage will be \$10.80.

31. Suppose Donald gets a 6% pay raise and his new wage is \$12.72. Find Donald's old wage. Explain the steps you used to find your answer.

Test Preparation

32. MULTI-STEP PROBLEM State a reason that makes the statement true.

a. If $4(x - 5 + 2x) = 0.5(12x - 16)$, then $4x - 20 + 8x = 6x - 8$.

b. If $4x - 20 + 8x = 6x - 8$, then $12x - 20 = 6x - 8$.

c. If $12x - 20 = 6x - 8$, then $6x - 20 = -8$.

d. If $6x - 20 = -8$, then $6x = 12$.

e. If $6x = 12$, then $x = 2$.

f. *Writing* Use parts (a) through (e) to provide an argument for "If $4(x - 5 + 2x) = 0.5(12x - 16)$, then $x = 2$."

★ Challenge

DETERMINING PROPERTIES Decide whether the relationship is *reflexive, symmetric,* or *transitive*. When the relationship does not have any of these properties, give a counterexample.

33. **Set:** students in a geometry class
Relationship: "earned the same grade as"
Example: Jim earned the same grade as Mario.

34. **Set:** letters of the alphabet
Relationship: "comes after"
Example: H comes after G.

EXTRA CHALLENGE
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MIXED REVIEW

USING THE DISTANCE FORMULA Find the distance between the two points. Round your result to two decimal places. (Review 1.3 for 2.5)

35. $A(4, 5), B(-3, -2)$ **36.** $E(-7, 6), F(2, 0)$ **37.** $J(1, 1), K(-1, 11)$

38. $P(8, -4), Q(1, -4)$ **39.** $S(9, -1), T(2, -6)$ **40.** $V(7, 10), W(1, 5)$

DETERMINING ENDPOINTS In Exercises 41–44, you are given an endpoint and the midpoint of a line segment. Find the coordinates of the other endpoint. Each midpoint is denoted by *M*(*x, y*). (Review 1.5 for 2.5)

41. $B(5, 7)$, $M(-1, 0)$ **42.** $C(-4, -5)$, $M(3, -6)$ **43.** $F(0, 9)$, $M(6, -2)$ **44.** $Q(-1, 14)$, $M(2, 7)$

45. Given that $m\angle A = 48°$, what are the measures of a complement and a supplement of $\angle A$? (Review 1.6)

ANALYZING STATEMENTS Use the diagram shown at the right to determine whether the statement is *true* or *false*. (Review 2.2)

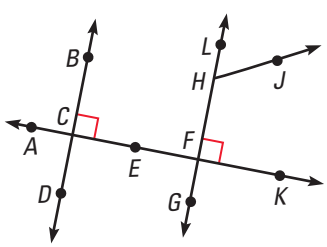
46. Points G, L, and J are collinear.

47. $\overline{BC} \perp \overline{FG}$

48. $\angle ECB \cong \angle ACD$

49. $\angle JHL$ and $\angle JHF$ are complementary.

50. $\overleftrightarrow{AK} \perp \overleftrightarrow{BD}$



2.5 Proving Statements about Segments

What you should learn

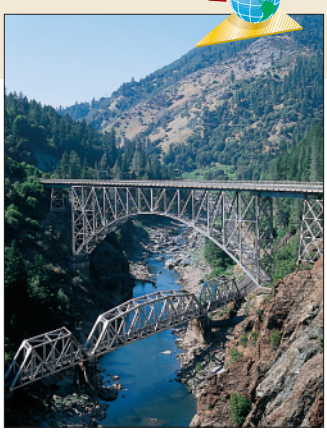
GOAL 1 Justify statements about congruent segments.

GOAL 2 Write reasons for steps in a proof.

Why you should learn it

▼ Properties of congruence allow you to justify segment relationships in **real life**, such as the segments in the trestle bridge shown and in **Exs. 3–5**.

REAL LIFE



GOAL 1 PROPERTIES OF CONGRUENT SEGMENTS

A true statement that follows as a result of other true statements is called a **theorem**. All theorems must be proved. You can prove a theorem using a *two-column proof*. A **two-column proof** has numbered statements and reasons that show the logical order of an argument.

THEOREM

THEOREM 2.1 ***Properties of Segment Congruence***

Segment congruence is reflexive, symmetric, and transitive. Here are some examples:

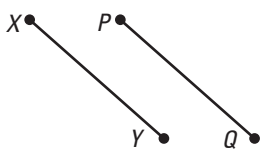
REFLEXIVE For any segment AB, $\overline{AB} \cong \overline{AB}$.

SYMMETRIC If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.

TRANSITIVE If $\overline{AB} \cong \overline{CD}$, and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.

EXAMPLE 1 *Symmetric Property of Segment Congruence*

You can prove the Symmetric Property of Segment Congruence as follows.



GIVEN ▶ $\overline{PQ} \cong \overline{XY}$

PROVE ▶ $\overline{XY} \cong \overline{PQ}$

Statements	Reasons
1. $\overline{PQ} \cong \overline{XY}$	1. Given
2. $PQ = XY$	2. Definition of congruent segments
3. $XY = PQ$	3. Symmetric property of equality
4. $\overline{XY} \cong \overline{PQ}$	4. Definition of congruent segments

STUDENT HELP

Study Tip

When writing a reason for a step in a proof, you must use one of the following: given information, a definition, a property, a postulate, or a previously proven theorem.

You are asked to complete proofs for the Reflexive and Transitive Properties of Segment Congruence in Exercises 6 and 7.

A proof can be written in paragraph form, called **paragraph proof**. Here is a paragraph proof for the Symmetric Property of Segment Congruence.

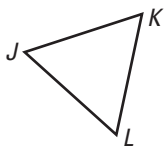
Paragraph Proof You are given that $\overline{PQ} \cong \overline{XY}$. By the definition of congruent segments, $PQ = XY$. By the symmetric property of equality, $XY = PQ$. Therefore, by the definition of congruent segments, it follows that $\overline{XY} \cong \overline{PQ}$.

GOAL 2 USING CONGRUENCE OF SEGMENTS

EXAMPLE 2 *Using Congruence*

Proof

Use the diagram and the given information to complete the missing steps and reasons in the proof.



GIVEN ▶ $LK = 5$, $JK = 5$, $\overline{JK} \cong \overline{JL}$

PROVE ▶ $\overline{LK} \cong \overline{JL}$

Statements	Reasons
1. __a.__	**1.** Given
2. __b.__	**2.** Given
3. $LK = JK$	**3.** Transitive property of equality
4. $\overline{LK} \cong \overline{JK}$	**4.** __c.__
5. $\overline{JK} \cong \overline{JL}$	**5.** Given
6. __d.__	**6.** Transitive Property of Congruence

SOLUTION

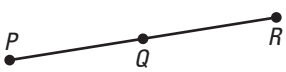
a. $LK = 5$ **b.** $JK = 5$ **c.** Definition of congruent segments **d.** $\overline{LK} \cong \overline{JL}$

EXAMPLE 3 *Using Segment Relationships*

Proof

In the diagram, Q is the midpoint of $\overline{PR}$.

Show that PQ and QR are each equal to $\frac{1}{2}PR$.



SOLUTION

Decide what you know and what you need to prove. Then write the proof.

GIVEN ▶ Q is the midpoint of $\overline{PR}$.

PROVE ▶ $PQ = \frac{1}{2}PR$ and $QR = \frac{1}{2}PR$.

Statements	Reasons
1. Q is the midpoint of $\overline{PR}$.	**1.** Given
2. $PQ = QR$	**2.** Definition of midpoint
3. $PQ + QR = PR$	**3.** Segment Addition Postulate
4. $PQ + PQ = PR$	**4.** Substitution property of equality
5. $2 \cdot PQ = PR$	**5.** Distributive property
6. $PQ = \frac{1}{2}PR$	**6.** Division property of equality
7. $QR = \frac{1}{2}PR$	**7.** Substitution property of equality

STUDENT HELP

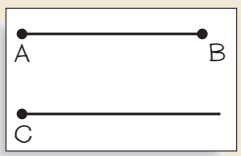
Study Tip
The distributive property can be used to simplify a sum, as in Step 5 of the proof. You can think of $PQ + PQ$ as follows: $1(PQ) + 1(PQ) = (1 + 1)(PQ) = 2 \cdot PQ$.

ACTIVITY

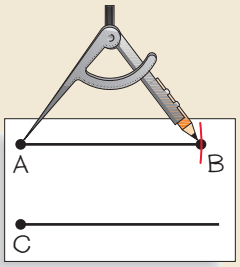
Construction

Copy a Segment

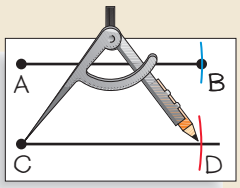
Use the following steps to construct a segment that is congruent to $\overline{AB}$.



1 Use a straightedge to draw a segment longer than $\overline{AB}$. Label the point C on the new segment.



2 Set your compass at the length of $\overline{AB}$.



3 Place the compass point at C and mark a second point, D, on the new segment. $\overline{CD}$ is congruent to $\overline{AB}$.

You will practice copying a segment in Exercises 12–15. It is an important construction because copying a segment is used in many constructions throughout this course.

GUIDED PRACTICE

Vocabulary Check ✓

1. An example of the Symmetric Property of Segment Congruence is "If $\overline{AB} \cong$ __?__, then $\overline{CD} \cong$ __?__."

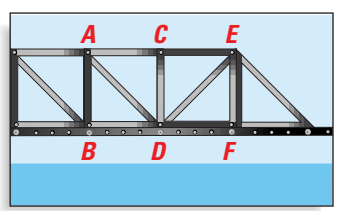
Concept Check ✓

2. **ERROR ANALYSIS** In the diagram below, $\overline{CB} \cong \overline{SR}$ and $\overline{CB} \cong \overline{QR}$. Explain what is wrong with Michael's argument.

~~Because $\overline{CB} \cong \overline{SR}$ and $\overline{CB} \cong \overline{QR}$, then $\overline{CB} \cong \overline{AC}$ by the Transitive Property of Segment Congruence.~~

Skill Check ✓

BRIDGES **The diagram below shows a portion of a trestle bridge, where $\overline{BF} \perp \overline{CD}$ and D is the midpoint of $\overline{BF}$.**



3. Give a reason why $\overline{BD}$ and $\overline{FD}$ are congruent.

4. Are $\angle CDE$ and $\angle FDE$ complementary? Explain.

5. If $\overline{CE}$ and $\overline{BD}$ are congruent, explain why $\overline{CE}$ and $\overline{FD}$ are congruent.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 806.

PROVING THEOREM 2.1 **Copy and complete the proof for two of the cases of the Properties of Segment Congruence Theorem.**

6. Reflexive Property of Segment Congruence

GIVEN ▶ EF is a line segment

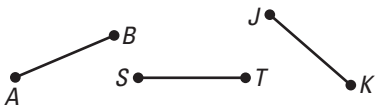
PROVE ▶ $\overline{EF} \cong \overline{EF}$

Statements	Reasons
1. $EF = EF$	**1.** ___?___
2. ___?___	**2.** Definition of congruent segments

7. Transitive Property of Segment Congruence

GIVEN ▶ $\overline{AB} \cong \overline{JK}$, $\overline{JK} \cong \overline{ST}$

PROVE ▶ $\overline{AB} \cong \overline{ST}$

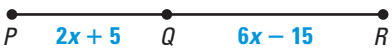


Statements	Reasons
1. $\overline{AB} \cong \overline{JK}$, $\overline{JK} \cong \overline{ST}$	**1.** ___?___
2. $AB = JK$, $JK = ST$	**2.** ___?___
3. $AB = ST$	**3.** ___?___
4. $\overline{AB} \cong \overline{ST}$	**4.** ___?___

USING ALGEBRA **Solve for the variable using the given information. Explain your steps.**

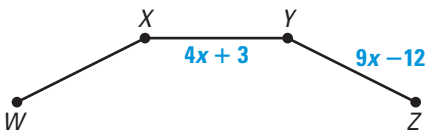
8. GIVEN ▶ $\overline{AB} \cong \overline{BC}$, $\overline{CD} \cong \overline{BC}$

9. GIVEN ▶ $PR = 46$

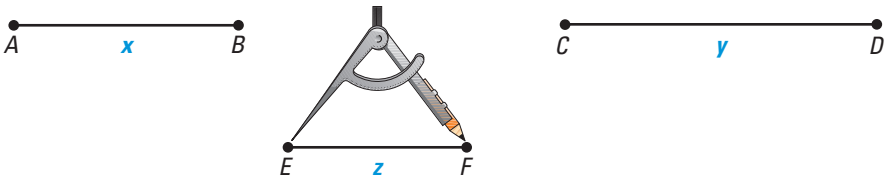


10. GIVEN ▶ $\overline{ST} \cong \overline{SR}$, $\overline{QR} \cong \overline{SR}$

11. GIVEN ▶ $\overline{XY} \cong \overline{WX}$, $\overline{YZ} \cong \overline{WX}$



CONSTRUCTION **In Exercises 12–15, use the segments, along with a straightedge and compass, to construct a segment with the given length.**



12. $x + y$ **13.** $y - z$ **14.** $3x - z$ **15.** $z + y - 2x$

STUDENT HELP

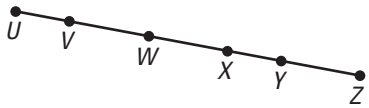
HOMEWORK HELP

Example 1: Exs. 6, 7
Example 2: Exs. 16–18
Example 3: Exs. 16–18

16. **DEVELOPING PROOF** Write a complete proof by rearranging the reasons listed on the pieces of paper.

GIVEN ▶ $\overline{UV} \cong \overline{XY}$, $\overline{VW} \cong \overline{WX}$, $\overline{WX} \cong \overline{YZ}$

PROVE ▶ $\overline{UW} \cong \overline{XZ}$

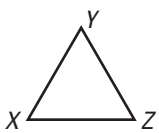


Statements	Reasons
1. $\overline{UV} \cong \overline{XY}$, $\overline{VW} \cong \overline{WX}$, $\overline{WX} \cong \overline{YZ}$	Transitive Property of Segment Congruence
2. $\overline{VW} \cong \overline{YZ}$	Addition property of equality
3. $UV = XY$, $VW = YZ$	Definition of congruent segments
4. $UV + VW = XY + YZ$	Given
5. $UV + VW = UW$, $XY + YZ = XZ$	Segment Addition Postulate
6. $UW = XZ$	Definition of congruent segments
7. $\overline{UW} \cong \overline{XZ}$	Substitution property of equality

TWO-COLUMN PROOF **Write a two-column proof.**

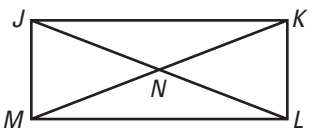
17. **GIVEN** ▶ $XY = 8$, $XZ = 8$, $\overline{XY} \cong \overline{ZY}$

PROVE ▶ $\overline{XZ} \cong \overline{ZY}$

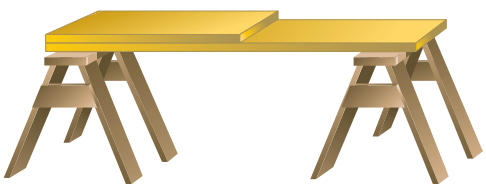


18. **GIVEN** ▶ $\overline{NK} \cong \overline{NL}$, $NK = 13$

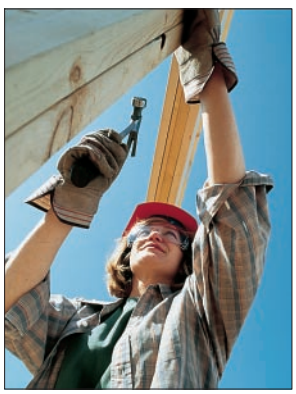
PROVE ▶ $NL = 13$



19. **CARPENTRY** You need to cut ten wood planks that are the same size. You measure and cut the first plank. You cut the second piece, using the first plank as a guide, as in the diagram below. The first plank is put aside and the second plank is used to cut a third plank. You follow this pattern for the rest of the planks. Is the last plank the same length as the first plank? Explain.



FOCUS ON CAREERS



CARPENTRY For many projects, carpenters need boards that are all the same length. For instance, equally-sized boards in the house frame above insure stability.

CAREER LINK
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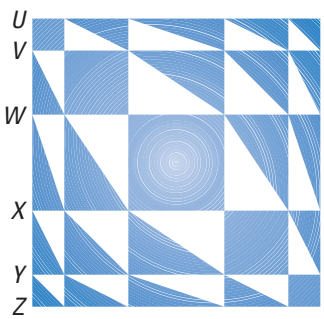
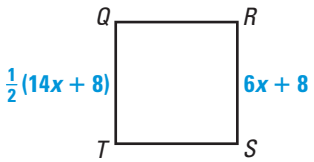
20. **OPTICAL ILLUSION** To create the illusion, a special grid was used. In the grid, corresponding row heights are the same measure. For instance, $\overline{UV}$ and $\overline{ZY}$ are congruent. You decide to make this design yourself. You draw the grid, but you need to make sure that the row heights are the same. You measure $\overline{UV}$, $\overline{UW}$, $\overline{ZY}$, and $\overline{ZX}$. You find that $\overline{UV} \cong \overline{ZY}$ and $\overline{UW} \cong \overline{ZX}$. Write an argument that allows you to conclude that $\overline{VW} \cong \overline{YX}$.



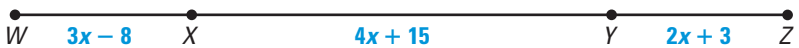
Test Preparation

21. MULTIPLE CHOICE In $QRST$, $\overline{QT} \cong \overline{TS}$ and $\overline{RS} \cong \overline{TS}$. What is x?



Ⓐ 1 Ⓑ 4 Ⓒ 12 Ⓓ 16 Ⓔ 32

22. MULTIPLE CHOICE In the figure shown below, $\overline{WX} \cong \overline{YZ}$. What is the length of $\overline{XZ}$?



Ⓐ 25 Ⓑ 34 Ⓒ 59 Ⓓ 60 Ⓔ 84

★ Challenge

REPRESENTING SEGMENT LENGTHS In Exercises 23–26, suppose point *T* is the midpoint of $\overline{RS}$ and point *W* is the midpoint of $\overline{RT}$. If $\overline{XY} \cong \overline{RT}$ and $\overline{TS}$ has a length of *z*, write the length of the segment in terms of *z*.

23. $\overline{RT}$ **24.** $\overline{XY}$ **25.** $\overline{RW}$ **26.** $\overline{WT}$

EXTRA CHALLENGE
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27. CRITICAL THINKING Suppose M is the midpoint of $\overline{AB}$, P is the midpoint of $\overline{AM}$, and Q is the midpoint of $\overline{PM}$. If a and b are the coordinates of points A and B on a number line, find the coordinates of P and Q in terms of a and b.

MIXED REVIEW

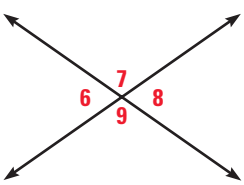
FINDING COUNTEREXAMPLES Find a counterexample that shows the statement is false. (Review 1.1)

28. For every number n, $2^n > n + 1$.

29. The sum of an even number and an odd number is always even.

30. If a number is divisible by 5, then it is divisible by 10.

FINDING ANGLE MEASURES In Exercises 31–34, use the diagram to find the angle measure. (Review 1.6 for 2.6)



31. If $m\angle 6 = 64°$, then $m\angle 7 =$ __?__.

32. If $m\angle 8 = 70°$, then $m\angle 6 =$ __?__.

33. If $m\angle 9 = 115°$, then $m\angle 8 =$ __?__.

34. If $m\angle 7 = 108°$, then $m\angle 8 =$ __?__.

35. Write the contrapositive of the conditional statement, "If Matthew wins this wrestling match, then he will win first place." (Review 2.1)

36. Is the converse of a true conditional statement always true? Explain. (Review 2.1)

USING SYMBOLIC NOTATION Let *p* be "the car is in the garage" and let *q* be "Mark is home." Write the statement in words and symbols. (Review 2.3)

37. The conditional statement $p \rightarrow q$ **38.** The converse of $p \rightarrow q$

39. The inverse of $p \rightarrow q$ **40.** The contrapositive of $p \rightarrow q$

2.6 Proving Statements about Angles

What you should learn

GOAL 1 Use angle congruence properties.

GOAL 2 Prove properties about special pairs of angles.

Why you should learn it

▼ Properties of special pairs of angles help you determine angles in wood-working projects, such as the corners in the piece of furniture below and in the picture frame in **Ex. 30**.

REAL LIFE

GOAL 1 CONGRUENCE OF ANGLES

In Lesson 2.5, you proved segment relationships. In this lesson, you will prove statements about angles.

THEOREM

THEOREM 2.2 ***Properties of Angle Congruence***

Angle congruence is reflexive, symmetric, and transitive.

Here are some examples.

REFLEXIVE	For any angle A, $\angle A \cong \angle A$.
SYMMETRIC	If $\angle A \cong \angle B$, then $\angle B \cong \angle A$.
TRANSITIVE	If $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.

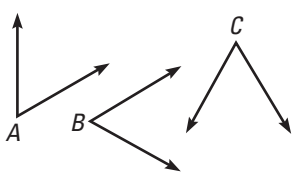
The Transitive Property of Angle Congruence is proven in Example 1. The Reflexive and Symmetric Properties are left for you to prove in Exercises 10 and 11.

EXAMPLE 1 *Transitive Property of Angle Congruence*

Prove the Transitive Property of Congruence for angles.

SOLUTION

To prove the Transitive Property of Congruence for angles, begin by drawing three congruent angles. Label the vertices as A, B, and C.



GIVEN ▶ $\angle A \cong \angle B$, $\angle B \cong \angle C$

PROVE ▶ $\angle A \cong \angle C$

Statements	Reasons
1. $\angle A \cong \angle B$, $\angle B \cong \angle C$	**1.** Given
2. $m\angle A = m\angle B$	**2.** Definition of congruent angles
3. $m\angle B = m\angle C$	**3.** Definition of congruent angles
4. $m\angle A = m\angle C$	**4.** Transitive property of equality
5. $\angle A \cong \angle C$	**5.** Definition of congruent angles

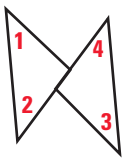
EXAMPLE 2 *Using the Transitive Property*

Proof

This two-column proof uses the Transitive Property.

GIVEN ▶ $m\angle 3 = 40°$, $\angle 1 \cong \angle 2$, $\angle 2 \cong \angle 3$

PROVE ▶ $m\angle 1 = 40°$



Statements	Reasons
1. $m\angle 3 = 40°$, $\angle 1 \cong \angle 2$, $\angle 2 \cong \angle 3$	**1.** Given
2. $\angle 1 \cong \angle 3$	**2.** Transitive Property of Congruence
3. $m\angle 1 = m\angle 3$	**3.** Definition of congruent angles
4. $m\angle 1 = 40°$	**4.** Substitution property of equality

THEOREM

THEOREM 2.3 ***Right Angle Congruence Theorem***

All right angles are congruent.

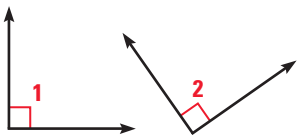
EXAMPLE 3 *Proving Theorem 2.3*

Proof

You can prove Theorem 2.3 as shown.

GIVEN ▶ $\angle 1$ and $\angle 2$ are right angles

PROVE ▶ $\angle 1 \cong \angle 2$



Statements	Reasons
1. $\angle 1$ and $\angle 2$ are right angles	**1.** Given
2. $m\angle 1 = 90°$, $m\angle 2 = 90°$	**2.** Definition of right angle
3. $m\angle 1 = m\angle 2$	**3.** Transitive property of equality
4. $\angle 1 \cong \angle 2$	**4.** Definition of congruent angles

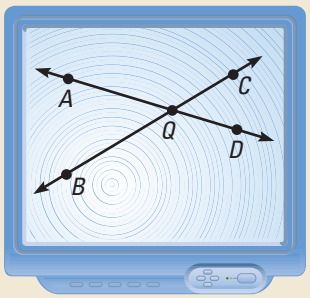
ACTIVITY

Using Technology

Investigating Supplementary Angles

Use geometry software to draw and label two intersecting lines.

1. What do you notice about the measures of $\angle AQB$ and $\angle AQC$? $\angle AQC$ and $\angle CQD$? $\angle AQB$ and $\angle CQD$?
2. Rotate $\overleftrightarrow{BC}$ to a different position. Do the angles retain the same relationship?
3. Make a conjecture about two angles supplementary to the same angle.



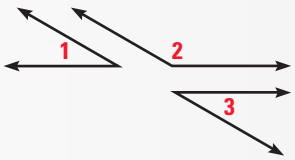
GOAL 2 PROPERTIES OF SPECIAL PAIRS OF ANGLES

THEOREMS

THEOREM 2.4 ***Congruent Supplements Theorem***

If two angles are supplementary to the same angle (or to congruent angles) then they are congruent.

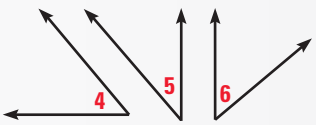
If $m\angle 1 + m\angle 2 = 180°$ and $m\angle 2 + m\angle 3 = 180°$, then $\angle 1 \cong \angle 3$.



THEOREM 2.5 ***Congruent Complements Theorem***

If two angles are complementary to the same angle (or to congruent angles) then the two angles are congruent.

If $m\angle 4 + m\angle 5 = 90°$ and $m\angle 5 + m\angle 6 = 90°$, then $\angle 4 \cong \angle 6$.



EXAMPLE 4 *Proving Theorem 2.4*

Proof

GIVEN ▶ $\angle 1$ and $\angle 2$ are supplements, $\angle 3$ and $\angle 4$ are supplements, $\angle 1 \cong \angle 4$

PROVE ▶ $\angle 2 \cong \angle 3$

Statements	Reasons
1. $\angle 1$ and $\angle 2$ are supplements, $\angle 3$ and $\angle 4$ are supplements, $\angle 1 \cong \angle 4$	**1.** Given
2. $m\angle 1 + m\angle 2 = 180°$ $m\angle 3 + m\angle 4 = 180°$	**2.** Definition of supplementary angles
3. $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	**3.** Transitive property of equality
4. $m\angle 1 = m\angle 4$	**4.** Definition of congruent angles
5. $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 1$	**5.** Substitution property of equality
6. $m\angle 2 = m\angle 3$	**6.** Subtraction property of equality
7. $\angle 2 \cong \angle 3$	**7.** Definition of congruent angles

POSTULATE

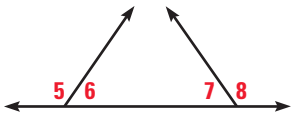
POSTULATE 12 ***Linear Pair Postulate***

If two angles form a linear pair, then they are supplementary.

$m\angle 1 + m\angle 2 = 180°$

EXAMPLE 5 *Using Linear Pairs*

In the diagram, $m\angle 8 = m\angle 5$ and $m\angle 5 = 125°$. Explain how to show $m\angle 7 = 55°$.



SOLUTION

Using the transitive property of equality, $m\angle 8 = 125°$. The diagram shows $m\angle 7 + m\angle 8 = 180°$. Substitute 125° for $m\angle 8$ to show $m\angle 7 = 55°$.

THEOREM

THEOREM 2.6 ***Vertical Angles Theorem***

Vertical angles are congruent.

$\angle 1 \cong \angle 3$, $\angle 2 \cong \angle 4$

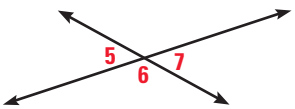
STUDENT HELP

Study Tip
Remember that previously proven theorems can be used as reasons in a proof, as in **Step 3** of the proof at the right.

EXAMPLE 6 *Proving Theorem 2.6*

GIVEN ▶ $\angle 5$ and $\angle 6$ are a linear pair, $\angle 6$ and $\angle 7$ are a linear pair

PROVE ▶ $\angle 5 \cong \angle 7$



Statements	Reasons
1. $\angle 5$ and $\angle 6$ are a linear pair, $\angle 6$ and $\angle 7$ are a linear pair	**1.** Given
2. $\angle 5$ and $\angle 6$ are supplementary, $\angle 6$ and $\angle 7$ are supplementary	**2.** Linear Pair Postulate
3. $\angle 5 \cong \angle 7$	**3.** Congruent Supplements Theorem

GUIDED PRACTICE

Vocabulary Check ✓

1. "If $\angle CDE \cong$ __?__ and $\angle QRS \cong \angle XYZ$, then $\angle CDE \cong \angle XYZ$," is an example of the __?__ Property of Angle Congruence.

Concept Check ✓

2. To close the blades of the scissors, you close the handles. Will the angle formed by the blades be the same as the angle formed by the handles? Explain.

Skill Check ✓

3. By the Transitive Property of Congruence, if $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then __?__ $\cong \angle C$.

In Exercises 4–9, $\angle 1$ and $\angle 3$ are a linear pair, $\angle 1$ and $\angle 4$ are a linear pair, and $\angle 1$ and $\angle 2$ are vertical angles. Is the statement true?

4. $\angle 1 \cong \angle 3$ **5.** $\angle 1 \cong \angle 2$ **6.** $\angle 1 \cong \angle 4$

7. $\angle 3 \cong \angle 2$ **8.** $\angle 3 \cong \angle 4$ **9.** $m\angle 2 + m\angle 3 = 180°$

PRACTICE AND APPLICATIONS

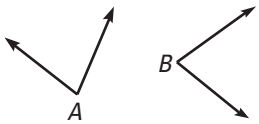
STUDENT HELP

Extra Practice to help you master skills is on p. 806.

10. PROVING THEOREM 2.2 Copy and complete the proof of the Symmetric Property of Congruence for angles.

GIVEN ▶ $\angle A \cong \angle B$

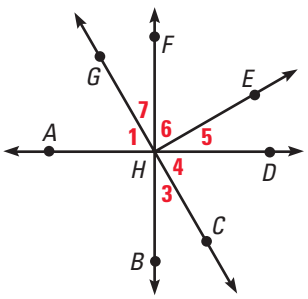
PROVE ▶ $\angle B \cong \angle A$



Statements	Reasons
1. $\angle A \cong \angle B$	**1.** ___?___
2. ___?___	**2.** Definition of congruent angles
3. $m\angle B = m\angle A$	**3.** ___?___
4. $\angle B \cong \angle A$	**4.** ___?___

11. PROVING THEOREM 2.2 Write a two-column proof for the Reflexive Property of Congruence for angles.

FINDING ANGLES **In Exercises 12–17, complete the statement given that $m\angle EHC = m\angle DHB = m\angle AHB = 90°$**



12. If $m\angle 7 = 28°$, then $m\angle 3 =$ ___?___.

13. If $m\angle EHB = 121°$, then $m\angle 7 =$ ___?___.

14. If $m\angle 3 = 34°$, then $m\angle 5 =$ ___?___.

15. If $m\angle GHB = 158°$, then $m\angle FHC =$ ___?___.

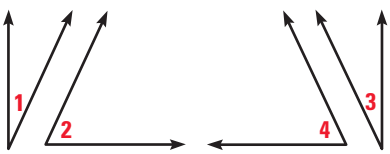
16. If $m\angle 7 = 31°$, then $m\angle 6 =$ ___?___.

17. If $m\angle GHD = 119°$, then $m\angle 4 =$ ___?___.

18. PROVING THEOREM 2.5 Copy and complete the proof of the Congruent Complements Theorem.

GIVEN ▶ $\angle 1$ and $\angle 2$ are complements, $\angle 3$ and $\angle 4$ are complements, $\angle 2 \cong \angle 4$

PROVE ▶ $\angle 1 \cong \angle 3$



Statements	Reasons
1. $\angle 1$ and $\angle 2$ are complements, $\angle 3$ and $\angle 4$ are complements, $\angle 2 \cong \angle 4$	**1.** ___?___
2. ___?___, ___?___	**2.** Def. of complementary angles
3. $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	**3.** Transitive property of equality
4. $m\angle 2 = m\angle 4$	**4.** ___?___
5. $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 2$	**5.** ___?___
6. $m\angle 1 = m\angle 3$	**6.** ___?___
7. ___?___	**7.** Definition of congruent angles

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 10, 11
Example 2: Exs. 12–17
Example 3: Exs. 12–17
Example 4: Exs. 19–22
Example 5: Exs. 23–28
Example 6: Exs. 23–28

FINDING CONGRUENT ANGLES **Make a sketch using the given information. Then, state all of the pairs of congruent angles.**

19. $\angle 1$ and $\angle 2$ are a linear pair. $\angle 2$ and $\angle 3$ are a linear pair. $\angle 3$ and $\angle 4$ are a linear pair.

20. $\angle XYZ$ and $\angle VYW$ are vertical angles. $\angle XYZ$ and $\angle ZYW$ are supplementary. $\angle VYW$ and $\angle XYV$ are supplementary.

21. $\angle 1$ and $\angle 3$ are complementary. $\angle 4$ and $\angle 2$ are complementary. $\angle 1$ and $\angle 2$ are vertical angles.

22. $\angle ABC$ and $\angle CBD$ are adjacent, complementary angles. $\angle CBD$ and $\angle DBF$ are adjacent, complementary angles.

STUDENT HELP

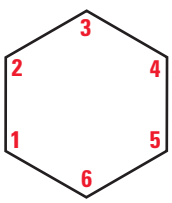
HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 23–26.

WRITING PROOFS **Write a two-column proof.**

23. GIVEN ▶ $m\angle 3 = 120°$, $\angle 1 \cong \angle 4$, $\angle 3 \cong \angle 4$

PROVE ▶ $m\angle 1 = 120°$

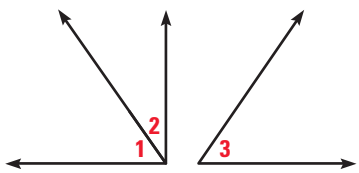
Plan for Proof First show that $\angle 1 \cong \angle 3$. Then use transitivity to show that $m\angle 1 = 120°$.



24. GIVEN ▶ $\angle 3$ and $\angle 2$ are complementary, $m\angle 1 + m\angle 2 = 90°$

PROVE ▶ $\angle 3 \cong \angle 1$

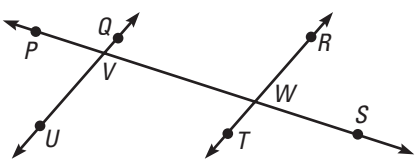
Plan for Proof First show that $\angle 1$ and $\angle 2$ are complementary. Then show that $\angle 3 \cong \angle 1$.



25. GIVEN ▶ $\angle QVW$ and $\angle RWV$ are supplementary

PROVE ▶ $\angle QVP \cong \angle RWV$

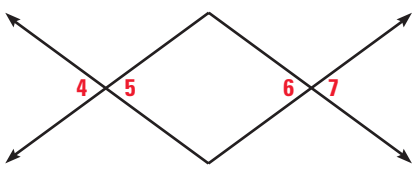
Plan for Proof First show that $\angle QVP$ and $\angle QVW$ are supplementary. Then show that $\angle QVP \cong \angle RWV$.



26. GIVEN ▶ $\angle 5 \cong \angle 6$

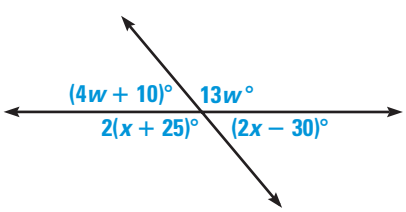
PROVE ▶ $\angle 4 \cong \angle 7$

Plan for Proof First show that $\angle 4 \cong \angle 5$ and $\angle 6 \cong \angle 7$. Then use transitivity to show that $\angle 4 \cong \angle 7$.

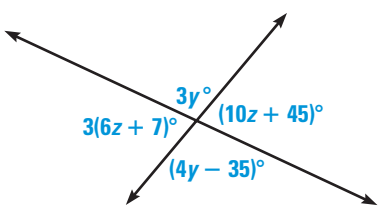


USING ALGEBRA **In Exercises 27 and 28, solve for each variable. Explain your reasoning.**

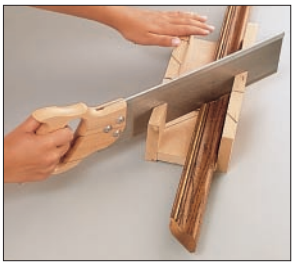
27.



28.

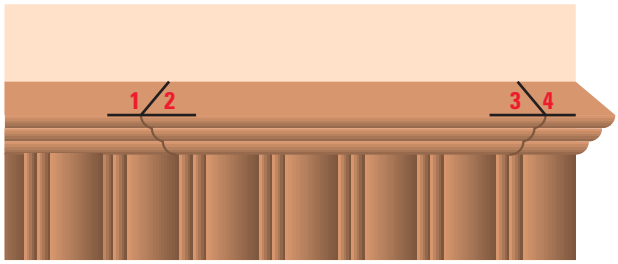


FOCUS ON APPLICATIONS

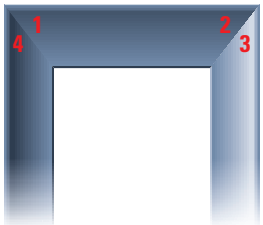


REAL LIFE **MITER BOX** This box has slotted sides to guide a saw when making angled cuts.

29. **WALL TRIM** A chair rail is a type of wall trim that is placed about three feet above the floor to protect the walls. Part of the chair rail below has been replaced because it was damaged. The edges of the replacement piece were angled for a better fit. In the diagram, $\angle 1$ and $\angle 2$ are supplementary, $\angle 3$ and $\angle 4$ are supplementary, and $\angle 2$ and $\angle 3$ each have measures of 50°. Is $\angle 1 \cong \angle 4$? Explain.



30. **PICTURE FRAMES** Suppose you are making a picture frame, as shown at the right. The corners are all right angles, and $m\angle 1 = m\angle 2 = 52°$. Is $\angle 4 \cong \angle 3$? Explain why or why not.



31. *Writing* Describe some instances of mitered, or angled, corners in the real world.

32. **TECHNOLOGY** Use geometry software to draw two overlapping right angles with a common vertex. Observe the measures of the three angles as one right angle is rotated about the other. What theorem does this illustrate?

Test Preparation

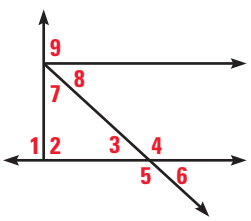
QUANTITATIVE COMPARISON Choose the statement that is true about the diagram. In the diagram, $\angle 9$ is a right angle and $m\angle 3 = 42°$.

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship can't be determined from the given information.



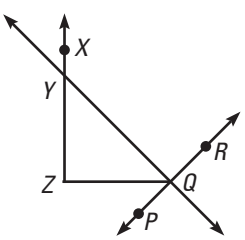
	Column A	Column B
33.	$m\angle 3 + m\angle 4$	$m\angle 1 + m\angle 2$
34.	$m\angle 3 + m\angle 6$	$m\angle 7 + m\angle 8$
35.	$m\angle 5$	$3(m\angle 3)$
36.	$m\angle 7 + m\angle 8$	$m\angle 9$

★ Challenge

37. **PROOF** Write a two-column proof.

GIVEN ▶ $m\angle ZYQ = 45°$, $m\angle ZQP = 45°$

PROVE ▶ $\angle ZQR \cong \angle XYQ$



MIXED REVIEW

FINDING ANGLE MEASURES **In Exercises 38–40, the measure of $\angle 1$ and the relationship of $\angle 1$ to $\angle 2$ is given. Find $m\angle 2$.** (Review 1.6 for 3.1)

38. $m\angle 1 = 62°$, complementary to $\angle 2$

39. $m\angle 1 = 8°$, supplementary to $\angle 2$

40. $m\angle 1 = 47°$, complementary to $\angle 2$

41. **PERPENDICULAR LINES** The definition of perpendicular lines states that if two lines are perpendicular, then they intersect to form a right angle. Is the converse true? Explain. (Review 2.2 for 3.1)

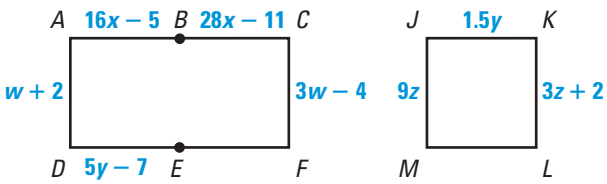
USING ALGEBRA **Use the diagram and the given information to solve for the variable.** (Review 2.5)

42. $\overline{AD} \cong \overline{EF}$, $\overline{EF} \cong \overline{CF}$

43. $\overline{AB} \cong \overline{EF}$, $\overline{EF} \cong \overline{BC}$

44. $\overline{DE} \cong \overline{EF}$, $\overline{EF} \cong \overline{JK}$

45. $\overline{JM} \cong \overline{ML}$, $\overline{ML} \cong \overline{KL}$



QUIZ 2

Self-Test for Lessons 2.4–2.6

Solve the equation and state a reason for each step. (Lesson 2.4)

1. $x - 3 = 7$

2. $x + 8 = 27$

3. $2x - 5 = 13$

4. $2x + 20 = 4x - 12$

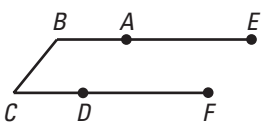
5. $3(3x - 7) = 6$

6. $-2(-2x + 4) = 16$

PROOF **In Exercises 7 and 8 write a two column proof.** (Lesson 2.5)

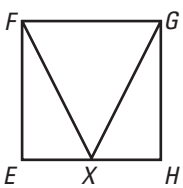
7. **GIVEN** ▶ $\overline{BA} \cong \overline{BC}$, $\overline{BC} \cong \overline{CD}$, $\overline{AE} \cong \overline{DF}$

PROVE ▶ $\overline{BE} \cong \overline{CF}$

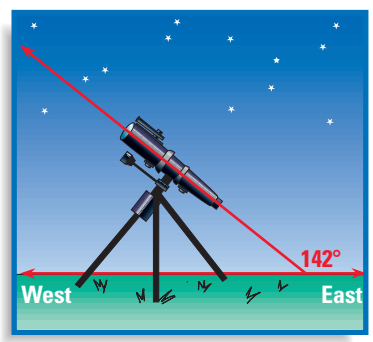


8. **GIVEN** ▶ $\overline{EH} \cong \overline{GH}$, $\overline{FG} \cong \overline{GH}$

PROVE ▶ $\overline{FG} \cong \overline{EH}$



9. **ASTRONOMY** While looking through a telescope one night, you begin looking due east. You rotate the telescope straight upward until you spot a comet. The telescope forms a 142° angle with due east, as shown. What is the angle of inclination of the telescope from due west? (Lesson 2.6)



2.1 Conditional Statements

What you should learn

GOAL 1 Recognize and analyze a conditional statement.

GOAL 2 Write postulates about points, lines, and planes using conditional statements.

Why you should learn it

▼ Point, line, and plane postulates help you analyze **real-life** objects, such as the research buggy below and in **Ex. 54**.

REAL LIFE

Coastal Research Amphibious Buggy

GOAL 1 RECOGNIZING CONDITIONAL STATEMENTS

In this lesson you will study a type of logical statement called a conditional statement. A **conditional statement** has two parts, a *hypothesis* and a *conclusion*. When the statement is written in **if-then form**, the "if" part contains the **hypothesis** and the "then" part contains the **conclusion**. Here is an example:

$$\text{If } \underbrace{\textbf{it is noon in Georgia}}_{\text{Hypothesis}}, \text{ then } \underbrace{\textbf{it is 9 A.M. in California}}_{\text{Conclusion}}.$$

EXAMPLE 1 *Rewriting in If-Then Form*

Rewrite the conditional statement in *if-then form.*

a. Two points are collinear if they lie on the same line.

b. All sharks have a boneless skeleton.

c. A number divisible by 9 is also divisible by 3.

SOLUTION

a. If two points lie on the same line, then they are collinear.

b. If a fish is a shark, then it has a boneless skeleton.

c. If a number is divisible by 9, then it is divisible by 3.

Conditional statements can be either true or false. To show that a conditional statement is true, you must present an argument that the conclusion follows for *all* cases that fulfill the hypothesis. To show that a conditional statement is false, describe a single counterexample that shows the statement is not always true.

EXAMPLE 2 *Writing a Counterexample*

Write a counterexample to show that the following conditional statement is false.

$$\text{If } x^2 = 16, \text{ then } x = 4.$$

SOLUTION

As a counterexample, let $x = -4$. The hypothesis is true, because $(-4)^2 = 16$. However, the conclusion is false. This implies that the given conditional statement is false.

The **converse** of a conditional statement is formed by switching the hypothesis and conclusion. Here is an example.

Statement: If you see lightning, then you hear thunder.

Converse: If you hear thunder, then you see lightning.

EXAMPLE 3 *Writing the Converse of a Conditional Statement*

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

Write the converse of the following conditional statement.

Statement: If two segments are congruent, then they have the same length.

SOLUTION

Converse: If two segments have the same length, then they are congruent.

. . . . . .

A statement can be altered by **negation**, that is, by writing the negative of the statement. Here are some examples.

STATEMENT	NEGATION
$m\angle A = 30°$	$m\angle A \neq 30°$
$\angle A$ is acute.	$\angle A$ is not acute.

When you negate the hypothesis and conclusion of a conditional statement, you form the **inverse**. When you negate the hypothesis and conclusion of the converse of a conditional statement, you form the **contrapositive**.

Original	If $m\angle A = 30°$, then $\angle A$ is acute.
Inverse	If $m\angle A \neq 30°$, then $\angle A$ is not acute.
Converse	If $\angle A$ is acute, then $m\angle A = 30°$.
Contrapositive	If $\angle A$ is not acute, then $m\angle A \neq 30°$.

Both false **Both true**

When two statements are both true or both false, they are called **equivalent statements**. A conditional statement is equivalent to its contrapositive. Similarly, the inverse and converse of any conditional statement are equivalent. This is shown in the table above.

FOCUS ON APPLICATIONS

REAL LIFE **CROCUS** There are some exceptions to the statement in Example 4. For instance, crocuses can bloom when snow is on the ground.

EXAMPLE 4 *Writing an Inverse, Converse, and Contrapositive*

Write the (**a**) inverse, (**b**) converse, and (**c**) contrapositive of the statement.

If there is snow on the ground, then flowers are not in bloom.

SOLUTION

a. Inverse: If there is no snow on the ground, then flowers are in bloom.

b. Converse: If flowers are not in bloom, then there is snow on the ground.

c. Contrapositive: If flowers are in bloom, then there is no snow on the ground.

GOAL 2 USING POINT, LINE, AND PLANE POSTULATES

In Chapter 1, you studied four postulates.

Ruler Postulate	(Lesson 1.3, page 17)
Segment Addition Postulate	(Lesson 1.3, page 18)
Protractor Postulate	(Lesson 1.4, page 27)
Angle Addition Postulate	(Lesson 1.4, page 27)

Remember that postulates are assumed to be true—they form the foundation on which other statements (called *theorems*) are built.

STUDENT HELP

Study Tip
There is a list of all the postulates in this course at the end of the book beginning on page 827.

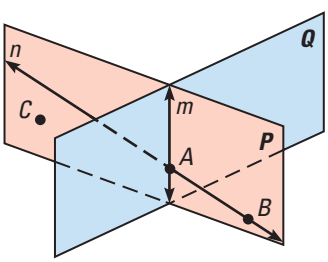
POINT, LINE, AND PLANE POSTULATES

POSTULATE 5	Through any two points there exists exactly one line.
POSTULATE 6	A line contains at least two points.
POSTULATE 7	If two lines intersect, then their intersection is exactly one point.
POSTULATE 8	Through any three noncollinear points there exists exactly one plane.
POSTULATE 9	A plane contains at least three noncollinear points.
POSTULATE 10	If two points lie in a plane, then the line containing them lies in the plane.
POSTULATE 11	If two planes intersect, then their intersection is a line.

EXAMPLE 5 *Identifying Postulates*

Logical Reasoning

Use the diagram at the right to give examples of Postulates 5 through 11.



SOLUTION

a. Postulate 5: There is exactly one line (line n) that passes through the points A and B.

b. Postulate 6: Line n contains at least two points. For instance, line n contains the points A and B.

c. Postulate 7: Lines m and n intersect at point A.

d. Postulate 8: Plane P passes through the noncollinear points A, B, and C.

e. Postulate 9: Plane P contains at least three noncollinear points, A, B, and C.

f. Postulate 10: Points A and B lie in plane P. So, line n, which contains points A and B, also lies in plane P.

g. Postulate 11: Planes P and Q intersect. So, they intersect in a line, labeled in the diagram as line m.

EXAMPLE 6 *Rewriting a Postulate*

a. Rewrite Postulate 5 in if-then form.

b. Write the inverse, converse, and contrapositive of Postulate 5.

SOLUTION

a. Postulate 5 can be rewritten in if-then form as follows:

> If two points are distinct, then there is exactly one line that passes through them.

b. **Inverse:** If two points are not distinct, then it is not true that there is exactly one line that passes through them.

Converse: If exactly one line passes through two points, then the two points are distinct.

Contrapositive: If it is not true that exactly one line passes through two points, then the two points are not distinct.

EXAMPLE 7 *Using Postulates and Counterexamples*

Logical Reasoning

Decide whether the statement is *true* or *false*. If it is false, give a counterexample.

a. A line can be in more than one plane.

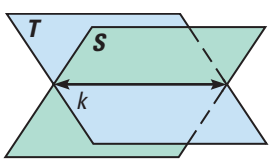
b. Four noncollinear points are always coplanar.

c. Two nonintersecting lines can be noncoplanar.

SOLUTION

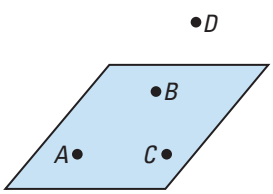
a. In the diagram at the right, line k is in plane S and line k is in plane T.

So, it is *true* that a line can be in more than one plane.



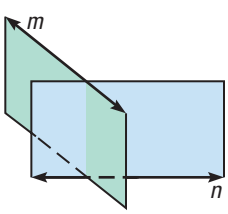
b. Consider the points A, B, C, and D at the right. The points A, B, and C lie in a plane, but there is no plane that contains all four points.

So, as shown in the counterexample at the right, it is *false* that four noncollinear points are always coplanar.



c. In the diagram at the right, line m and line n are nonintersecting and are also noncoplanar.

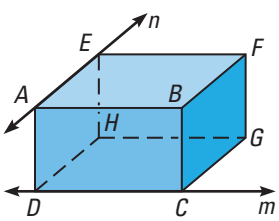
So, it is *true* that two nonintersecting lines can be noncoplanar.



STUDENT HELP

Study Tip

A box can be used to help visualize points and lines in space. For instance, the diagram shows that $\overleftrightarrow{AE}$ and $\overleftrightarrow{DC}$ are noncoplanar.



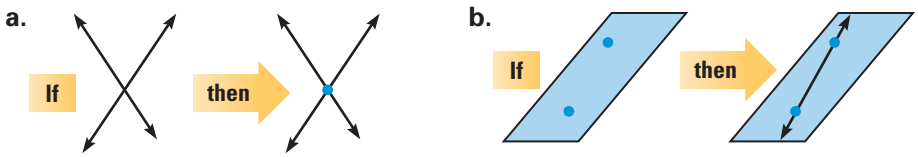
GUIDED PRACTICE

Vocabulary Check ✓

1. The ___?___ of a conditional statement is found by switching the hypothesis and conclusion.

Concept Check ✓

2. State the postulate described in each diagram.



Skill Check ✓

3. Write the hypothesis and conclusion of the statement, "If the dew point equals the air temperature, then it will rain."

In Exercises 4 and 5, write the statement in if-then form.

4. When threatened, the African ball python protects itself by coiling into a ball with its head in the middle.
5. The measure of a right angle is 90°.
6. Write the inverse, converse, and contrapositive of the conditional statement, "If a cactus is of the *cereus* variety, then its flowers open at night."

Decide whether the statement is *true* or *false*. Make a sketch to help you decide.

7. Through three noncollinear points there exists exactly one line.
8. If a line and a plane intersect, and the line does not lie in the plane, then their intersection is a point.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 805.

REWRITING STATEMENTS **Rewrite the conditional statement in if-then form.**

9. An object weighs one ton if it weighs 2000 pounds.
10. An object weighs 16 ounces if it weighs one pound.
11. Three points are collinear if they lie on the same line.
12. Blue trunkfish live in the waters of a coral reef.
13. Hagfish live in salt water.

STUDENT HELP

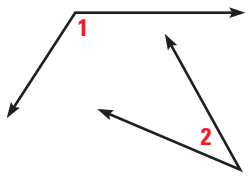
HOMEWORK HELP

Example 1: Exs. 9–13
Example 2: Exs. 14–17
Example 3: Exs. 18–21
Example 4: Exs. 46–52
Example 5: Exs. 25–34
Example 6: Exs. 22–24
Example 7: Exs. 35–38

ANALYZING STATEMENTS **Decide whether the statement is *true* or *false*. If false, provide a counterexample.**

14. A point may lie in more than one plane.
15. If x^4 equals 81, then x must equal 3.
16. If it is snowing, then the temperature is below freezing.
17. If four points are collinear, then they are coplanar.

WRITING CONVERSES **Write the converse of the statement.**



18. If $\angle 1$ measures 123°, then $\angle 1$ is obtuse.

19. If $\angle 2$ measures 38°, then $\angle 2$ is acute.

20. I will go to the mall if it is not raining.

21. I will go to the movies if it is raining.

REWRITING POSTULATES **Rewrite the postulate in if-then form. Then write the inverse, converse, and contrapositive of the conditional statement.**

22. A line contains at least two points.

23. Through any three noncollinear points there exists exactly one plane.

24. A plane contains at least three noncollinear points.

ILLUSTRATING POSTULATES **Fill in the blank. Then draw a sketch that helps illustrate your answer.**

25. If two lines intersect, then their intersection is ___?___ point(s).

26. Through any ___?___ points there exists exactly one line.

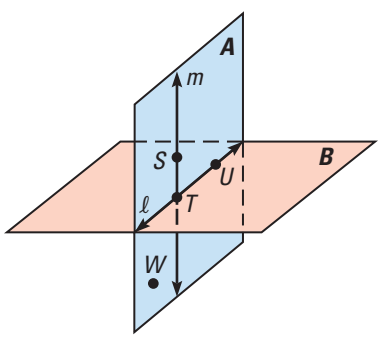
27. If two points lie in a plane, then the ___?___ containing them lies in the plane.

28. If two planes intersect, then their intersection is ___?___.

STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 29–34.

LINKING POSTULATES **Use the diagram to state the postulate(s) that verifies the truth of the statement.**



29. The points U and T lie on line ℓ.

30. Line ℓ contains points U and T.

31. The points W, S, and T lie in plane A.

32. The points S and T lie in plane A. Therefore, line m lies in plane A.

33. The planes A and B intersect in line ℓ.

34. Lines m and ℓ intersect at point T.

USING POSTULATES **In Exercises 35–38, state the postulate that shows that the statement is false.**

35. A line contains only one point.

36. Two planes intersect in exactly one point.

37. Three points, A, B, and C, are noncollinear, and two planes, M and N, each contain points A, B, and C.

38. Two points, P and Q, are collinear and two different lines, $\overleftrightarrow{RS}$ and $\overleftrightarrow{XY}$, each pass through points P and Q.

39. *Writing* Give an example of a true conditional statement with a true converse.

POINTS AND LINES IN SPACE **Think of the intersection of the ceiling and the front wall of your classroom as line *k*. Think of the center of the floor as point *A* and the center of the ceiling as point *B*.**

40. Is there more than one line that contains both points A and B?

41. Is there more than one plane that contains both points A and B?

42. Is there a plane that contains line k and point A?

43. Is there a plane that contains points A, B, and a point on the front wall?

USING ALGEBRA **Find the inverse, converse, and contrapositive of the statement.**

44. If $x = y$, then $5x = 5y$.

45. $6x - 6 = x + 14$ if $x = 4$.

QUOTES OF WISDOM **Rewrite the statement in if-then form. Then (a) determine the hypothesis and conclusion, and (b) find the inverse of the conditional statement.**

46. "If you tell the truth, you don't have to remember anything." — Mark Twain

47. "One can never consent to creep when one feels the impulse to soar." — Helen Keller

48. "Freedom is not worth having if it does not include the freedom to make mistakes." — Mahatma Ghandi

49. "Early to bed and early to rise, makes a man healthy, wealthy, and wise." — Benjamin Franklin

ADVERTISING **In Exercises 50–52, use the following advertising slogan: "You want a great selection of used cars? Come and see Bargain Bob's Used Cars!"**

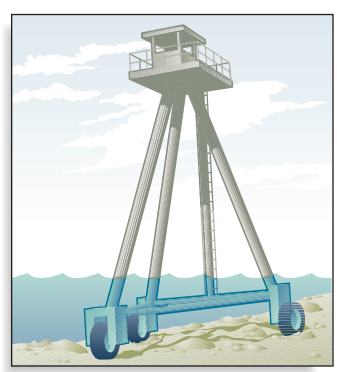
50. Write the slogan in if-then form. What are the hypothesis and conclusion of the conditional statement?

51. Write the inverse, converse, and contrapositive of the conditional statement.

52. *Writing* Find a real-life advertisement or slogan similar to the one given. Then repeat Exercises 50 and 51 using the advertisement or slogan.

53. **TECHNOLOGY** Use geometry software to draw a segment with endpoints A and C. Draw a third point B not on $\overline{AC}$. Measure $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. Move B closer to $\overline{AC}$ and observe the measures of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.

54. **RESEARCH BUGGY** The diagram at the right shows the 35 foot tall Coastal Research Amphibious Buggy, also known as CRAB. This vehicle moves along the ocean floor collecting data that are used to make an accurate map of the ocean floor. Using the postulates you have learned, make a conjecture about why the CRAB was built with three legs instead of four.



FOCUS ON CAREERS

REAL LIFE ADVERTISING COPYWRITER

Advertising copywriters write the advertisements you see and hear everyday. These ads appear in many forms including Internet home pages.

INTERNET CAREER LINK

www.mcdougallittell.com

Test Preparation

55. MULTIPLE CHOICE Use the conditional statement "If the measure of an angle is 44°, then the angle is acute" to decide which of the following are true.

I. The statement is true.
II. The converse of the statement is true.
III. The contrapositive of the statement is true.

Ⓐ I only Ⓑ II only Ⓒ I and II Ⓓ I and III Ⓔ I, II, and III

56. MULTIPLE CHOICE Which one of the following statements is *not* true?

Ⓐ If $x = 2$, then $x^2 = 4$.

Ⓑ If $x = -2$, then $x^2 = 4$.

Ⓒ If $x^3 = -8$, then $x = -2$.

Ⓓ If $x^2 = 4$, then $x = 2$.

Ⓔ If $x = -2$, then $x^3 = -8$.

★ Challenge

MAKING A CONJECTURE Sketch a line *k* and a point *P* not on line *k*. Make a conjecture about how many planes can be drawn through line *k* and point *P*, and then answer the following questions.

57. Which postulate allows you to state that there are two points, R and S, on line k?

58. Which postulate allows you to conclude that exactly one plane X can be drawn to contain points P, R, and S?

59. Which postulate guarantees that line k is contained in plane X?

60. Was your conjecture correct?

EXTRA CHALLENGE
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MIXED REVIEW

DRAWING ANGLES Plot the points in a coordinate plane. Then classify $\angle ABC$. (Review 1.4 for 2.2)

61. $A(0, 7)$, $B(2, 2)$, $C(6, -1)$

62. $A(-1, 0)$, $B(-6, 4)$, $C(-6, -1)$

63. $A(1, 3)$, $B(1, -5)$, $C(-5, -5)$

64. $A(-3, -1)$, $B(2, 5)$, $C(3, -2)$

FINDING THE MIDPOINT Find the coordinates of the midpoint of the segment joining the two points. (Review 1.5)

65. $A(-2, 8)$, $B(4, -12)$

66. $A(8, 8)$, $B(-6, 1)$

67. $A(-7, -4)$, $B(4, 7)$

68. $A(0, -9)$, $B(-8, 5)$

69. $A(1, 4)$, $B(11, -6)$

70. $A(-10, -10)$, $B(2, 12)$

FINDING PERIMETER AND AREA Find the area and perimeter (or circumference) of the figure described. (Use $\pi \approx 3.14$ when necessary.) (Review 1.7 for 2.2)

71. circle, radius = 6 m

72. square, side = 11 cm

73. square, side = 38.75 mm

74. circle, diameter = 23 ft

3.1 Lines and Angles

What you should learn

GOAL 1 Identify relationships between lines.

GOAL 2 Identify angles formed by transversals.

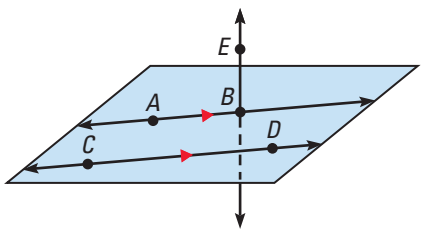
Why you should learn it

▼ To describe and understand **real-life** objects, such as the escalator in **Exs. 32–36**.

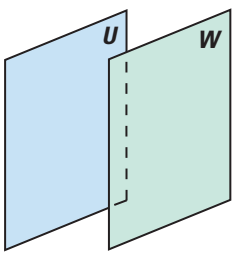
REAL LIFE

GOAL 1 RELATIONSHIPS BETWEEN LINES

Two lines are **parallel lines** if they are coplanar and do not intersect. Lines that do not intersect and are not coplanar are called **skew lines**. Similarly, two planes that do not intersect are called **parallel planes**.



$\overleftrightarrow{AB}$ **and** $\overleftrightarrow{CD}$ **are parallel lines.**
$\overleftrightarrow{CD}$ **and** $\overleftrightarrow{BE}$ **are skew lines.**



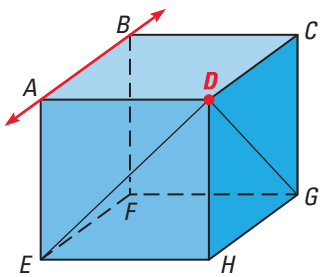
Planes *U* and *W* are parallel planes.

To write "$\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$," you write $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Triangles like those on $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are used on diagrams to indicate that lines are parallel.

Segments and rays are parallel if they lie on parallel lines. For example, $\overline{AB} \parallel \overline{CD}$.

EXAMPLE 1 *Identifying Relationships in Space*

Think of each segment in the diagram as part of a line. Which of the lines appear to fit the description?



a. parallel to $\overleftrightarrow{AB}$ and contains D

b. perpendicular to $\overleftrightarrow{AB}$ and contains D

c. skew to $\overleftrightarrow{AB}$ and contains D

d. Name the plane(s) that contain D and appear to be parallel to plane ABE.

SOLUTION

a. $\overleftrightarrow{CD}$, $\overleftrightarrow{GH}$, and $\overleftrightarrow{EF}$ are all parallel to $\overleftrightarrow{AB}$, but only $\overleftrightarrow{CD}$ passes through D and is parallel to $\overleftrightarrow{AB}$.

b. $\overleftrightarrow{BC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{AE}$, and $\overleftrightarrow{BF}$ are all perpendicular to $\overleftrightarrow{AB}$, but only $\overleftrightarrow{AD}$ passes through D and is perpendicular to $\overleftrightarrow{AB}$.

c. $\overleftrightarrow{DG}$, $\overleftrightarrow{DH}$, and $\overleftrightarrow{DE}$ all pass through D and are skew to $\overleftrightarrow{AB}$.

d. Only plane DCH contains D and is parallel to plane ABE.

STUDENT HELP

Look Back
For help identifying perpendicular lines, see p. 79.

Notice in Example 1 that, although there are many lines through D that are skew to $\overleftrightarrow{AB}$, there is only one line through D that is parallel to $\overleftrightarrow{AB}$ and there is only one line through D that is perpendicular to $\overleftrightarrow{AB}$.

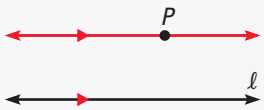
STUDENT HELP

APPLICATION LINK Visit our Web site www.mcdougallittell.com for more information about the parallel postulate.

PARALLEL AND PERPENDICULAR POSTULATES

POSTULATE 13 ***Parallel Postulate***

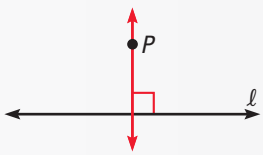
If there is a line and a point not on the line, then there is exactly one line through the point parallel to the given line.



There is exactly one line through *P* parallel to ℓ.

POSTULATE 14 ***Perpendicular Postulate***

If there is a line and a point not on the line, then there is exactly one line through the point perpendicular to the given line.



There is exactly one line through *P* perpendicular to ℓ.

You can use a compass and a straightedge to construct the line that passes through a given point and is perpendicular to a given line. In Lesson 6.6, you will learn why this construction works.

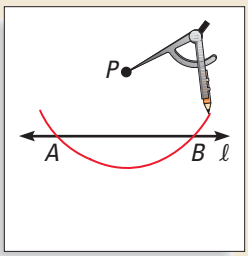
You will learn how to construct a parallel line in Lesson 3.5.

ACTIVITY

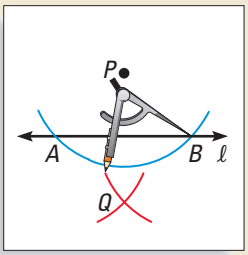
Construction

A Perpendicular to a Line

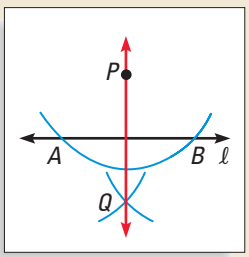
Use the following steps to construct a line that passes through a given point P and is perpendicular to a given line ℓ.



1 Place the compass point at P and draw an arc that intersects line ℓ twice. Label the intersections A and B.



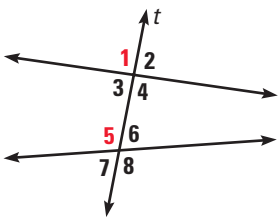
2 Draw an arc with center A. Using the same radius, draw an arc with center B. Label the intersection of the arcs Q.



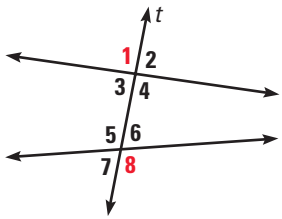
3 Use a straightedge to draw $\overleftrightarrow{PQ}$. $\overleftrightarrow{PQ} \perp \ell$.

GOAL 2 IDENTIFYING ANGLES FORMED BY TRANSVERSALS

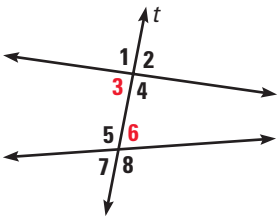
A **transversal** is a line that intersects two or more coplanar lines at different points. For instance, in the diagrams below, line t is a transversal. The angles formed by two lines and a transversal are given special names.



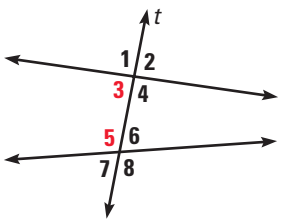
Two angles are **corresponding angles** if they occupy corresponding positions. For example, angles **1** and **5** are corresponding angles.



Two angles are **alternate exterior angles** if they lie outside the two lines on opposite sides of the transversal. Angles **1** and **8** are alternate exterior angles.



Two angles are **alternate interior angles** if they lie between the two lines on opposite sides of the transversal. Angles **3** and **6** are alternate interior angles.



Two angles are **consecutive interior angles** if they lie between the two lines on the same side of the transversal. Angles **3** and **5** are consecutive interior angles.

Consecutive interior angles are sometimes called **same side interior angles**.

EXAMPLE 2 *Identifying Angle Relationships*

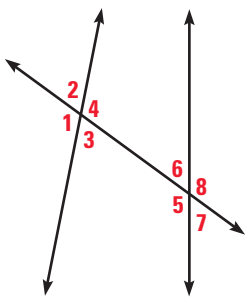
STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

List all pairs of angles that fit the description.

a. corresponding **b.** alternate exterior

c. alternate interior **d.** consecutive interior



SOLUTION

a. $\angle 1$ and $\angle 5$
$\angle 2$ and $\angle 6$
$\angle 3$ and $\angle 7$
$\angle 4$ and $\angle 8$

b. $\angle 1$ and $\angle 8$
$\angle 2$ and $\angle 7$

c. $\angle 3$ and $\angle 6$
$\angle 4$ and $\angle 5$

d. $\angle 3$ and $\angle 5$
$\angle 4$ and $\angle 6$

GUIDED PRACTICE

Vocabulary Check ✓ 1. Draw two lines and a transversal. Identify a pair of alternate interior angles.

Concept Check ✓ 2. How are skew lines and parallel lines alike? How are they different?

Skill Check ✓ **Match the photo with the corresponding description of the chopsticks.**

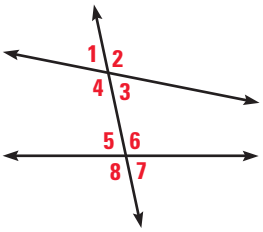
A. skew **B.** parallel **C.** intersecting

3.

4.

5.

In Exercises 6–9, use the diagram at the right.



6. Name a pair of corresponding angles.

7. Name a pair of alternate interior angles.

8. Name a pair of alternate exterior angles.

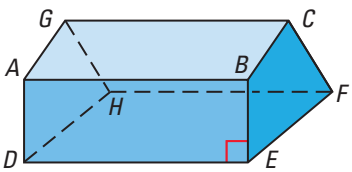
9. Name a pair of consecutive interior angles.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 807.

LINE RELATIONSHIPS **Think of each segment in the diagram as part of a line. Fill in the blank with *parallel, skew,* or *perpendicular*.**



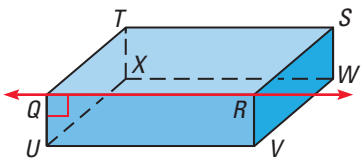
10. $\overleftrightarrow{DE}$, $\overleftrightarrow{AB}$, and $\overleftrightarrow{GC}$ are ___?___.

11. $\overleftrightarrow{DE}$ and $\overleftrightarrow{BE}$ are ___?___.

12. $\overleftrightarrow{BE}$ and $\overleftrightarrow{GC}$ are ___?___.

13. Plane *GAD* and plane *CBE* are ___?___.

IDENTIFYING RELATIONSHIPS **Think of each segment in the diagram as part of a line. There may be more than one right answer.**



14. Name a line parallel to $\overleftrightarrow{QR}$.

15. Name a line perpendicular to $\overleftrightarrow{QR}$.

16. Name a line skew to $\overleftrightarrow{QR}$.

17. Name a plane parallel to plane *QRS*.

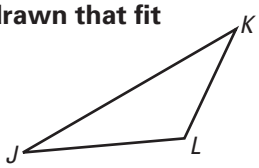
STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 10–20, 27–36

Example 2: Exs. 21–26

APPLYING POSTULATES **How many lines can be drawn that fit the description?**



18. through *L* parallel to $\overleftrightarrow{JK}$

19. through *L* perpendicular to $\overleftrightarrow{JK}$

FOCUS ON PEOPLE

PHILIPPE PETIT walked more than 2000 feet up an inclined cable to the Eiffel Tower. The photo above is from a performance in New York City.

20. **TIGHTROPE WALKING** Philippe Petit sometimes uses a long pole to help him balance on the tightrope. Are the rope and the pole at the left *intersecting*, *perpendicular*, *parallel*, or *skew*?

ANGLE RELATIONSHIPS **Complete the statement with *corresponding, alternate interior, alternate exterior,* or *consecutive interior*.**

21. $\angle 8$ and $\angle 12$ are ___?___ angles.

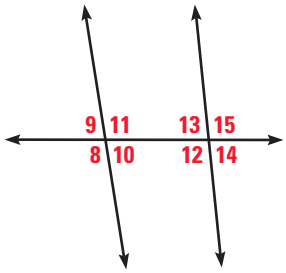
22. $\angle 9$ and $\angle 14$ are ___?___ angles.

23. $\angle 10$ and $\angle 12$ are ___?___ angles.

24. $\angle 11$ and $\angle 12$ are ___?___ angles.

25. $\angle 8$ and $\angle 15$ are ___?___ angles.

26. $\angle 10$ and $\angle 14$ are ___?___ angles.



ROMAN NUMERALS **Write the Roman numeral that consists of the indicated segments. Then write the base ten value of the Roman numeral. For example, the base ten value of XII is 10 + 1 + 1 = 12.**

Roman numeral	I	V	X	L	M
Base ten value	1	5	10	50	1000

27. Three parallel segments

28. Two non-congruent perpendicular segments

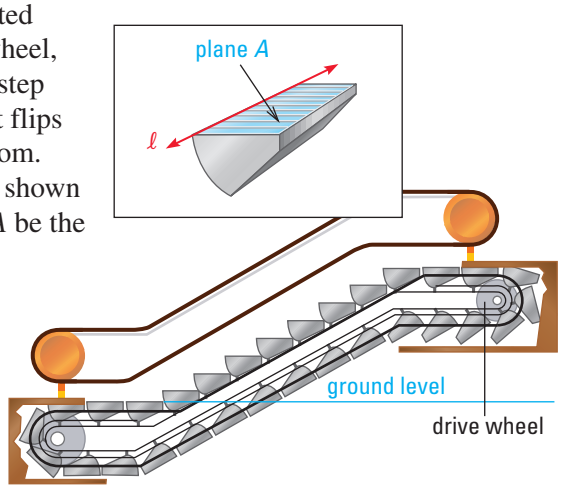
29. Two congruent segments that intersect to form only one angle

30. Two intersecting segments that form vertical angles

31. Four segments, two of which are parallel

ESCALATORS **In Exercises 32–36, use the following information.**

The steps of an escalator are connected to a chain that runs around a drive wheel, which moves continuously. When a step on an up-escalator reaches the top, it flips over and goes back down to the bottom. Each step is shaped like a wedge, as shown at the right. On each step, let plane A be the plane you stand on.



32. As each step moves around the escalator, is plane A always parallel to ground level?

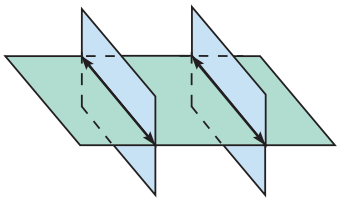
33. When a person is standing on plane A, is it parallel to ground level?

34. Is line ℓ on any step always parallel to ℓ on any other step?

35. Is plane A on any step always parallel to plane A on any other step?

36. As each step moves around the escalator, how many positions are there at which plane A is perpendicular to ground level?

37. **LOGICAL REASONING** If two parallel planes are cut by a third plane, explain why the lines of intersection are parallel.



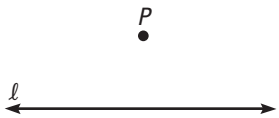
38. *Writing* What does "two lines intersect" mean?

39. **CONSTRUCTION** Draw a horizontal line ℓ and a point P above ℓ. Construct a line through P perpendicular to ℓ.

40. **CONSTRUCTION** Draw a diagonal line m and a point Q below m. Construct a line through Q perpendicular to m.

Test Preparation

41. **MULTIPLE CHOICE** In the diagram at the right, how many lines can be drawn through point P that are perpendicular to line ℓ?



(A) 0 (B) 1 (C) 2

(D) 3 (E) More than 3

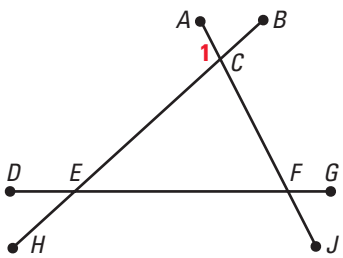
42. **MULTIPLE CHOICE** If two lines intersect, then they must be ___?___.

(A) perpendicular (B) parallel (C) coplanar

(D) skew (E) None of these

★ Challenge

ANGLE RELATIONSHIPS **Complete each statement. List all possible correct answers.**



43. $\angle 1$ and ___?___ are corresponding angles.

44. $\angle 1$ and ___?___ are consecutive interior angles.

45. $\angle 1$ and ___?___ are alternate interior angles.

46. $\angle 1$ and ___?___ are alternate exterior angles.

EXTRA CHALLENGE
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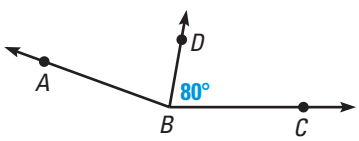
MIXED REVIEW

47. **ANGLE BISECTOR** The ray $\overrightarrow{BD}$ bisects $\angle ABC$, as shown at the right. Find $m\angle ABD$ and $m\angle ABC$. **(Review 1.5 for 3.2)**



COMPLEMENTS AND SUPPLEMENTS **Find the measures of a complement and a supplement of the angle.** **(Review 1.6 for 3.2)**

48. 71° **49.** 13° **50.** 56°

51. 88° **52.** 27° **53.** 68°

54. 1° **55.** 60° **56.** 45°

WRITING REASONS **Solve the equation and state a reason for each step.** **(Review 2.4 for 3.2)**

57. $x + 13 = 23$ **58.** $x - 8 = 17$ **59.** $4x + 11 = 31$

60. $2x + 9 = 4x - 29$ **61.** $2(x - 1) + 3 = 17$ **62.** $5x + 7(x - 10) = -94$

3.2 Proof and Perpendicular Lines

What you should learn

GOAL 1 Write different types of proofs.

GOAL 2 Prove results about perpendicular lines.

Why you should learn it

▼ Make conclusions from things you see in **real life**, such as the reflection in **Ex. 28**.

REAL LIFE

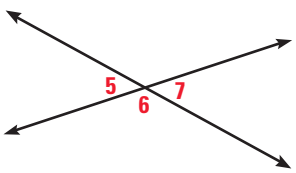
GOAL 1 COMPARING TYPES OF PROOFS

There is more than one way to write a proof. The two-column proof below is from Lesson 2.6. It can also be written as a paragraph proof or as a *flow* proof. A **flow proof** uses arrows to show the flow of the logical argument. Each reason in a flow proof is written below the statement it justifies.

EXAMPLE 1 *Comparing Types of Proof*

GIVEN ▶ $\angle 5$ and $\angle 6$ are a linear pair.
$\angle 6$ and $\angle 7$ are a linear pair.

PROVE ▶ $\angle 5 \cong \angle 7$



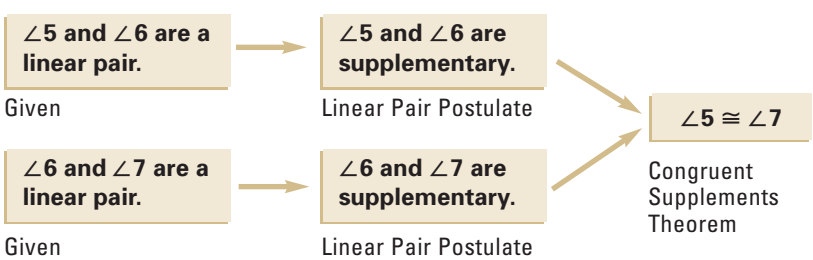
Method 1 *Two-column Proof*

Statements	Reasons
1. $\angle 5$ and $\angle 6$ are a linear pair. $\angle 6$ and $\angle 7$ are a linear pair.	**1.** Given
2. $\angle 5$ and $\angle 6$ are supplementary. $\angle 6$ and $\angle 7$ are supplementary.	**2.** Linear Pair Postulate
3. $\angle 5 \cong \angle 7$	**3.** Congruent Supplements Theorem

Method 2 *Paragraph Proof*

Because $\angle 5$ and $\angle 6$ are a linear pair, the Linear Pair Postulate says that $\angle 5$ and $\angle 6$ are supplementary. The same reasoning shows that $\angle 6$ and $\angle 7$ are supplementary. Because $\angle 5$ and $\angle 7$ are both supplementary to $\angle 6$, the Congruent Supplements Theorem says that $\angle 5 \cong \angle 7$.

Method 3 *Flow Proof*

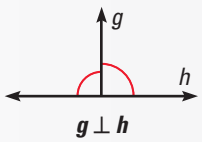


GOAL 2 PROVING RESULTS ABOUT PERPENDICULAR LINES

THEOREMS

THEOREM 3.1

If two lines intersect to form a linear pair of congruent angles, then the lines are perpendicular.

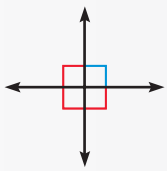


THEOREM 3.2

If two sides of two adjacent acute angles are perpendicular, then the angles are complementary.

THEOREM 3.3

If two lines are perpendicular, then they intersect to form four right angles.



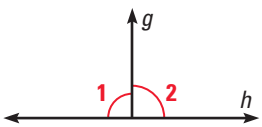
You will prove Theorem 3.2 and Theorem 3.3 in Exercises 17–19.

EXAMPLE 2 *Proof of Theorem 3.1*

Proof

Write a proof of Theorem 3.1.

SOLUTION



GIVEN ▶ $\angle 1 \cong \angle 2$, $\angle 1$ and $\angle 2$ are a linear pair.

PROVE ▶ $g \perp h$

Plan for Proof Use $m\angle 1 + m\angle 2 = 180°$ and $m\angle 1 = m\angle 2$ to show $m\angle 1 = 90°$.

$\angle 1$ and $\angle 2$ are a linear pair.
Given ↓
$\angle 1$ and $\angle 2$ are supplementary.
Linear Pair Postulate ↓
$m\angle 1 + m\angle 2 = 180°$
Def. of supplementary ∠s

$\angle 1 \cong \angle 2$
Given ↓
$m\angle 1 = m\angle 2$
Def. of ≅ angles

$m\angle 1 + m\angle 1 = 180°$
Substitution prop. of equality → **$2 \cdot (m\angle 1) = 180°$**
Distributive prop.

$m\angle 1 = 90°$
Div. prop. of equality → **$\angle 1$ is a right ∠.**
Def. of right angle → **$g \perp h$**
Def. of ⊥ lines

STUDENT HELP

Study Tip
When you write a complicated proof, it may help to write a plan first. The plan will also help others to understand your proof.

CONCEPT SUMMARY **TYPES OF PROOFS**

You have now studied three types of proofs.

1. **TWO-COLUMN PROOF** This is the most formal type of proof. It lists numbered statements in the left column and a reason for each statement in the right column.
2. **PARAGRAPH PROOF** This type of proof describes the logical argument with sentences. It is more conversational than a two-column proof.
3. **FLOW PROOF** This type of proof uses the same statements and reasons as a two-column proof, but the logical flow connecting the statements is indicated by arrows.

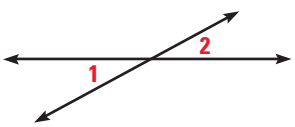
GUIDED PRACTICE

Vocabulary Check ✓ 1. Define *perpendicular lines*.

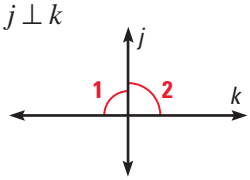
Concept Check ✓ 2. Which postulate or theorem guarantees that there is only one line that can be constructed perpendicular to a given line from a given point not on the line?

Skill Check ✓ **Write the postulate or theorem that justifies the statement about the diagram.**

3. $\angle 1 \cong \angle 2$

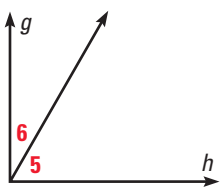


4. $j \perp k$

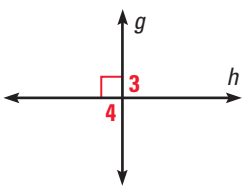


Write the postulate or theorem that justifies the statement, given that $g \perp h$.

5. $m\angle 5 + m\angle 6 = 90°$

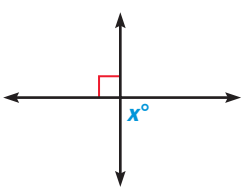


6. $\angle 3$ and $\angle 4$ are right angles.



Find the value of x.

7.



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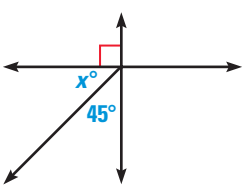
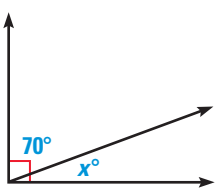


9.



10. **ERROR ANALYSIS** It is given that $\angle ABC \cong \angle CBD$. A student concludes that because $\angle ABC$ and $\angle CBD$ are congruent adjacent angles, $\overleftrightarrow{AB} \perp \overleftrightarrow{CB}$. What is wrong with this reasoning? Draw a diagram to support your answer.

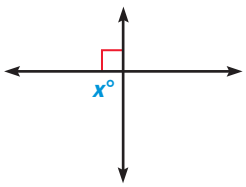
PRACTICE AND APPLICATIONS

STUDENT HELP

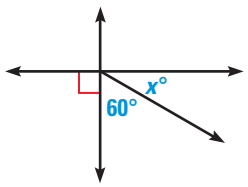
Extra Practice to help you master skills is on p. 807.

USING ALGEBRA **Find the value of *x*.**

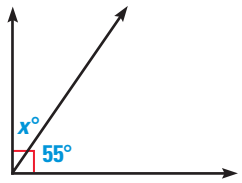
11.



12.

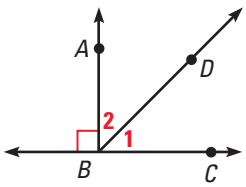


13.

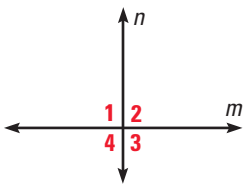


LOGICAL REASONING **What can you conclude about the labeled angles?**

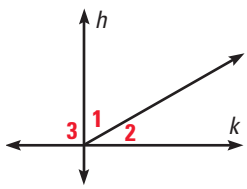
14. $\overline{AB} \perp \overline{CB}$



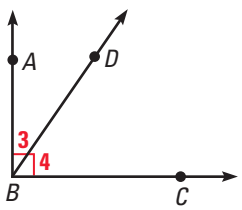
15. $n \perp m$



16. $h \perp k$



17. **DEVELOPING PARAGRAPH PROOF** Fill in the lettered blanks to complete the proof of Theorem 3.2.



GIVEN ▶ $\overrightarrow{BA} \perp \overrightarrow{BC}$

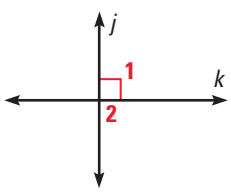
PROVE ▶ $\angle 3$ and $\angle 4$ are complementary.

Because $\overrightarrow{BA} \perp \overrightarrow{BC}$, $\angle ABC$ is a ___**a.**___ and $m\angle ABC =$ ___**b.**___. According to the ___**c.**___ Postulate, $m\angle 3 + m\angle 4 = m\angle ABC$. So, by the substitution property of equality, ___**d.**___ + ___**e.**___ = ___**f.**___. By definition, $\angle 3$ and $\angle 4$ are complementary.

18. **DEVELOPING FLOW PROOF** Fill in the lettered blanks to complete the proof of part of Theorem 3.3. Because the lines are perpendicular, they intersect to form a right angle. Call that $\angle 1$.

GIVEN ▶ $j \perp k$, $\angle 1$ and $\angle 2$ are a linear pair.

PROVE ▶ $\angle 2$ is a right angle.



$\angle 1$ and $\angle 2$ are a linear pair.
Given
↓
___a.___
Linear Pair Postulate
↓
$m\angle 1 + m\angle 2 = 180°$
___b.___

$j \perp k$
Given
↓
$\angle 1$ is a right $\angle$.
Def. of $\perp$ lines
↓
$m\angle 1 = 90°$
___c.___

$90° + m\angle 2 = 180°$
___d.___
→
___e.___
Subtr. prop. of equality
→
$\angle 2$ is a right $\angle$.
___f.___

STUDENT HELP

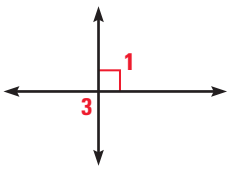
HOMEWORK HELP

Example 1: Exs. 17–23
Example 2: Exs. 11–19, 24, 25

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with writing proofs in Exs. 17–24.

19. DEVELOPING TWO-COLUMN PROOF Fill in the blanks to complete the proof of part of Theorem 3.3.



GIVEN ▶ $\angle 1$ is a right angle.

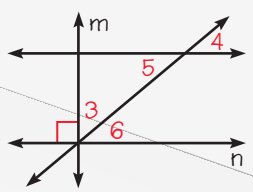
PROVE ▶ $\angle 3$ is a right angle.

Statements	Reasons
1. $\angle 1$ and $\angle 3$ are vertical angles.	**1.** Definition of vertical angles
2. ___?___	**2.** Vertical Angles Theorem
3. $m\angle 1 = m\angle 3$	**3.** ___?___
4. $\angle 1$ is a right angle.	**4.** ___?___
5. ___?___	**5.** Definition of right angle
6. ___?___	**6.** Substitution prop. of equality
7. ___?___	**7.** Definition of right angle

DEVELOPING PROOF In Exercises 20–23, use the following information. Dan is trying to figure out how to prove that $\angle 5 \cong \angle 6$ below. First he wrote everything that he knew about the diagram, as shown below in **blue**.

GIVEN $m \perp n$, $\angle 3$ and $\angle 4$ are complementary.

PROVE $\angle 5 \cong \angle 6$



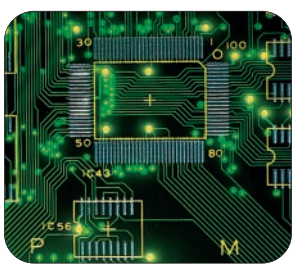
$m \perp n \rightarrow \angle 3$ and $\angle 6$ are complementary.

$\angle 3$ and $\angle 4$ are complementary.

$\angle 4$ and $\angle 5$ are vertical angles. $\rightarrow \angle 4 \cong \angle 5$

$\angle 4 \cong \angle 6 \rightarrow \angle 5 \cong \angle 6$

FOCUS ON APPLICATIONS



REAL LIFE CIRCUIT BOARDS The lines on circuit boards are made of metal and carry electricity. The lines must not touch each other or the electricity will flow to the wrong place, creating a *short circuit*.

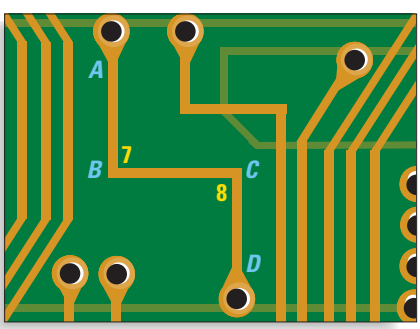
20. Write a justification for each statement Dan wrote in blue.

21. After writing all he knew, Dan wrote what he was supposed to prove in **red**. He also wrote $\angle 4 \cong \angle 6$ because he knew that if $\angle 4 \cong \angle 6$ and $\angle 4 \cong \angle 5$, then $\angle 5 \cong \angle 6$. Write a justification for this step.

22. How can you use Dan's blue statements to prove that $\angle 4 \cong \angle 6$?

23. Copy and complete Dan's flow proof.

24. CIRCUIT BOARDS The diagram shows part of a circuit board. Write any type of proof.

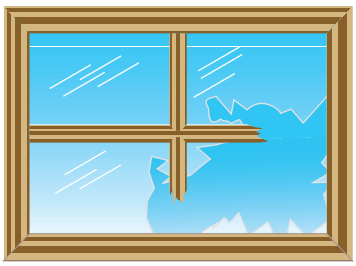


GIVEN ▶ $\overline{AB} \perp \overline{BC}$, $\overline{BC} \perp \overline{CD}$

PROVE ▶ $\angle 7 \cong \angle 8$

Plan for Proof Show that $\angle 7$ and $\angle 8$ are both right angles.

25. **WINDOW REPAIR** Cathy is fixing a window frame. She fit two strips of wood together to make the crosspieces. For the glass panes to fit, each angle of the crosspieces must be a right angle. Must Cathy measure all four angles to be sure they are all right angles? Explain.



Test Preparation

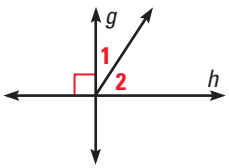
26. **MULTIPLE CHOICE** Which of the following is true if $g \perp h$?

Ⓐ $m\angle 1 + m\angle 2 > 180°$

Ⓑ $m\angle 1 + m\angle 2 < 180°$

Ⓒ $m\angle 1 + m\angle 2 = 180°$

Ⓓ Cannot be determined



27. **MULTIPLE CHOICE** Which of the following must be true if $m\angle ACD = 90°$?

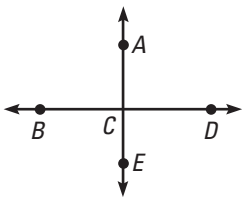
I. $\angle BCE$ is a right angle.

II. $\overleftrightarrow{AE} \perp \overleftrightarrow{BD}$

III. $\angle BCA$ and $\angle BCE$ are complementary.

Ⓐ I only Ⓑ I and II only Ⓒ III only

Ⓓ I, II, and III Ⓔ None of these



Challenge

28. **REFLECTIONS** Ann has a full-length mirror resting against the wall of her room. Ann notices that the floor and its reflection do not form a straight angle. She concludes that the mirror is not perpendicular to the floor. Explain her reasoning.

EXTRA CHALLENGE
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MIXED REVIEW

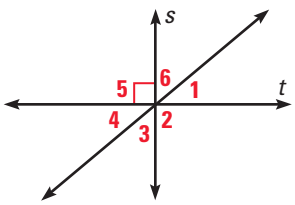
ANGLE MEASURES **Complete the statement given that $s \perp t$.** (Review 2.6 for 3.3)

29. If $m\angle 1 = 38°$, then $m\angle 4 =$ ___?___.

30. $m\angle 2 =$ ___?___

31. If $m\angle 6 = 51°$, then $m\angle 1 =$ ___?___.

32. If $m\angle 3 = 42°$, then $m\angle 1 =$ ___?___.



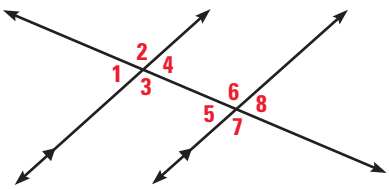
ANGLES **List all pairs of angles that fit the description.** (Review 3.1)

33. Corresponding angles

34. Alternate interior angles

35. Alternate exterior angles

36. Consecutive interior angles



3.3 Parallel Lines and Transversals

What you should learn

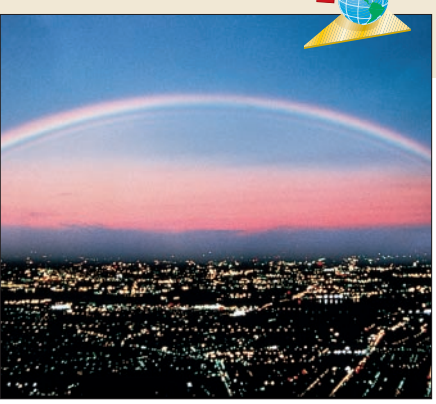
GOAL 1 Prove and use results about parallel lines and transversals.

GOAL 2 Use properties of parallel lines to solve **real-life** problems, such as estimating Earth's circumference in **Example 5**.

Why you should learn it

▼ Properties of parallel lines help you understand how rainbows are formed, as in **Ex. 30**.

REAL LIFE



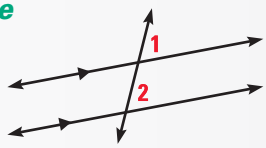
GOAL 1 PROPERTIES OF PARALLEL LINES

In the activity on page 142, you may have discovered the following results.

POSTULATE

POSTULATE 15 ***Corresponding Angles Postulate***

If two parallel lines are cut by a transversal, then the pairs of corresponding angles are congruent.



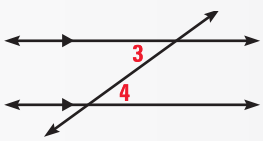
$\angle 1 \cong \angle 2$

You are asked to prove Theorems 3.5, 3.6, and 3.7 in Exercises 27–29.

THEOREMS ABOUT PARALLEL LINES

THEOREM 3.4 ***Alternate Interior Angles***

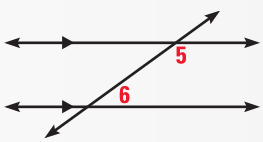
If two parallel lines are cut by a transversal, then the pairs of alternate interior angles are congruent.



$\angle 3 \cong \angle 4$

THEOREM 3.5 ***Consecutive Interior Angles***

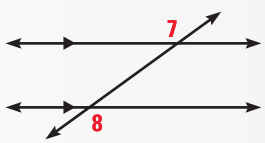
If two parallel lines are cut by a transversal, then the pairs of consecutive interior angles are supplementary.



$m\angle 5 + m\angle 6 = 180°$

THEOREM 3.6 ***Alternate Exterior Angles***

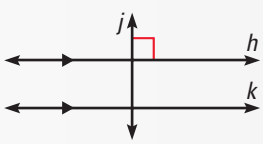
If two parallel lines are cut by a transversal, then the pairs of alternate exterior angles are congruent.



$\angle 7 \cong \angle 8$

THEOREM 3.7 ***Perpendicular Transversal***

If a transversal is perpendicular to one of two parallel lines, then it is perpendicular to the other.



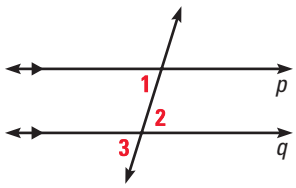
$j \perp k$

STUDENT HELP

Study Tip
When you prove a theorem, the hypotheses of the theorem becomes the GIVEN, and the conclusion is what you must PROVE.

EXAMPLE 1 *Proving the Alternate Interior Angles Theorem*

Prove the Alternate Interior Angles Theorem.



SOLUTION

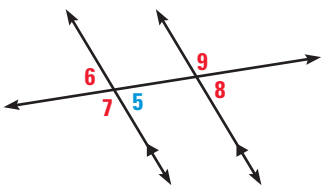
GIVEN ▶ $p \parallel q$

PROVE ▶ $\angle 1 \cong \angle 2$

Statements	Reasons
1. $p \parallel q$	**1.** Given
2. $\angle 1 \cong \angle 3$	**2.** Corresponding Angles Postulate
3. $\angle 3 \cong \angle 2$	**3.** Vertical Angles Theorem
4. $\angle 1 \cong \angle 2$	**4.** Transitive Property of Congruence

EXAMPLE 2 *Using Properties of Parallel Lines*

Given that $m\angle 5 = 65°$, find each measure. Tell which postulate or theorem you use.



a. $m\angle 6$ **b.** $m\angle 7$

c. $m\angle 8$ **d.** $m\angle 9$

SOLUTION

a. $m\angle 6 = m\angle 5 = 65°$ — **Vertical Angles Theorem**

b. $m\angle 7 = 180° - m\angle 5 = 115°$ — **Linear Pair Postulate**

c. $m\angle 8 = m\angle 5 = 65°$ — **Corresponding Angles Postulate**

d. $m\angle 9 = m\angle 7 = 115°$ — **Alternate Exterior Angles Theorem**

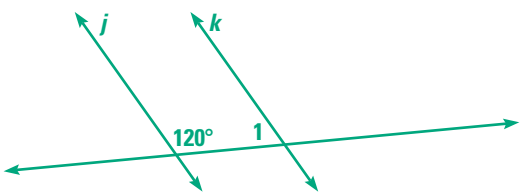
FOCUS ON CAREERS

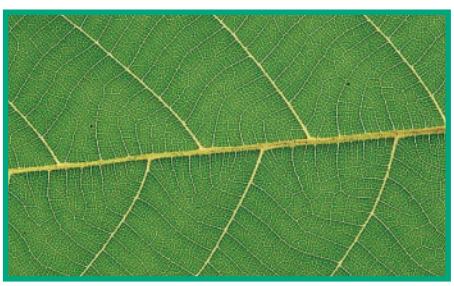
BOTANY Botanists study plants and environmental issues such as conservation, weed control, and re-vegetation.

CAREER LINK
www.mcdougallittell.com

EXAMPLE 3 *Classifying Leaves*

BOTANY Some plants are classified by the arrangement of the veins in their leaves. In the diagram of the leaf, $j \parallel k$. What is $m\angle 1$?





SOLUTION

$m\angle 1 + 120° = 180°$ — **Consecutive Interior Angles Theorem**

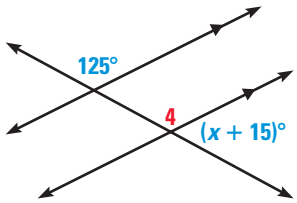
$m\angle 1 = 60°$ — **Subtract.**

GOAL 2 PROPERTIES OF SPECIAL PAIRS OF ANGLES

EXAMPLE 4 *Using Properties of Parallel Lines*

Using Algebra

Use properties of parallel lines to find the value of x.



SOLUTION

$m\angle 4 = 125°$	**Corresponding Angles Postulate**
$m\angle 4 + (x + 15)° = 180°$	**Linear Pair Postulate**
$125° + (x + 15)° = 180°$	**Substitute.**
$x = 40$	**Subtract.**

EXAMPLE 5 *Estimating Earth's Circumference*

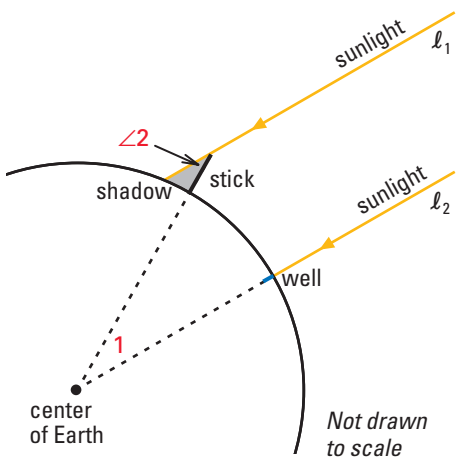
HISTORY CONNECTION Eratosthenes was a Greek scholar. Over 2000 years ago, he estimated Earth's circumference by using the fact that the Sun's rays are parallel.

Eratosthenes chose a day when the Sun shone exactly down a vertical well in Syene at noon. On that day, he measured the angle the Sun's rays made with a vertical stick in Alexandria at noon. He discovered that

$$m\angle 2 \approx \frac{1}{50} \text{ of a circle.}$$

By using properties of parallel lines, he knew that $m\angle 1 = m\angle 2$. So he reasoned that

$$m\angle 1 \approx \frac{1}{50} \text{ of a circle.}$$



At the time, the distance from Syene to Alexandria was believed to be **575 miles**.

$$\frac{1}{50} \text{ of a circle} \approx \frac{575 \text{ miles}}{\text{Earth's circumference}}$$

$$\text{Earth's circumference} \approx 50(575 \text{ miles}) \longleftarrow \textbf{Use cross product property.}$$

$$\approx 29{,}000 \text{ miles}$$

How did Eratosthenes know that $m\angle 1 = m\angle 2$?

SOLUTION

Because the Sun's rays are parallel, $\ell_1 \parallel \ell_2$. Angles 1 and 2 are alternate interior angles, so $\angle 1 \cong \angle 2$. By the definition of congruent angles, $m\angle 1 = m\angle 2$.

STUDENT HELP

INTERNET **APPLICATION LINK** Visit our Web site www.mcdougallittell.com for more information about Eratosthenes' estimate in Example 5.

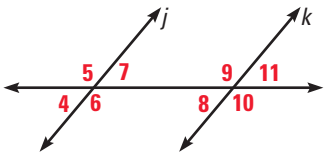
GUIDED PRACTICE

Vocabulary Check ✓

1. Sketch two parallel lines cut by a transversal. Label a pair of consecutive interior angles.

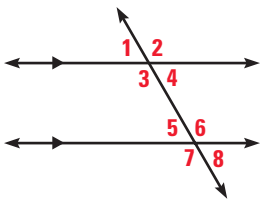
Concept Check ✓

2. In the figure at the right, $j \parallel k$. How many angle measures must be given in order to find the measure of every angle? Explain your reasoning.

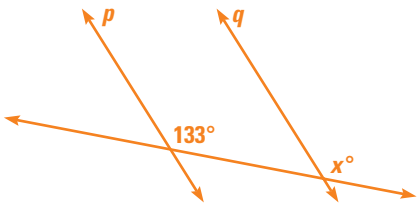


Skill Check ✓

State the postulate or theorem that justifies the statement.



3. $\angle 2 \cong \angle 7$
4. $\angle 4 \cong \angle 5$
5. $m\angle 3 + m\angle 5 = 180°$
6. $\angle 2 \cong \angle 6$
7. In the diagram of the feather below, lines p and q are parallel. What is the value of x?



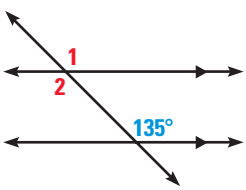
PRACTICE AND APPLICATIONS

STUDENT HELP

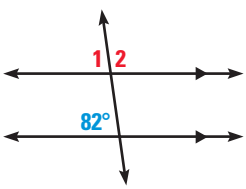
Extra Practice to help you master skills is on p. 808.

USING PARALLEL LINES **Find $m\angle 1$ and $m\angle 2$. Explain your reasoning.**

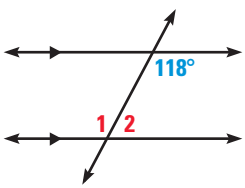
8.



9.

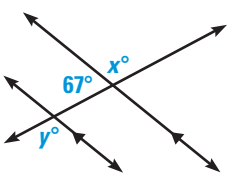


10.

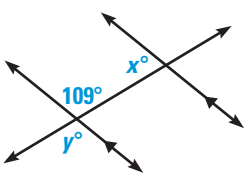


USING PARALLEL LINES **Find the values of x and y. Explain your reasoning.**

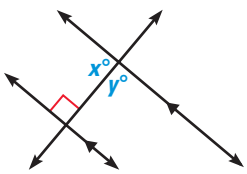
11.



12.



13.

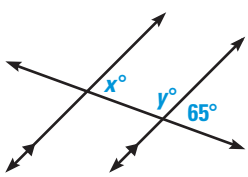


STUDENT HELP

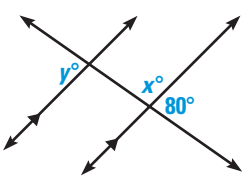
HOMEWORK HELP

Example 1: Exs. 27–29
Example 2: Exs. 8–17
Example 3: Exs. 8–17
Example 4: Exs. 18–26
Example 5: Ex. 30

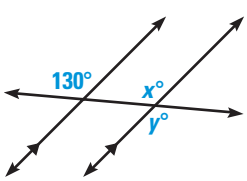
14.



15.

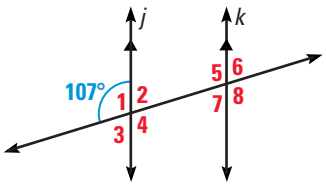


16.



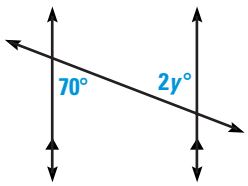
17. USING PROPERTIES OF PARALLEL LINES Use the given information to find the measures of the other seven angles in the figure at the right.

GIVEN ▶ $j \parallel k$, $m\angle 1 = 107°$

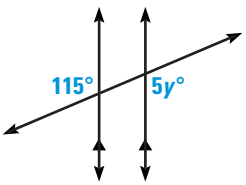


USING ALGEBRA **Find the value of *y*.**

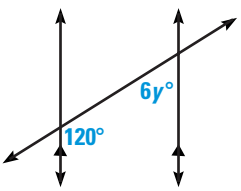
18.



19.

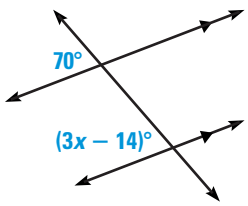


20.

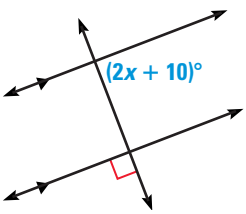


USING ALGEBRA **Find the value of *x*.**

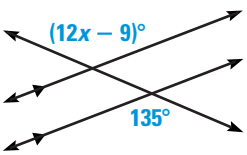
21.



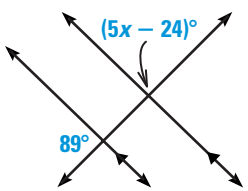
22.



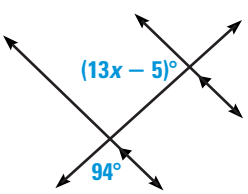
23.



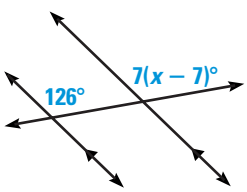
24.



25.



26.



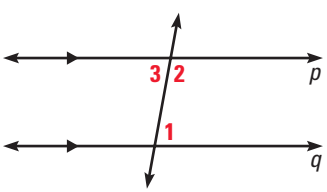
STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with proving theorems in Exs. 27–29.

27. DEVELOPING PROOF Complete the proof of the Consecutive Interior Angles Theorem.

GIVEN ▶ $p \parallel q$

PROVE ▶ $\angle 1$ and $\angle 2$ are supplementary.



Statements	Reasons
1. ___?___	**1.** Given
2. $\angle 1 \cong \angle 3$	**2.** ___?___
3. ___?___	**3.** Definition of congruent angles
4. ___?___	**4.** Definition of linear pair
5. $m\angle 3 + m\angle 2 = 180°$	**5.** ___?___
6. ___?___	**6.** Substitution prop. of equality
7. $\angle 1$ and $\angle 2$ are supplementary.	**7.** ___?___

STUDENT HELP

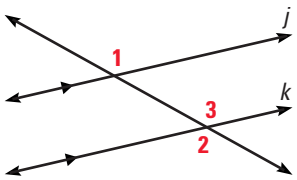
Study Tip
When you prove a theorem you may use any previous theorem, but you may not use the one you're proving.

PROVING THEOREMS 3.6 AND 3.7 **In Exercises 28 and 29, complete the proof.**

28. To prove the Alternate Exterior Angles Theorem, first show that $\angle 1 \cong \angle 3$. Then show that $\angle 3 \cong \angle 2$. Finally, show that $\angle 1 \cong \angle 2$.

GIVEN ▶ $j \parallel k$

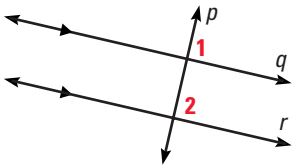
PROVE ▶ $\angle 1 \cong \angle 2$



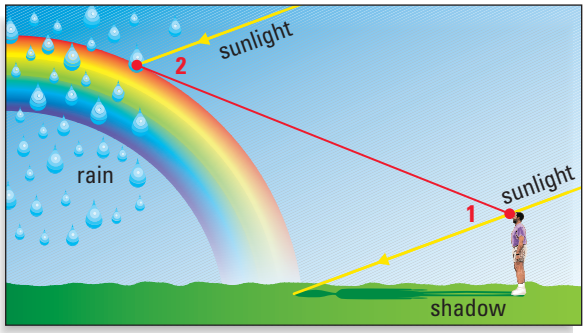
29. To prove the Perpendicular Transversal Theorem, show that $\angle 1$ is a right angle, $\angle 1 \cong \angle 2$, $\angle 2$ is a right angle, and finally that $p \perp r$.

GIVEN ▶ $p \perp q$, $q \parallel r$

PROVE ▶ $p \perp r$

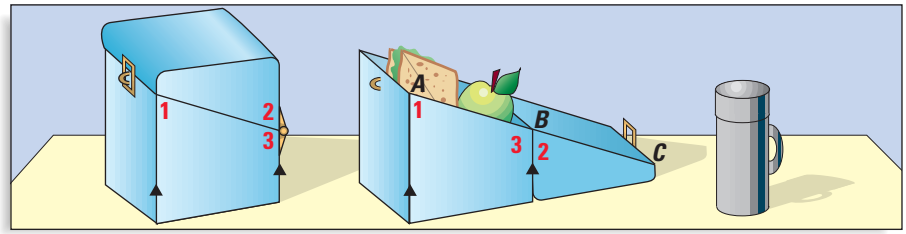


30. **FORMING RAINBOWS** When sunlight enters a drop of rain, different colors leave the drop at different angles. That's what makes a rainbow. For red light, $m\angle 2 = 42°$. What is $m\angle 1$? How do you know?



Test Preparation

31. **MULTI-STEP PROBLEM** You are designing a lunch box like the one below.



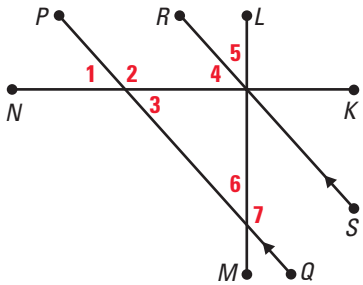
a. The measure of $\angle 1$ is 70°. What is the measure of $\angle 2$? What is the measure of $\angle 3$?

b. *Writing* Explain why $\angle ABC$ is a straight angle.

★ Challenge

32. **USING PROPERTIES OF PARALLEL LINES** Use the given information to find the measures of the other labeled angles in the figure. For each angle, tell which postulate or theorem you used.

GIVEN ▶ $\overline{PQ} \parallel \overline{RS}$, $\overline{LM} \perp \overline{NK}$, $m\angle 1 = 48°$



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

ANGLE MEASURES **$\angle 1$ and $\angle 2$ are supplementary. Find $m\angle 2$.** (Review 1.6)

33. $m\angle 1 = 50°$ **34.** $m\angle 1 = 73°$ **35.** $m\angle 1 = 101°$

36. $m\angle 1 = 107°$ **37.** $m\angle 1 = 111°$ **38.** $m\angle 1 = 118°$

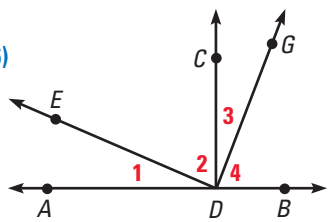
CONVERSES **Write the converse of the statement.** (Review 2.1 for 3.4)

39. If the measure of an angle is 19°, then the angle is acute.

40. I will go to the park if you go with me.

41. I will go fishing if I do not have to work.

FINDING ANGLES **Complete the statement, given that $\overrightarrow{DE} \perp \overrightarrow{DG}$ and $\overleftrightarrow{AB} \perp \overrightarrow{DC}$.** (Review 2.6)



42. If $m\angle 1 = 23°$, then $m\angle 2 =$ __?__.

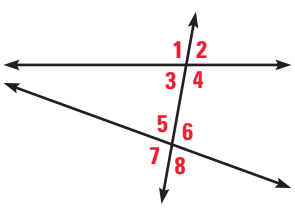
43. If $m\angle 4 = 69°$, then $m\angle 3 =$ __?__.

44. If $m\angle 2 = 70°$, then $m\angle 4 =$ __?__.

QUIZ 1

Self-Test for Lessons 3.1–3.3

Complete the statement. (Lesson 3.1)



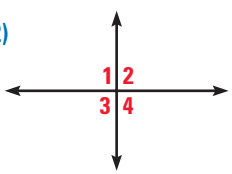
1. $\angle 2$ and __?__ are corresponding angles.

2. $\angle 3$ and __?__ are consecutive interior angles.

3. $\angle 3$ and __?__ are alternate interior angles.

4. $\angle 2$ and __?__ are alternate exterior angles.

5. **PROOF** Write a plan for a proof. (Lesson 3.2)

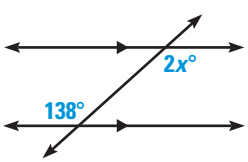


GIVEN ▶ $\angle 1 \cong \angle 2$

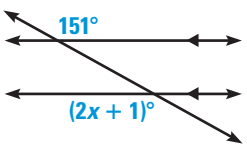
PROVE ▶ $\angle 3$ and $\angle 4$ are right angles.

Find the value of x. (Lesson 3.3)

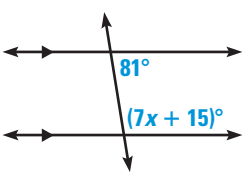
6.



7.



8.



9. **FLAG OF PUERTO RICO** Sketch the flag of Puerto Rico shown at the right. Given that $m\angle 3 = 55°$, determine the measure of $\angle 1$. Justify each step in your argument. (Lesson 3.3)

3.4 Proving Lines are Parallel

What you should learn

GOAL 1 Prove that two lines are parallel.

GOAL 2 Use properties of parallel lines to solve **real-life** problems, such as proving that prehistoric mounds are parallel in **Ex. 19**.

Why you should learn it

▼ Properties of parallel lines help you predict the paths of boats sailing into the wind, as in **Example 4**.

REAL LIFE

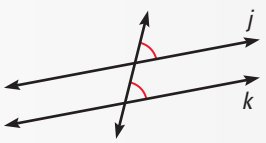
GOAL 1 PROVING LINES ARE PARALLEL

To use the theorems you learned in Lesson 3.3, you must first know that two lines are parallel. You can use the following postulate and theorems to prove that two lines are parallel.

POSTULATE

POSTULATE 16 ***Corresponding Angles Converse***

If two lines are cut by a transversal so that corresponding angles are congruent, then the lines are parallel.



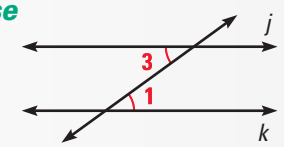
$j \parallel k$

The following theorems are converses of those in Lesson 3.3. Remember that the converse of a true conditional statement is not necessarily true. Thus, each of the following must be proved to be true. Theorems 3.8 and 3.9 are proved in Examples 1 and 2. You are asked to prove Theorem 3.10 in Exercise 30.

THEOREMS ABOUT TRANSVERSALS

THEOREM 3.8 ***Alternate Interior Angles Converse***

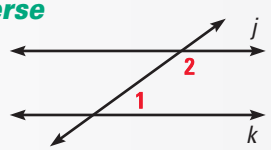
If two lines are cut by a transversal so that alternate interior angles are congruent, then the lines are parallel.



If $\angle 1 \cong \angle 3$, then $j \parallel k$.

THEOREM 3.9 ***Consecutive Interior Angles Converse***

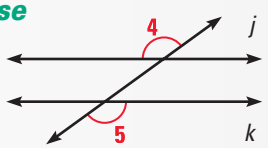
If two lines are cut by a transversal so that consecutive interior angles are supplementary, then the lines are parallel.



If $m\angle 1 + m\angle 2 = 180°$, then $j \parallel k$.

THEOREM 3.10 ***Alternate Exterior Angles Converse***

If two lines are cut by a transversal so that alternate exterior angles are congruent, then the lines are parallel.

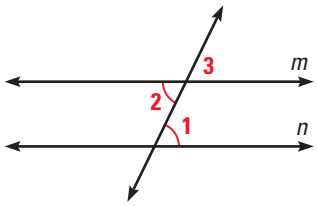


If $\angle 4 \cong \angle 5$, then $j \parallel k$.

EXAMPLE 1 *Proof of the Alternate Interior Angles Converse*

Proof

Prove the Alternate Interior Angles Converse.



SOLUTION

GIVEN ▶ $\angle 1 \cong \angle 2$

PROVE ▶ $m \parallel n$

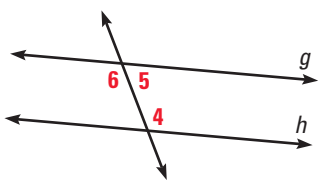
Statements	Reasons
1. $\angle 1 \cong \angle 2$	**1.** Given
2. $\angle 2 \cong \angle 3$	**2.** Vertical Angles Theorem
3. $\angle 1 \cong \angle 3$	**3.** Transitive Property of Congruence
4. $m \parallel n$	**4.** Corresponding Angles Converse

When you prove a theorem you may use only earlier results. For example, to prove Theorem 3.9, you may use Theorem 3.8 and Postulate 16, but you may not use Theorem 3.9 itself or Theorem 3.10.

EXAMPLE 2 *Proof of the Consecutive Interior Angles Converse*

Proof

Prove the Consecutive Interior Angles Converse.



SOLUTION

GIVEN ▶ $\angle 4$ and $\angle 5$ are supplementary.

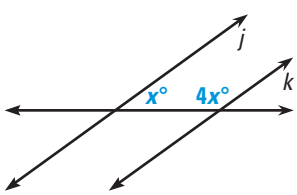
PROVE ▶ $g \parallel h$

Paragraph Proof You are given that $\angle 4$ and $\angle 5$ are supplementary. By the Linear Pair Postulate, $\angle 5$ and $\angle 6$ are also supplementary because they form a linear pair. By the Congruent Supplements Theorem, it follows that $\angle 4 \cong \angle 6$. Therefore, by the Alternate Interior Angles Converse, g and h are parallel.

EXAMPLE 3 *Applying the Consecutive Interior Angles Converse*

Using Algebra

Find the value of x that makes $j \parallel k$.



SOLUTION

Lines j and k will be parallel if the marked angles are supplementary.

$$x° + 4x° = 180°$$

$$5x = 180$$

$$x = 36$$

▶ So, if $x = 36$, then $j \parallel k$.

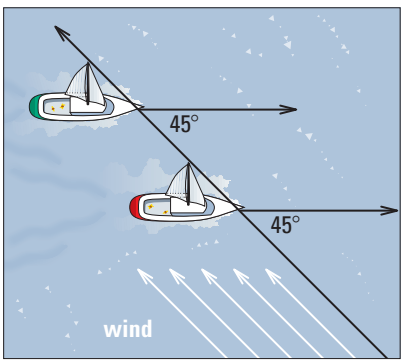
GOAL 2 USING THE PARALLEL CONVERSES

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

EXAMPLE 4 *Using the Corresponding Angles Converse*

REAL LIFE **SAILING** If two boats sail at a 45° angle to the wind as shown, and the wind is constant, will their paths ever cross? Explain.



SOLUTION

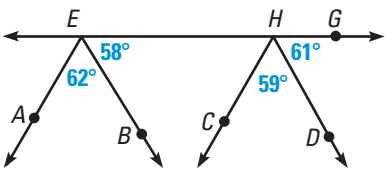
Because corresponding angles are congruent, the boats' paths are parallel. Parallel lines do not intersect, so the boats' paths will not cross.

EXAMPLE 5 *Identifying Parallel Lines*

Decide which rays are parallel.

a. Is $\overrightarrow{EB}$ parallel to $\overrightarrow{HD}$?

b. Is $\overrightarrow{EA}$ parallel to $\overrightarrow{HC}$?

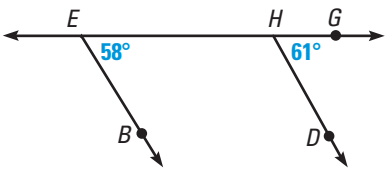


SOLUTION

a. Decide whether $\overrightarrow{EB} \parallel \overrightarrow{HD}$.

$m\angle BEH = 58°$

$m\angle DHG = 61°$



▶ $\angle BEH$ and $\angle DHG$ are corresponding angles, but they are not congruent, so $\overrightarrow{EB}$ and $\overrightarrow{HD}$ are not parallel.

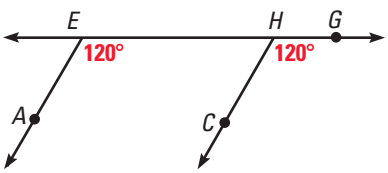
b. Decide whether $\overrightarrow{EA} \parallel \overrightarrow{HC}$.

$m\angle AEH = 62° + 58°$

$= 120°$

$m\angle CHG = 59° + 61°$

$= 120°$



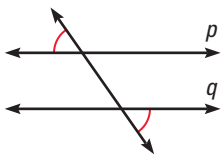
▶ $\angle AEH$ and $\angle CHG$ are congruent corresponding angles, so $\overrightarrow{EA} \parallel \overrightarrow{HC}$.

GUIDED PRACTICE

Vocabulary Check ✓ 1. What are *parallel lines*?

Concept Check ✓ 2. Write the converse of Theorem 3.8. Is the converse true?

Skill Check ✓ **Can you prove that lines *p* and *q* are parallel? If so, describe how.**

3.


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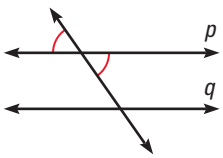

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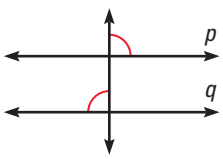

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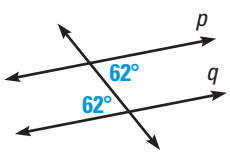

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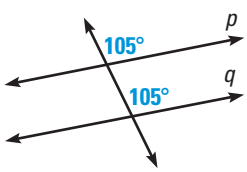

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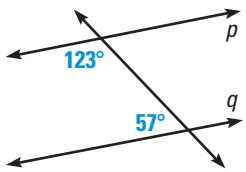
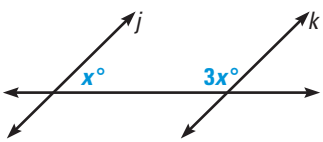


9. Find the value of x that makes $j \parallel k$. Which postulate or theorem about parallel lines supports your answer?

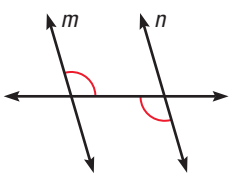


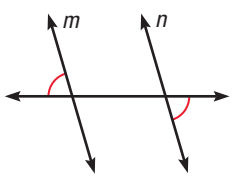
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 808.

LOGICAL REASONING **Is it possible to prove that lines *m* and *n* are parallel? If so, state the postulate or theorem you would use.**

10.


11.


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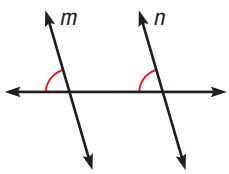
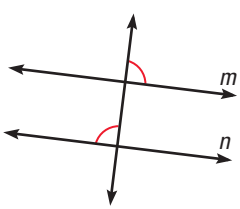


13.


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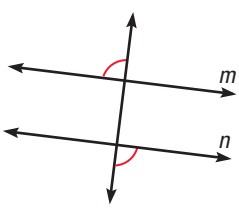

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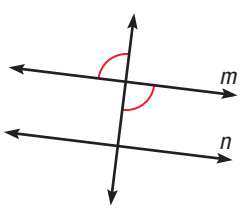
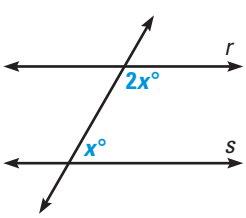


USING ALGEBRA **Find the value of *x* that makes $r \parallel s$.**

16.


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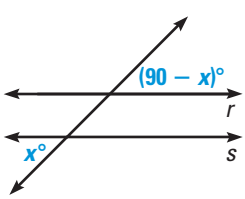

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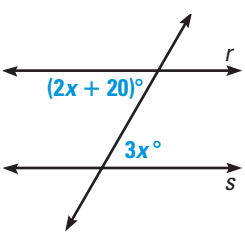


STUDENT HELP

HOMEWORK HELP

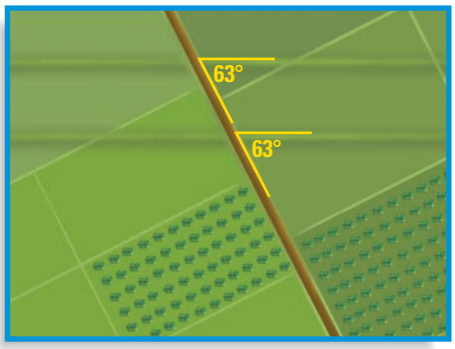
Example 1: Exs. 28, 30
Example 2: Exs. 28, 30
Example 3: Exs. 10–18
Example 4: Exs. 19, 29, 31
Example 5: Exs. 20–27

FOCUS ON APPLICATIONS

THE GREAT SERPENT MOUND, an archaeological mound near Hillsboro, Ohio, is 2 to 5 feet high, and is nearly 20 feet wide. It is over $\frac{1}{4}$ mile long.

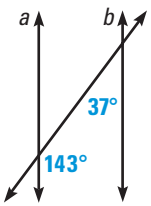
APPLICATION LINK
www.mcdougallittell.com

19. **ARCHAEOLOGY** A farm lane in Ohio crosses two long, straight earthen mounds that may have been built about 2000 years ago. The mounds are about 200 feet apart, and both form a 63° angle with the lane, as shown. Are the mounds parallel? How do you know?

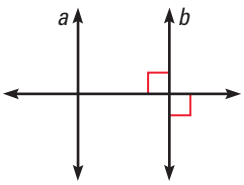


LOGICAL REASONING **Is it possible to prove that lines *a* and *b* are parallel? If so, explain how.**

20.



21.



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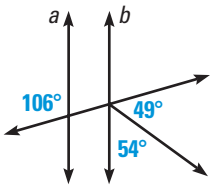
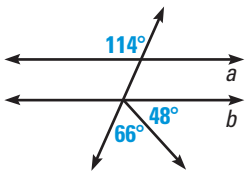
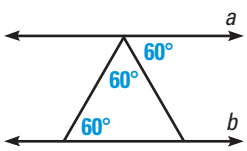


23.



24.



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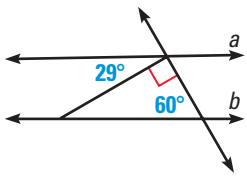
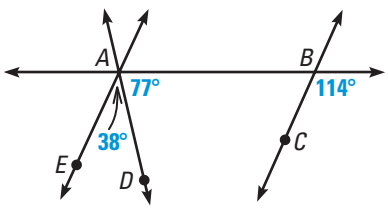
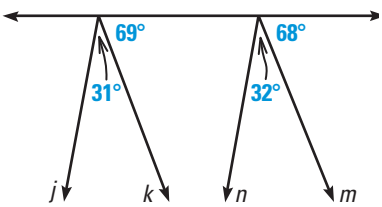


LOGICAL REASONING **Which lines, if any, are parallel? Explain.**

26.



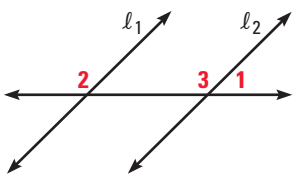
27.



28. **PROOF** Complete the proof.

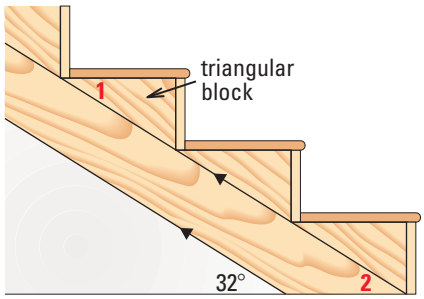
GIVEN ▶ $\angle 1$ and $\angle 2$ are supplementary.

PROVE ▶ $\ell_1 \parallel \ell_2$

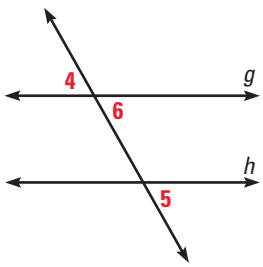


Statements	Reasons
1. $\angle 1$ and $\angle 2$ are supplementary.	**1.** ___?___
2. $\angle 1$ and $\angle 3$ are a linear pair.	**2.** Definition of linear pair
3. ___?___	**3.** Linear Pair Postulate
4. ___?___	**4.** Congruent Supplements Theorem
5. $\ell_1 \parallel \ell_2$	**5.** ___?___

29. **BUILDING STAIRS** One way to build stairs is to attach triangular blocks to an angled support, as shown at the right. If the support makes a 32° angle with the floor, what must $m\angle 1$ be so the step will be parallel to the floor? The sides of the angled support are parallel.



30. **PROVING THEOREM 3.10** Write a two-column proof for the Alternate Exterior Angles Converse: If two lines are cut by a transversal so that alternate exterior angles are congruent, then the lines are parallel.

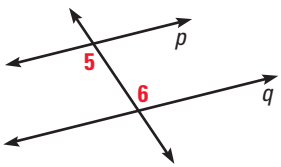


GIVEN ▶ $\angle 4 \cong \angle 5$

PROVE ▶ $g \parallel h$

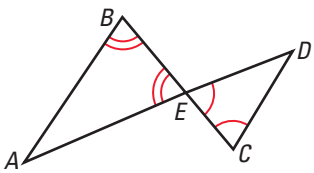
Plan for Proof Show that $\angle 4$ is congruent to $\angle 6$, show that $\angle 6$ is congruent to $\angle 5$, and then use the Corresponding Angles Converse.

31. *Writing* In the diagram at the right, $m\angle 5 = 110°$ and $m\angle 6 = 110°$. Explain why $p \parallel q$.

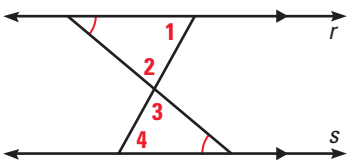


LOGICAL REASONING **Use the information given in the diagram.**

32. What can you prove about $\overline{AB}$ and $\overline{CD}$? Explain.



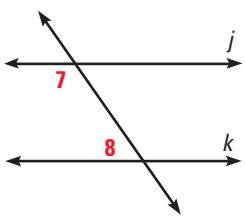
33. What can you prove about $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$? Explain.



PROOF **Write a proof.**

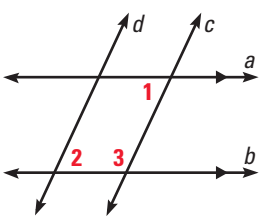
34. **GIVEN** ▶ $m\angle 7 = 125°$, $m\angle 8 = 55°$

PROVE ▶ $j \parallel k$



35. **GIVEN** ▶ $a \parallel b$, $\angle 1 \cong \angle 2$

PROVE ▶ $c \parallel d$



STUDENT HELP

SOFTWARE HELP Visit our Web site www.mcdougallittell.com to see instructions for several software applications.

36. **TECHNOLOGY** Use geometry software to construct a line ℓ, a point P not on ℓ, and a line n through P parallel to ℓ. Construct a point Q on ℓ and construct $\overleftrightarrow{PQ}$. Choose a pair of alternate interior angles and construct their angle bisectors. Are the bisectors parallel? Make a conjecture. Write a plan for a proof of your conjecture.

Test Preparation

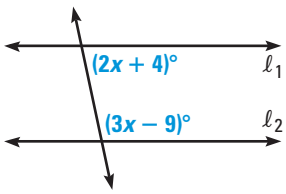
37. MULTIPLE CHOICE What is the converse of the following statement?

If $\angle 1 \cong \angle 2$, then $n \parallel m$.

Ⓐ $\angle 1 \cong \angle 2$ if and only if $n \parallel m$. Ⓑ If $\angle 2 \cong \angle 1$, then $m \parallel n$.

Ⓒ $\angle 1 \cong \angle 2$ if $n \parallel m$. Ⓓ $\angle 1 \cong \angle 2$ only if $n \parallel m$.

38. MULTIPLE CHOICE What value of x would make lines ℓ_1 and ℓ_2 parallel?



Ⓐ 13 Ⓑ 35 Ⓒ 37

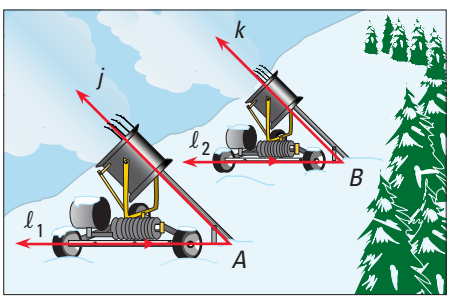
Ⓓ 78 Ⓔ 102

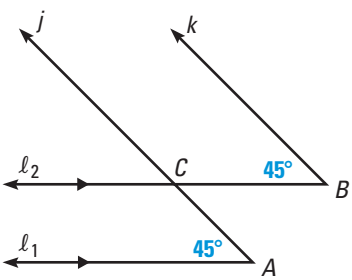
★ Challenge

39. SNOW MAKING To shoot the snow as far as possible, each snowmaker below is set at a 45° angle. The axles of the snowmakers are all parallel. It is possible to prove that the barrels of the snowmakers are also parallel, but the proof is difficult in 3 dimensions. To simplify the problem, think of the illustration as a flat image on a piece of paper. The axles and barrels are represented in the diagram on the right. Lines j and ℓ_2 intersect at C.

GIVEN ▶ $\ell_1 \parallel \ell_2$, $m\angle A = m\angle B = 45°$

PROVE ▶ $j \parallel k$





EXTRA CHALLENGE
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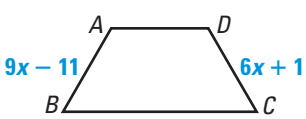
MIXED REVIEW

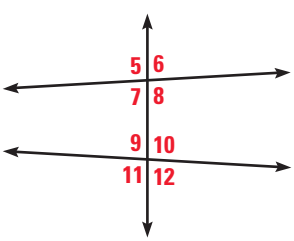
FINDING THE MIDPOINT Use a ruler to draw a line segment with the given length. Then use a compass and straightedge to construct the midpoint of the line segment. (Review 1.5 for 3.5)

40. 3 inches **41.** 8 centimeters **42.** 5 centimeters **43.** 1 inch

44. CONGRUENT SEGMENTS Find the value of x if $\overline{AB} \cong \overline{AD}$ and $\overline{CD} \cong \overline{AD}$. Explain your steps. (Review 2.5)



IDENTIFYING ANGLES Use the diagram to complete the statement. (Review 3.1)



45. $\angle 12$ and ___?___ are alternate exterior angles.

46. $\angle 10$ and ___?___ are corresponding angles.

47. $\angle 10$ and ___?___ are alternate interior angles.

48. $\angle 9$ and ___?___ are consecutive interior angles.

3.5 Using Properties of Parallel Lines

What you should learn

GOAL 1 Use properties of parallel lines in **real-life** situations, such as building a CD rack in **Example 3**.

GOAL 2 Construct parallel lines using straightedge and compass.

Why you should learn it

▼ To understand how light bends when it passes through glass or water, as in **Ex. 42**.

REAL LIFE

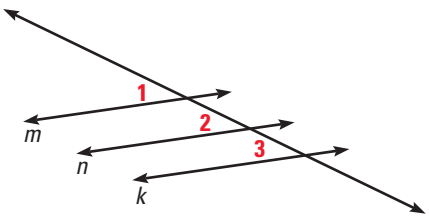
GOAL 1 USING PARALLEL LINES IN REAL LIFE

When a team of rowers competes, each rower keeps his or her oars parallel to the adjacent rower's oars. If any two *adjacent* oars on the same side of the boat are parallel, does this imply that *any two* oars on that side are parallel? This question is examined below.

Example 1 justifies Theorem 3.11, and you will prove Theorem 3.12 in Exercise 38.

EXAMPLE 1 *Proving Two Lines are Parallel*

Lines m, n, and k represent three of the oars above. $m \parallel n$ and $n \parallel k$. Prove that $m \parallel k$.



SOLUTION

GIVEN ▶ $m \parallel n$, $n \parallel k$

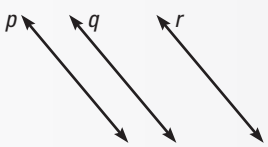
PROVE ▶ $m \parallel k$

Statements	Reasons
1. $m \parallel n$	**1.** Given
2. $\angle 1 \cong \angle 2$	**2.** Corresponding Angles Postulate
3. $n \parallel k$	**3.** Given
4. $\angle 2 \cong \angle 3$	**4.** Corresponding Angles Postulate
5. $\angle 1 \cong \angle 3$	**5.** Transitive Property of Congruence
6. $m \parallel k$	**6.** Corresponding Angles Converse

THEOREMS ABOUT PARALLEL AND PERPENDICULAR LINES

THEOREM 3.11

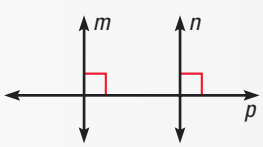
If two lines are parallel to the same line, then they are parallel to each other.



If $p \parallel q$ and $q \parallel r$, then $p \parallel r$.

THEOREM 3.12

In a plane, if two lines are perpendicular to the same line, then they are parallel to each other.

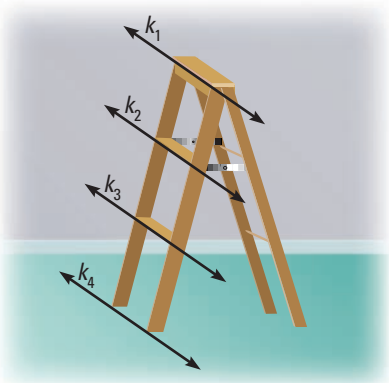


If $m \perp p$ and $n \perp p$, then $m \parallel n$.

Logical Reasoning

EXAMPLE 2 *Explaining Why Steps are Parallel*

In the diagram at the right, each step is parallel to the step immediately below it and the bottom step is parallel to the floor. Explain why the top step is parallel to the floor.



SOLUTION

You are given that $k_1 \parallel k_2$ and $k_2 \parallel k_3$.
By transitivity of parallel lines, $k_1 \parallel k_3$.
Since $k_1 \parallel k_3$ and $k_3 \parallel k_4$, it follows that $k_1 \parallel k_4$.
So, the top step is parallel to the floor.

EXAMPLE 3 *Building a CD Rack*

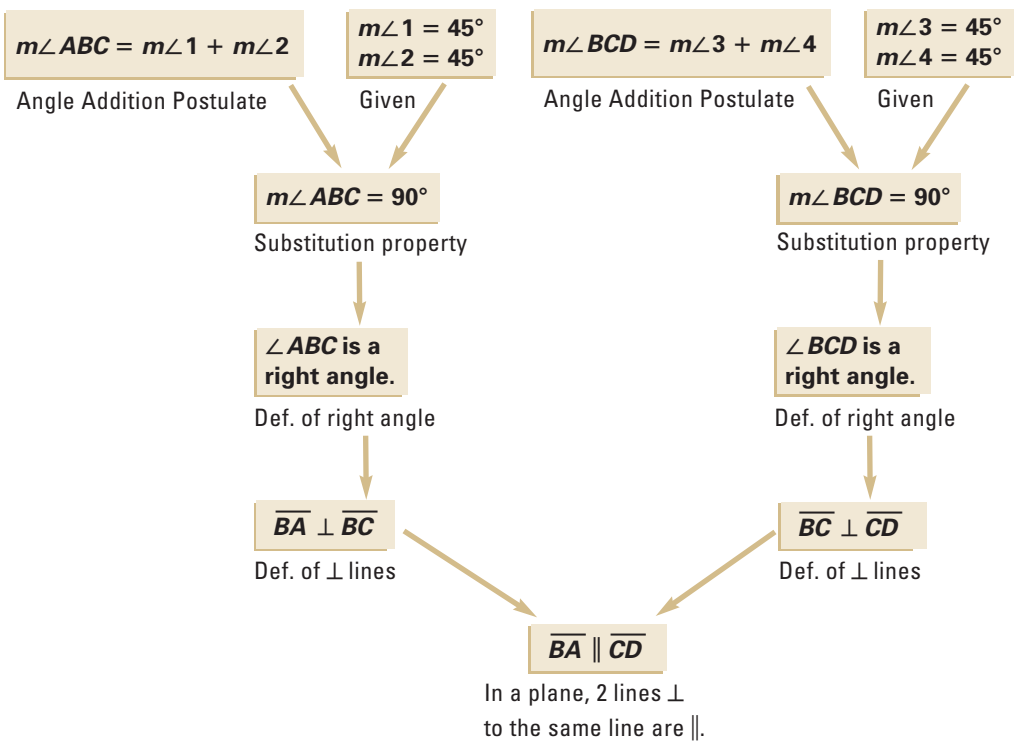
You are building a CD rack. You cut the sides, bottom, and top so that each corner is composed of two 45° angles. Prove that the top and bottom front edges of the CD rack are parallel.

Proof

SOLUTION

GIVEN ▶ $m\angle 1 = 45°$, $m\angle 2 = 45°$
$m\angle 3 = 45°$, $m\angle 4 = 45°$

PROVE ▶ $\overline{BA} \parallel \overline{CD}$



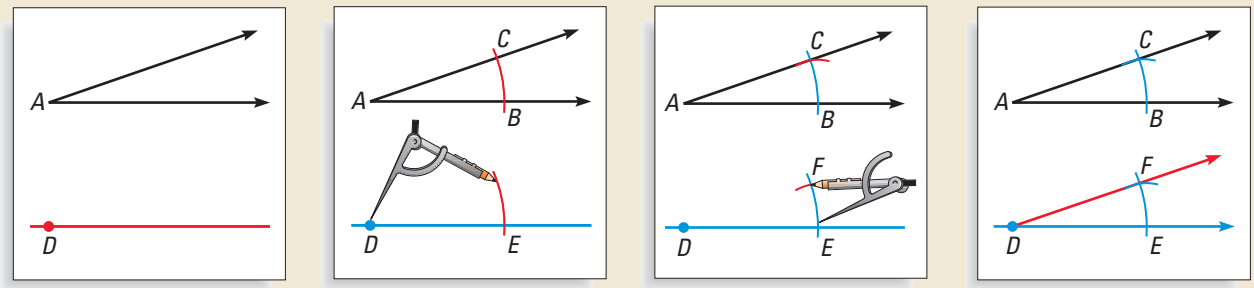
GOAL 2 CONSTRUCTING PARALLEL LINES

To construct parallel lines, you first need to know how to copy an angle.

ACTIVITY Construction

Copying an Angle

Use these steps to construct an angle that is congruent to a given $\angle A$.



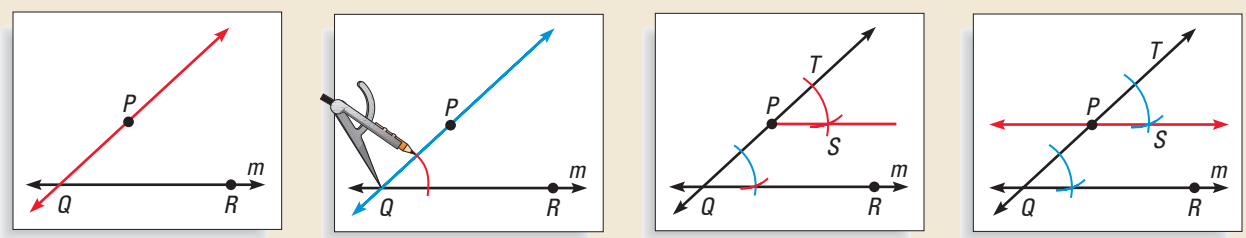
1. Draw a line. Label a point on the line D.
2. Draw an arc with center A. Label B and C. With the same radius, draw an arc with center D. Label E.
3. Draw an arc with radius BC and center E. Label the intersection F.
4. Draw $\overrightarrow{DF}$. $\angle EDF \cong \angle BAC$.

In Chapter 4, you will learn why the *Copying an Angle* construction works. You can use the *Copying an Angle* construction to construct two congruent corresponding angles. If you do, the sides of the angles will be parallel.

ACTIVITY Construction

Parallel Lines

Use these steps to construct a line that passes through a given point P and is parallel to a given line m.



1. Draw points Q and R on m. Draw $\overleftrightarrow{PQ}$.
2. Draw an arc with the compass point at Q so that it crosses $\overrightarrow{QP}$ and $\overrightarrow{QR}$.
3. Copy $\angle PQR$ on $\overleftrightarrow{QP}$ as shown. Be sure the two angles are corresponding. Label the new angle $\angle TPS$ as shown.
4. Draw $\overleftrightarrow{PS}$. Because $\angle TPS$ and $\angle PQR$ are congruent corresponding angles, $\overleftrightarrow{PS} \parallel \overleftrightarrow{QR}$.

GUIDED PRACTICE

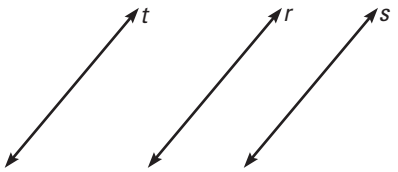
Concept Check ✓

Skill Check ✓

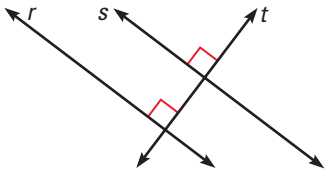
1. Name two ways, from this lesson, to prove that two lines are parallel.
if they are ∥ to the same line, if they are ⊥ to the same line

State the theorem that you can use to prove that *r* is parallel to *s*.

2. GIVEN ▶ $r \parallel t$, $t \parallel s$

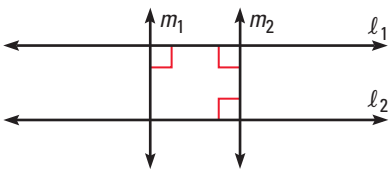


3. GIVEN ▶ $r \perp t$, $t \perp s$

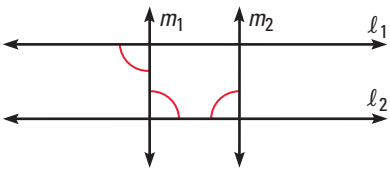


Determine which lines, if any, must be parallel. Explain your reasoning.

4.



5.



6. Draw any angle $\angle A$. Then construct $\angle B$ congruent to $\angle A$.

7. Given a line ℓ and a point P not on ℓ, describe how to construct a line through P parallel to ℓ.

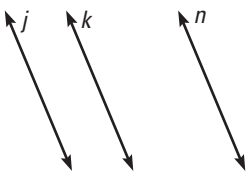
PRACTICE AND APPLICATIONS

STUDENT HELP

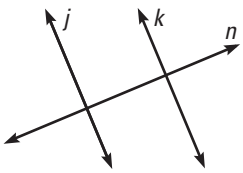
Extra Practice to help you master skills is on p. 808.

LOGICAL REASONING **State the postulate or theorem that allows you to conclude that *j* ∥ *k*.**

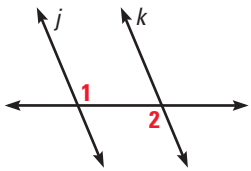
8. GIVEN ▶ $j \parallel n$, $k \parallel n$



9. GIVEN ▶ $j \perp n$, $k \perp n$

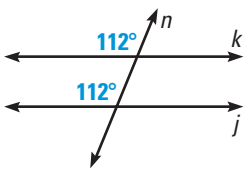


10. GIVEN ▶ $\angle 1 \cong \angle 2$

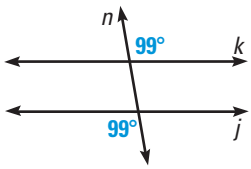


SHOWING LINES ARE PARALLEL **Explain how you would show that *k* ∥ *j*. State any theorems or postulates that you would use.**

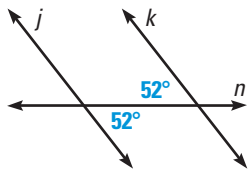
11.



12.



13.



14. *Writing* Make a list of all the ways you know to prove that two lines are parallel.

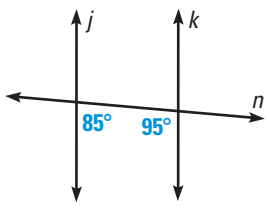
STUDENT HELP

HOMEWORK HELP

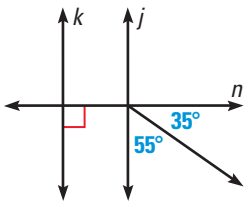
Example 1: Exs. 8–24
Example 2: Exs. 8–24
Example 3: Exs. 8–24

SHOWING LINES ARE PARALLEL **Explain how you would show that $k \parallel j$.**

15.



16.



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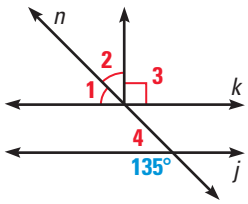
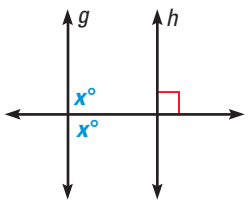


USING ALGEBRA **Explain how you would show that $g \parallel h$.**

18.



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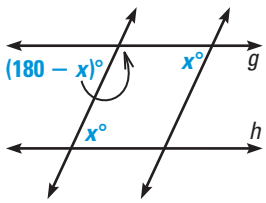

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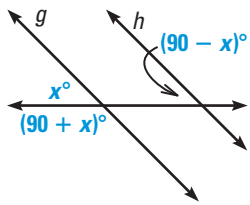
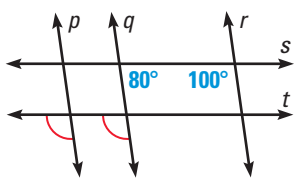
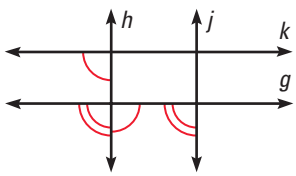


NAMING PARALLEL LINES **Determine which lines, if any, must be parallel. Explain your reasoning.**

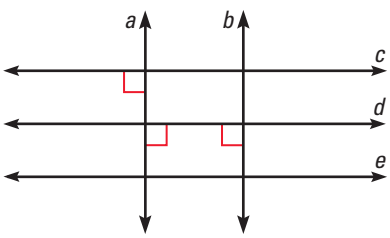
21.



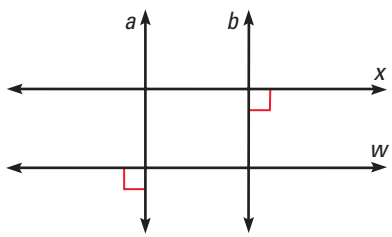
22.



23.



24.



CONSTRUCTIONS **Use a straightedge to draw an angle that fits the description. Then use the *Copying an Angle* construction on page 159 to copy the angle.**

25. An acute angle

26. An obtuse angle

27. CONSTRUCTING PARALLEL LINES Draw a horizontal line and construct a line parallel to it through a point above the line.

28. CONSTRUCTING PARALLEL LINES Draw a diagonal line and construct a line parallel to it through a point to the right of the line.

29. JUSTIFYING A CONSTRUCTION Explain why the lines in Exercise 28 are parallel. Use a postulate or theorem from Lesson 3.4 to support your answer.

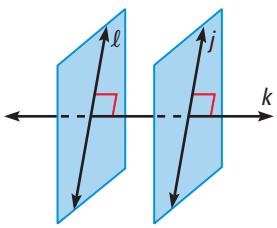
STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with constructions in Exs. 25–29.

30. **FOOTBALL FIELD** The white lines along the long edges of a football field are called *sidelines*. *Yard lines* are perpendicular to the sidelines and cross the field every five yards. Explain why you can conclude that the yard lines are parallel.

31. **HANGING WALLPAPER** When you hang wallpaper, you use a tool called a *plumb line* to make sure one edge of the first strip of wallpaper is vertical. If the edges of each strip of wallpaper are parallel and there are no gaps between the strips, how do you know that the rest of the strips of wallpaper will be parallel to the first?

32. **ERROR ANALYSIS** It is given that $j \perp k$ and $k \perp \ell$. A student reasons that lines j and ℓ must be parallel. What is wrong with this reasoning? Sketch a counterexample to support your answer.



CATEGORIZING **Tell whether the statement is *sometimes, always,* or *never* true.**

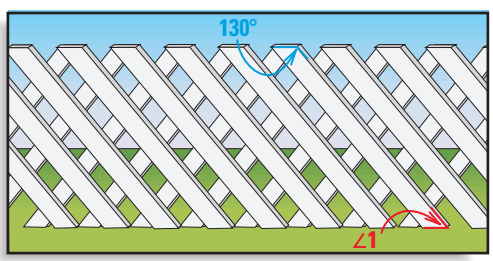
33. Two lines that are parallel to the same line are parallel to each other.

34. *In a plane*, two lines that are perpendicular to the same line are parallel to each other.

35. Two *noncoplanar* lines that are perpendicular to the same line are parallel to each other.

36. Through a point not on a line you can construct a parallel line.

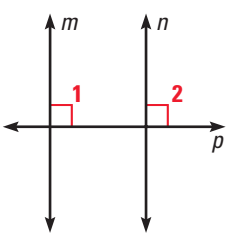
37. **LATTICEWORK** You are making a lattice fence out of pieces of wood called slats. You want the top of each slat to be parallel to the bottom. At what angle should you cut $\angle 1$?



38. **PROVING THEOREM 3.12** Rearrange the statements to write a flow proof of Theorem 3.12. Remember to include a reason for each statement.

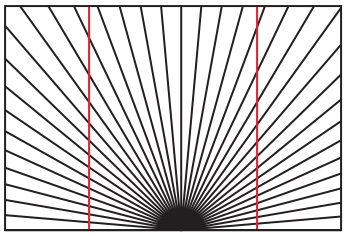
GIVEN ▶ $m \perp p$, $n \perp p$

PROVE ▶ $m \parallel n$



$\angle 1 \cong \angle 2$	$n \perp p$	$\angle 1$ is a right $\angle$.
$m \parallel n$	$m \perp p$	$\angle 2$ is a right $\angle$.

39. OPTICAL ILLUSION The radiating lines make it hard to tell if the red lines are straight. Explain how you can answer the question using only a straightedge and a protractor.



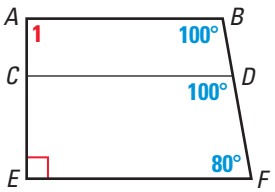
a. Are the red lines straight?

b. Are the red lines parallel?

40. CONSTRUCTING WITH PERPENDICULARS Draw a horizontal line ℓ and a point P not on ℓ. Construct a line m through P perpendicular to ℓ. Draw a point Q not on m or ℓ. Construct a line n through Q perpendicular to m. What postulate or theorem guarantees that the lines ℓ and n are parallel?

Test Preparation

41. MULTI-STEP PROBLEM Use the information given in the diagram at the right.



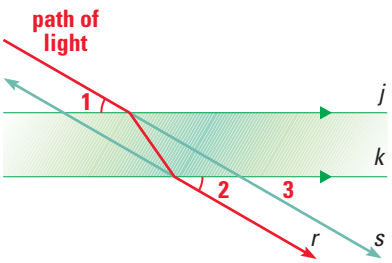
a. Explain why $\overline{AB} \parallel \overline{CD}$.

b. Explain why $\overline{CD} \parallel \overline{EF}$.

c. *Writing* What is $m\angle 1$? How do you know?

★ Challenge

42. SCIENCE CONNECTION When light enters glass, the light bends. When it leaves glass, it bends again. If both sides of a pane of glass are parallel, light leaves the pane at the same angle at which it entered. Prove that the path of the exiting light is parallel to the path of the entering light.



GIVEN ▶ $\angle 1 \cong \angle 2$, $j \parallel k$

PROVE ▶ $r \parallel s$

INTERNET APPLICATION LINK
www.mcdougallittell.com

MIXED REVIEW

USING THE DISTANCE FORMULA **Find the distance between the two points.** (Review 1.3 for 3.6)

43. $A(0, -6)$, $B(14, 0)$ **44.** $A(-3, -8)$, $B(2, -1)$ **45.** $A(0, -7)$, $B(6, 3)$

46. $A(-9, -5)$, $B(-1, 11)$ **47.** $A(5, -7)$, $B(-11, 6)$ **48.** $A(4, 4)$, $B(-3, -3)$

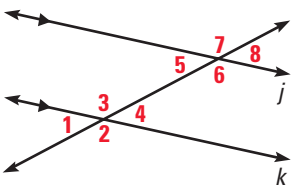
FINDING COUNTEREXAMPLES **Give a counterexample that demonstrates that the converse of the statement is false.** (Review 2.2)

49. If an angle measures 42°, then it is acute.

50. If two angles measure 150° and 30°, then they are supplementary.

51. If a polygon is a rectangle, then it contains four right angles.

52. USING PROPERTIES OF PARALLEL LINES Use the given information to find the measures of the other seven angles in the figure shown at the right. (Review 3.3)

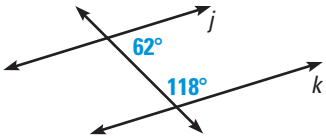


GIVEN ▶ $j \parallel k$, $m\angle 1 = 33°$

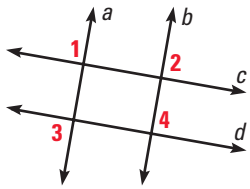
QUIZ 2

Self-Test for Lessons 3.4 and 3.5

1. In the diagram shown at the right, determine whether you can prove that lines j and k are parallel. If you can, state the postulate or theorem that you would use. **(Lesson 3.4)**



Use the given information and the diagram to determine which lines must be parallel. **(Lesson 3.5)**

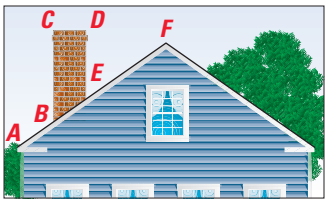


2. $\angle 1$ and $\angle 2$ are right angles.

3. $\angle 4 \cong \angle 3$

4. $\angle 2 \cong \angle 3$, $\angle 3 \cong \angle 4$.

5. **FIREPLACE CHIMNEY** In the illustration at the right, $\angle ABC$ and $\angle DEF$ are supplementary. Explain how you know that the left and right edges of the chimney are parallel. **(Lesson 3.4)**



MATH & History — Measuring Earth's Circumference

APPLICATION LINK
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THEN

AROUND 230 B.C., the Greek scholar Eratosthenes estimated Earth's circumference. In the late 15th century, Christopher Columbus used a smaller estimate to convince the king and queen of Spain that his proposed voyage to India would take only 30 days.

NOW

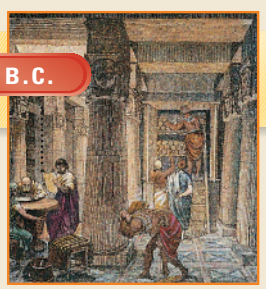
TODAY, satellites and other tools are used to determine Earth's circumference with great accuracy.

1. The actual distance from Syene to Alexandria is about 500 miles. Use this value and the information on page 145 to estimate Earth's circumference. How close is your value to the modern day measurement in the table at the right?

Measuring Earth's Circumference	
Circumference estimated by Eratosthenes (230 B.C.)	About 29,000 mi
Circumference assumed by Columbus (about 1492)	About 17,600 mi
Modern day measurement	24,902 mi

Eratosthenes becomes the head of the library in Alexandria.

235 B.C.



1492

A replica of one of the ships used by Christopher Columbus.

1999

Photograph of Earth from space.

3.6 Parallel Lines in the Coordinate Plane

What you should learn

GOAL 1 Find slopes of lines and use slope to identify parallel lines in a coordinate plane.

GOAL 2 Write equations of parallel lines in a coordinate plane.

Why you should learn it

▼ To describe steepness in **real-life**, such as the cog railway in **Example 1** and the zip line in **Ex. 46**.

REAL LIFE

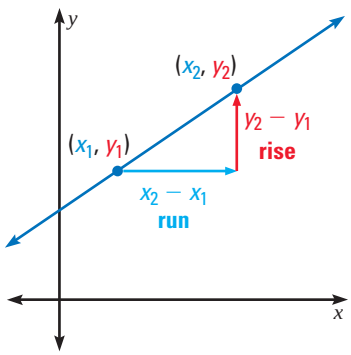
GOAL 1 SLOPE OF PARALLEL LINES

In algebra, you learned that the slope of a nonvertical line is the ratio of the vertical change (the rise) to the horizontal change (the run). If the line passes through the points (x_1, y_1) and (x_2, y_2), then the slope is given by

$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

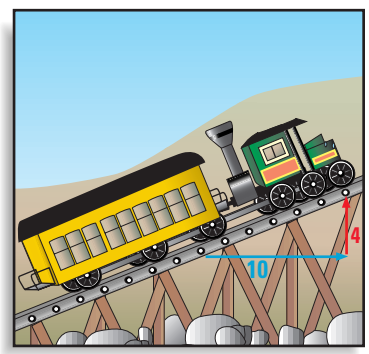
Slope is usually represented by the variable m.



EXAMPLE 1 *Finding the Slope of Train Tracks*

REAL LIFE

COG RAILWAY A cog railway goes up the side of Mount Washington, the tallest mountain in New England. At the steepest section, the train goes up about 4 feet for each 10 feet it goes forward. What is the slope of this section?

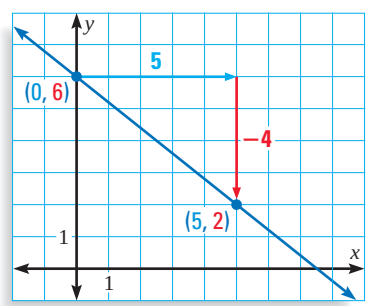


SOLUTION

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{4 \text{ feet}}{10 \text{ feet}} = 0.4$$

EXAMPLE 2 *Finding the Slope of a Line*

Find the slope of the line that passes through the points (0, 6) and (5, 2).



SOLUTION

Let $(x_1, y_1) = (0, 6)$ and $(x_2, y_2) = (5, 2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 6}{5 - 0} = -\frac{4}{5}$$

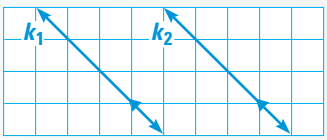
▶ The slope of the line is $-\frac{4}{5}$.

You can use the slopes of two lines to tell whether the lines are parallel.

POSTULATE

POSTULATE 17 ***Slopes of Parallel Lines***

In a coordinate plane, two nonvertical lines are parallel if and only if they have the same slope. Any two vertical lines are parallel.



Lines k_1 and k_2 have the same slope.

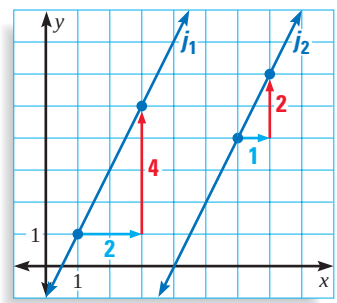
STUDENT HELP

Study Tip

To find the slope in Example 3, you can either use the slope formula or you can count units on the graph.

EXAMPLE 3 *Deciding Whether Lines are Parallel*

Find the slope of each line. Is $j_1 \parallel j_2$?



SOLUTION

Line j_1 has a slope of

$$m_1 = \frac{4}{2} = 2$$

Line j_2 has a slope of

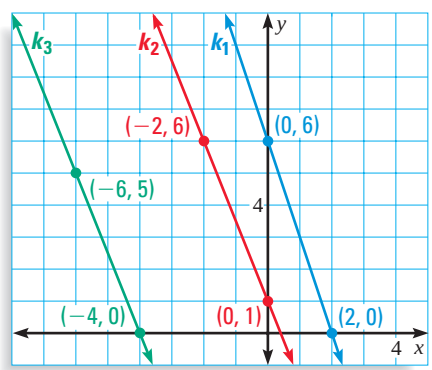
$$m_2 = \frac{2}{1} = 2$$

▶ Because the lines have the same slope, $j_1 \parallel j_2$.

Using Algebra

EXAMPLE 4 *Identifying Parallel Lines*

Find the slope of each line. Which lines are parallel?



SOLUTION

Find the slope of k_1. Line k_1 passes through (0, 6) and (2, 0).

$$m_1 = \frac{0-6}{2-0} = \frac{-6}{2} = -3$$

Find the slope of k_2. Line k_2 passes through (−2, 6) and (0, 1).

$$m_2 = \frac{1-6}{0-(-2)} = \frac{-5}{0+2} = -\frac{5}{2}$$

Find the slope of k_3. Line k_3 passes through (−6, 5) and (−4, 0).

$$m_3 = \frac{0-5}{-4-(-6)} = \frac{-5}{-4+6} = -\frac{5}{2}$$

▶ Compare the slopes. Because k_2 and k_3 have the same slope, they are parallel. Line k_1 has a different slope, so it is not parallel to either of the other lines.

GOAL 2 WRITING EQUATIONS OF PARALLEL LINES

In algebra, you learned that you can use the slope m of a nonvertical line to write an equation of the line in *slope-intercept form*.

$$y = \underset{\text{slope}}{m}x + \underset{y\text{-intercept}}{b}$$

The y-intercept is the y-coordinate of the point where the line crosses the y-axis.

EXAMPLE 5 *Writing an Equation of a Line*

Using Algebra

Write an equation of the line through the point (2, 3) that has a slope of 5.

SOLUTION

Solve for b. Use $(x, y) = (2, 3)$ and $m = 5$.

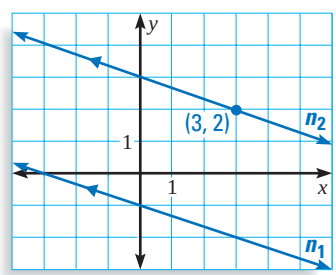
$y = mx + b$	**Slope-intercept form**
$3 = 5(2) + b$	**Substitute 2 for x, 3 for y, and 5 for m.**
$3 = 10 + b$	**Simplify.**
$-7 = b$	**Subtract.**

▶ Write an equation. Since $m = 5$ and $b = -7$, an equation of the line is $y = 5x - 7$.

EXAMPLE 6 *Writing an Equation of a Parallel Line*

Line n_1 has the equation $y = -\frac{1}{3}x - 1$.

Line n_2 is parallel to n_1 and passes through the point (3, 2). Write an equation of n_2.



SOLUTION

Find the slope.

The slope of n_1 is $-\frac{1}{3}$. Because parallel lines have the same slope, the slope of n_2 is also $-\frac{1}{3}$.

Solve for b. Use $(x, y) = (3, 2)$ and $m = -\frac{1}{3}$.

$$y = mx + b$$
$$2 = -\frac{1}{3}(3) + b$$
$$2 = -1 + b$$
$$3 = b$$

Write an equation.

▶ Because $m = -\frac{1}{3}$ and $b = 3$, an equation of n_2 is $y = -\frac{1}{3}x + 3$.

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

GUIDED PRACTICE

Vocabulary Check ✓

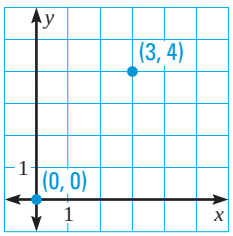
1. What does *intercept* mean in the expression *slope-intercept form*?

Concept Check ✓

2. The slope of line j is 2 and $j \parallel k$. What is the slope of line k?

3. What is the slope of a horizontal line? What is the slope of a vertical line?

Skill Check ✓ **Find the slope of the line that passes through the labeled points.**

4.


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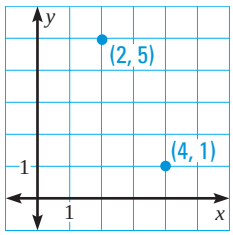
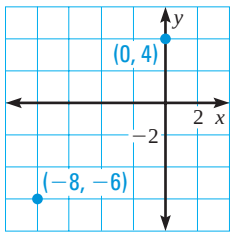
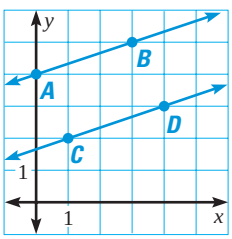


6.


Determine whether the two lines shown in the graph are parallel. If they are parallel, explain how you know.

7.


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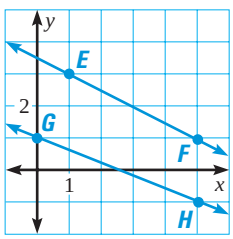

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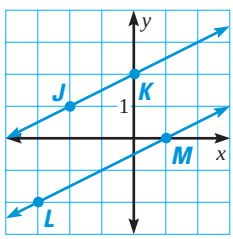


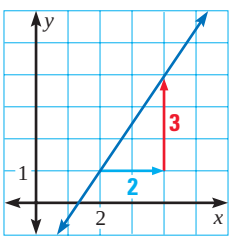
10. Write an equation of the line that passes through the point $(2, -3)$ and has a slope of -1.

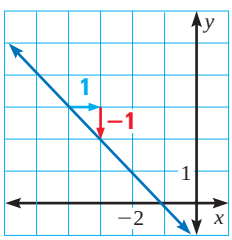
PRACTICE AND APPLICATIONS

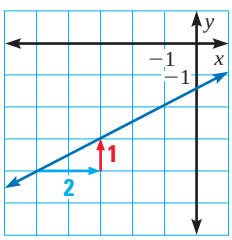
STUDENT HELP

Extra Practice to help you master skills is on p. 808.

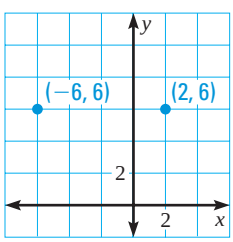
CALCULATING SLOPE **What is the slope of the line?**

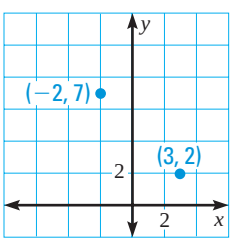
11.


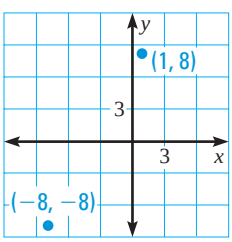
12.


13.


CALCULATING SLOPE **Find the slope of the line that passes through the labeled points on the graph.**

14.


15.


16.


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 11–16, 23, 46, 49–52

Example 2: Exs. 11–16

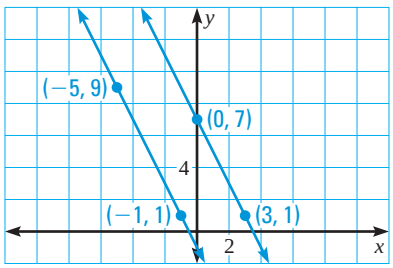
STUDENT HELP

HOMEWORK HELP

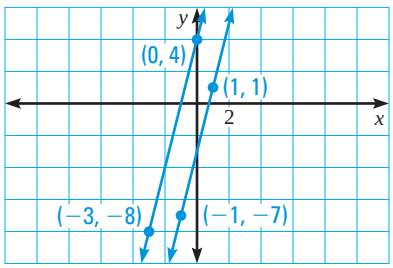
Example 3: Exs. 17–22
Example 4: Exs. 24–26, 47, 48
Example 5: Exs. 27–41
Example 6: Exs. 42–45

IDENTIFYING PARALLELS **Find the slope of each line. Are the lines parallel?**

17.



18.



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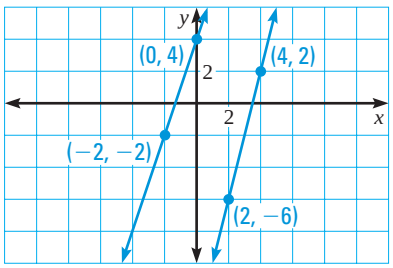
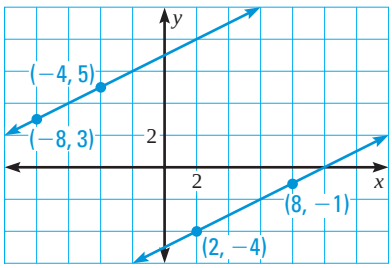
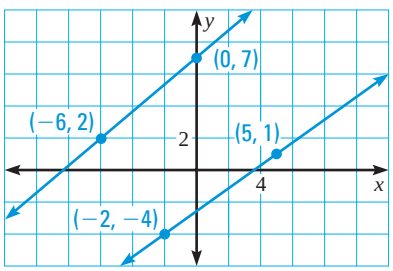


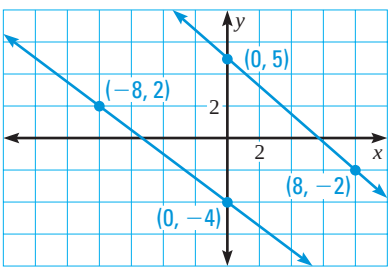
20.



21.



22.



23. **UNDERGROUND RAILROAD** The photo at the right shows a monument in Oberlin, Ohio, that is dedicated to the Underground Railroad. The slope of each of the rails is about $-\frac{3}{5}$ and the sculpture is about 12 feet long. What is the height of the ends of the rails? Explain how you found your answer.

FOCUS ON PEOPLE

REAL LIFE **UNDERGROUND RAILROAD** is the name given to the network of people who helped some slaves to freedom. Harriet Tubman, a former slave, helped about 300 escape.

IDENTIFYING PARALLELS **Find the slopes of $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{EF}$. Which lines are parallel, if any?**

24. $A(0, -6)$, $B(4, -4)$
$C(0, 2)$, $D(2, 3)$
$E(0, -4)$, $F(1, -7)$

25. $A(2, 6)$, $B(4, 7)$
$C(0, -1)$, $D(6, 2)$
$E(4, -5)$, $F(8, -2)$

26. $A(-4, 10)$, $B(-8, 7)$
$C(-5, 7)$, $D(-2, 4)$
$E(2, -3)$, $F(6, -7)$

WRITING EQUATIONS **Write an equation of the line.**

27. slope $= 3$
y-intercept $= 2$

28. slope $= \frac{1}{3}$
y-intercept $= -4$

29. slope $= -\frac{2}{9}$
y-intercept $= 0$

30. slope $= \frac{1}{2}$
y-intercept $= 6$

31. slope $= 0$
y-intercept $= -3$

32. slope $= -\frac{2}{9}$
y-intercept $= -\frac{3}{5}$

WRITING EQUATIONS **Write an equation of the line that has a *y*-intercept of 3 and is parallel to the line whose equation is given.**

33. $y = -6x + 2$ **34.** $y = x - 8$ **35.** $y = -\frac{4}{3}x$

WRITING EQUATIONS **Write an equation of the line that passes through the given point *P* and has the given slope.**

36. $P(0, -6)$, $m = -2$ **37.** $P(-3, 9)$, $m = -1$ **38.** $P\left(\frac{3}{2}, 4\right)$, $m = \frac{1}{2}$

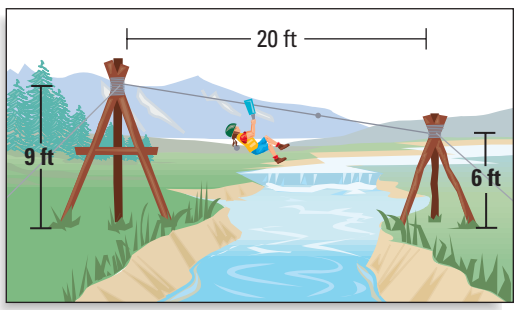
39. $P(2, -4)$, $m = 0$ **40.** $P(-7, -5)$, $m = \frac{3}{4}$ **41.** $P(6, 1)$, undefined slope

USING ALGEBRA **Write an equation of the line that passes through point *P* and is parallel to the line with the given equation.**

42. $P(-3, 6)$, $y = -x - 5$ **43.** $P(1, -2)$, $y = \frac{5}{4}x - 8$ **44.** $P(8, 7)$, $y = 3$

45. **USING ALGEBRA** Write an equation of a line parallel to $y = \frac{1}{3}x - 16$.

46. **ZIP LINE** A zip line is a taut rope or cable that you can ride down on a pulley. The zip line at the right goes from a 9 foot tall tower to a 6 foot tall tower. The towers are 20 feet apart. What is the slope of the zip line?



COORDINATE GEOMETRY **In Exercises 47 and 48, use the five points: *P*(0, 0), *Q*(1, 3), *R*(4, 0), *S*(8, 2), and *T*(9, 5).**

47. Plot and label the points. Connect every pair of points with a segment.

48. Which segments are parallel? How can you verify this?

CIVIL ENGINEERING **In Exercises 49–52, use the following information.**
The slope of a road is called the road's *grade*. Grades are measured in percents. For example, if the slope of a road is $\frac{1}{20}$, the grade is 5%. A warning sign is needed before any hill that fits one of the following descriptions.

5% grade and more than 3000 feet long
6% grade and more than 2000 feet long
7% grade and more than 1000 feet long
8% grade and more than 750 feet long
9% grade and more than 500 feet long

▶ Source: U.S. Department of Transportation

What is the grade of the hill to the nearest percent? Is a sign needed?

49. The hill is 1400 feet long and drops 70 feet.

50. The hill is 2200 feet long and drops 140 feet.

51. The hill is 600 feet long and drops 55 feet.

52. The hill is 450 feet long and drops 40 feet.

FOCUS ON CAREERS

CIVIL ENGINEERING
Civil engineers design and supervise the construction of roads, buildings, tunnels, bridges, and water supply systems.

CAREER LINK
www.mcdougallittell.com

TECHNOLOGY **Using a square viewing screen on a graphing calculator, graph a line that passes through the origin and has a slope of 1.**

53. Write an equation of the line you graphed. Approximately what angle does the line form with the x-axis?

54. Graph a line that passes through the origin and has a slope of 2. Write an equation of the line. When you doubled the slope, did the measure of the angle formed with the x-axis double?

Test Preparation

55. **MULTIPLE CHOICE** If two different lines with equations $y = m_1x + b_1$ and $y = m_2x + b_2$ are parallel, which of the following must be true?

Ⓐ $b_1 = b_2$ and $m_1 \neq m_2$ Ⓑ $b_1 \neq b_2$ and $m_1 \neq m_2$

Ⓒ $b_1 \neq b_2$ and $m_1 = m_2$ Ⓓ $b_1 = b_2$ and $m_1 = m_2$

Ⓔ None of these

56. **MULTIPLE CHOICE** Which of the following is an equation of a line parallel to $y - 4 = -\frac{1}{2}x$?

Ⓐ $y = \frac{1}{2}x - 6$ Ⓑ $y = 2x + 1$ Ⓒ $y = -2x + 3$

Ⓓ $y = \frac{7}{2}x - 1$ Ⓔ $y = -\frac{1}{2}x - 8$

★ Challenge

57. **USING ALGEBRA** Find a value for k so that the line through $(4, k)$ and $(-2, -1)$ is parallel to $y = -2x + \frac{3}{2}$.

58. **USING ALGEBRA** Find a value for k so that the line through $(k, -10)$ and $(5, -6)$ is parallel to $y = -\frac{1}{4}x + 3$.

MIXED REVIEW

RECIPROCALS **Find the reciprocal of the number.** (Skills Review, p. 788)

59. 20 **60.** -3 **61.** -11 **62.** 340

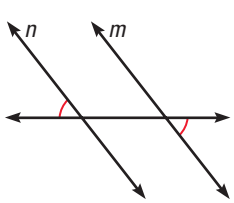
63. $\frac{3}{7}$ **64.** $-\frac{13}{3}$ **65.** $-\frac{1}{2}$ **66.** 0.25

MULTIPLYING NUMBERS **Evaluate the expression.** (Skills Review, p. 785)

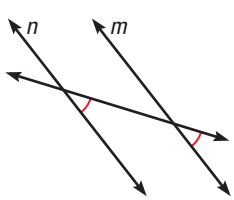
67. $\frac{3}{4} \cdot (-12)$ **68.** $-\frac{3}{2} \cdot \left(-\frac{8}{3}\right)$ **69.** $-10 \cdot \frac{7}{6}$ **70.** $-\frac{2}{9} \cdot (-33)$

PROVING LINES PARALLEL **Can you prove that lines *m* and *n* are parallel? If so, state the postulate or theorem you would use.** (Review 3.4)

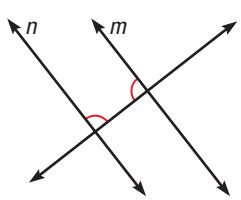
71.



72.



73.



3.7 Perpendicular Lines in the Coordinate Plane

What you should learn

GOAL 1 Use slope to identify perpendicular lines in a coordinate plane.

GOAL 2 Write equations of perpendicular lines, as applied in **Ex. 46**.

Why you should learn it

▼ Equations of perpendicular lines are used by ray tracing software to create realistic reflections, as in the illustration below and in **Example 6**.

REAL LIFE

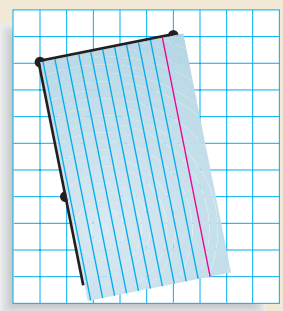
GOAL 1 SLOPE OF PERPENDICULAR LINES

In the activity below, you will trace a piece of paper to draw perpendicular lines on a coordinate grid. Points where grid lines cross are called *lattice points*.

ACTIVITY Developing Concepts

Investigating Slopes of Perpendicular Lines

1. Put the corner of a piece of paper on a lattice point. Rotate the corner so each edge passes through another lattice point but neither edge is vertical. Trace the edges.
2. Find the slope of each line.
3. Multiply the slopes.
4. Repeat Steps 1–3 with the paper at a different angle.



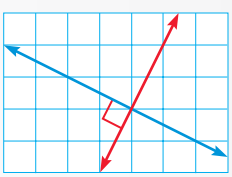
In the activity, you may have discovered the following.

POSTULATE

POSTULATE 18 ***Slopes of Perpendicular Lines***

In a coordinate plane, two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

Vertical and horizontal lines are perpendicular.



product of slopes $= 2\left(-\frac{1}{2}\right) = -1$

EXAMPLE 1 ***Deciding Whether Lines are Perpendicular***

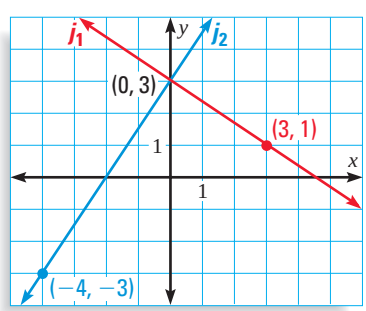
Find each slope.

$$\text{Slope of } j_1 = \frac{3-1}{0-3} = -\frac{2}{3}$$

$$\text{Slope of } j_2 = \frac{3-(-3)}{0-(-4)} = \frac{6}{4} = \frac{3}{2}$$

Multiply the slopes.

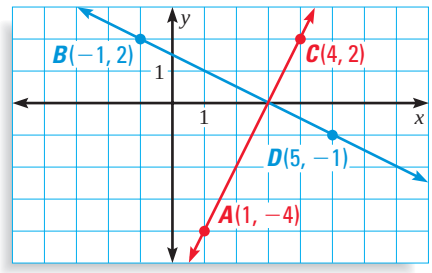
The product is $\left(-\frac{2}{3}\right)\left(\frac{3}{2}\right) = -1$, so $j_1 \perp j_2$.



EXAMPLE 2 *Deciding Whether Lines are Perpendicular*

Logical Reasoning

Decide whether $\overleftrightarrow{AC}$ and $\overleftrightarrow{DB}$ are perpendicular.



SOLUTION

$$\text{Slope of } \overleftrightarrow{AC} = \frac{2-(-4)}{4-1} = \frac{6}{3} = 2$$

$$\text{Slope of } \overleftrightarrow{DB} = \frac{2-(-1)}{-1-5} = \frac{3}{-6} = -\frac{1}{2}$$

The product is $2\left(-\frac{1}{2}\right) = -1$, so $\overleftrightarrow{AC} \perp \overleftrightarrow{DB}$.

EXAMPLE 3 *Deciding Whether Lines are Perpendicular*

Decide whether the lines are perpendicular.

line *h*: $y = \frac{3}{4}x + 2$ **line *j*:** $y = -\frac{4}{3}x - 3$

SOLUTION

The slope of line h is $\frac{3}{4}$. The slope of line j is $-\frac{4}{3}$.

The product is $\left(\frac{3}{4}\right)\left(-\frac{4}{3}\right) = -1$, so the lines are perpendicular.

EXAMPLE 4 *Deciding Whether Lines are Perpendicular*

Using Algebra

Decide whether the lines are perpendicular.

line *r*: $4x + 5y = 2$ **line *s*:** $5x + 4y = 3$

SOLUTION

Rewrite each equation in slope-intercept form to find the slope.

line *r*:

$$4x + 5y = 2$$
$$5y = -4x + 2$$
$$y = -\frac{4}{5}x + \frac{2}{5}$$

slope $= -\frac{4}{5}$

line *s*:

$$5x + 4y = 3$$
$$4y = -5x + 3$$
$$y = -\frac{5}{4}x + \frac{3}{4}$$

slope $= -\frac{5}{4}$

Multiply the slopes to see if the lines are perpendicular.

$$\left(-\frac{4}{5}\right)\left(-\frac{5}{4}\right) = 1$$

▶ The product of the slopes is *not* -1. So, r and s are *not* perpendicular.

GOAL 2 WRITING EQUATIONS OF PERPENDICULAR LINES

STUDENT HELP

Study Tip

You can check m_2 by multiplying $m_1 \cdot m_2$.

$(-2)\left(\frac{1}{2}\right) = -1$ ✓

EXAMPLE 5 *Writing the Equation of a Perpendicular Line*

Line ℓ_1 has equation $y = -2x + 1$. Find an equation of the line ℓ_2 that passes through $P(4, 0)$ and is perpendicular to ℓ_1. First you must find the slope, m_2.

$m_1 \cdot m_2 = -1$	**The product of the slopes of ⊥ lines is −1.**
$-2 \cdot m_2 = -1$	**The slope of ℓ_1 is −2.**
$m_2 = \frac{1}{2}$	**Divide both sides by −2.**

Then use $m = \frac{1}{2}$ and $(x, y) = (4, 0)$ to find b.

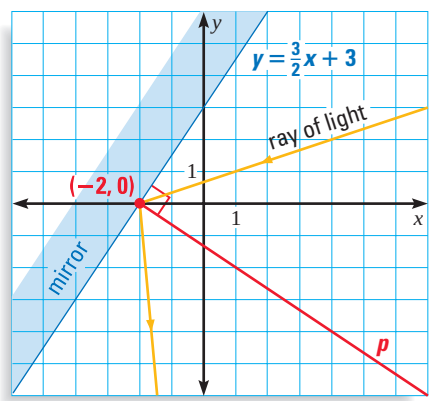
$y = mx + b$	**Slope-intercept form**
$0 = \frac{1}{2}(4) + b$	**Substitute 0 for y, $\frac{1}{2}$ for m, and 4 for x.**
$-2 = b$	**Simplify.**

▶ So, an equation of ℓ_2 is $y = \frac{1}{2}x - 2$.

RAY TRACING Computer illustrators use *ray tracing* to make accurate reflections. To figure out what to show in the mirror, the computer traces a ray of light as it reflects off the mirror. This calculation has many steps. One of the first steps is to find the equation of a line perpendicular to the mirror.

EXAMPLE 6 *Writing the Equation of a Perpendicular Line*

The equation $y = \frac{3}{2}x + 3$ represents a mirror. A ray of light hits the mirror at $(-2, 0)$. What is the equation of the line p that is perpendicular to the mirror at this point?



Top view of mirror

SOLUTION

The mirror's slope is $\frac{3}{2}$, so the slope of p is $-\frac{2}{3}$.

Use $m = -\frac{2}{3}$ and $(x, y) = (-2, 0)$ to find b.

$$0 = -\frac{2}{3}(-2) + b$$

$$-\frac{4}{3} = b$$

▶ So, an equation for p is $y = -\frac{2}{3}x - \frac{4}{3}$.

FOCUS ON CAREERS

REAL LIFE GRAPHIC ARTS Many graphic artists use computer software to design images.

INTERNET CAREER LINK
www.mcdougallittell.com

GUIDED PRACTICE

Vocabulary Check ✓

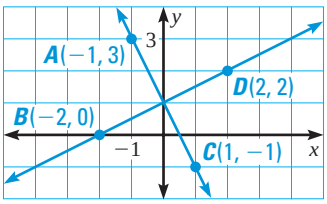
Concept Check ✓

Skill Check ✓

1. Define *slope of a line*.

2. The slope of line m is $-\frac{1}{5}$. What is the slope of a line perpendicular to m?

3. In the coordinate plane shown at the right, is $\overleftrightarrow{AC}$ perpendicular to $\overleftrightarrow{BD}$? Explain.



4. Decide whether the lines with the equations $y = 2x - 1$ and $y = -2x + 1$ are perpendicular.

5. Decide whether the lines with the equations $5y - x = 15$ and $y + 5x = 2$ are perpendicular.

6. The line ℓ_1 has the equation $y = 3x$. The line ℓ_2 is perpendicular to ℓ_1 and passes through the point $P(0, 0)$. Write an equation of ℓ_2.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 808.

SLOPES OF PERPENDICULAR LINES **The slopes of two lines are given. Are the lines perpendicular?**

7. $m_1 = 2, m_2 = -\frac{1}{2}$ **8.** $m_1 = \frac{2}{3}, m_2 = \frac{3}{2}$ **9.** $m_1 = \frac{1}{4}, m_2 = -4$

10. $m_1 = \frac{5}{7}, m_2 = -\frac{7}{5}$ **11.** $m_1 = -\frac{1}{2}, m_2 = -\frac{1}{2}$ **12.** $m_1 = -1, m_2 = 1$

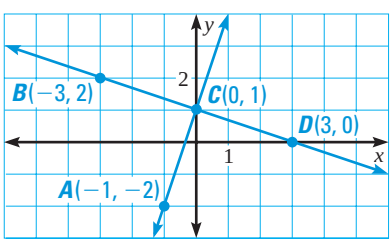
SLOPES OF PERPENDICULAR LINES **Lines *j* and *n* are perpendicular. The slope of line *j* is given. What is the slope of line *n*? Check your answer.**

13. 2 **14.** 5 **15.** -3 **16.** -7

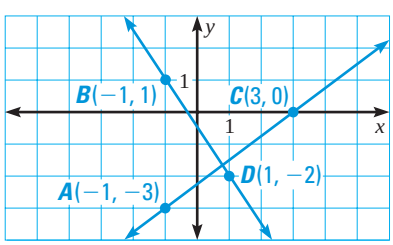
17. $\frac{2}{3}$ **18.** $\frac{1}{5}$ **19.** $-\frac{1}{3}$ **20.** $-\frac{4}{3}$

IDENTIFYING PERPENDICULAR LINES **Find the slope of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$. Decide whether $\overleftrightarrow{AC}$ is perpendicular to $\overleftrightarrow{BD}$.**

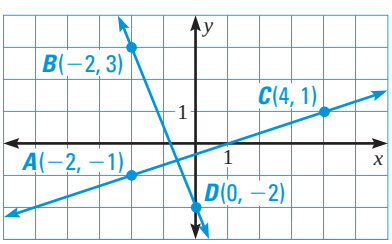
21.



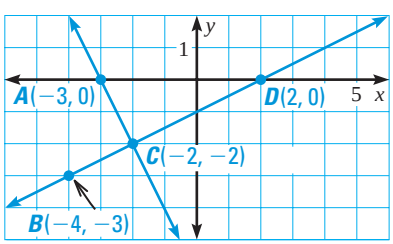
22.



23.



24.



STUDENT HELP

HOMEWORK HELP

Example 1:	Exs. 7–20
Example 2:	Exs. 21–24, 33–37
Example 3:	Exs. 25–28, 47–50
Example 4:	Exs. 29–32
Example 5:	Exs. 38–41
Example 6:	Exs. 42–46

USING ALGEBRA **Decide whether lines k_1 and k_2 are perpendicular. Then graph the lines to check your answer.**

25. line k_1: $y = 3x$
line k_2: $y = -\frac{1}{3}x - 2$

26. line k_1: $y = -\frac{4}{5}x - 2$
line k_2: $y = \frac{1}{5}x + 4$

27. line k_1: $y = -\frac{3}{4}x + 2$
line k_2: $y = \frac{4}{3}x + 5$

28. line k_1: $y = \frac{1}{3}x - 10$
line k_2: $y = 3x$

USING ALGEBRA **Decide whether lines p_1 and p_2 are perpendicular.**

29. line p_1: $3y - 4x = 3$
line p_2: $4y + 3x = -12$

30. line p_1: $y - 6x = 2$
line p_2: $6y - x = 12$

31. line p_1: $3y + 2x = -36$
line p_2: $4y - 3x = 16$

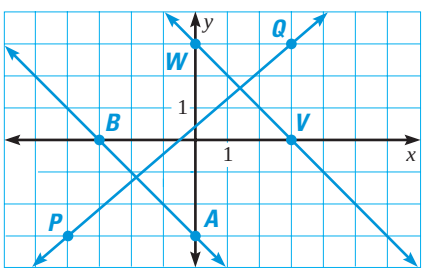
32. line p_1: $5y + 3x = -15$
line p_2: $3y - 5x = -33$

STUDENT HELP

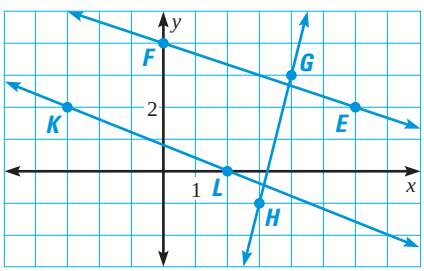
INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 33–36.

LINE RELATIONSHIPS **Find the slope of each line. Identify any parallel or perpendicular lines.**

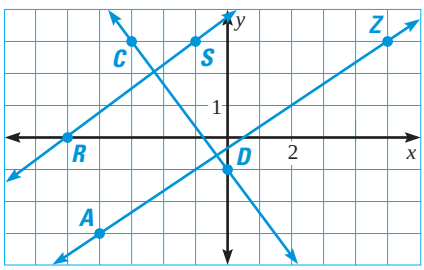
33.



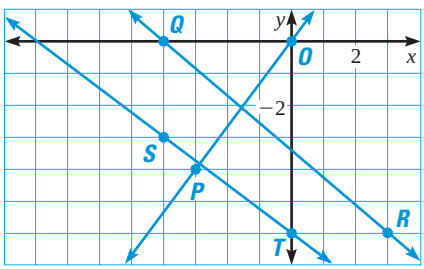
34.



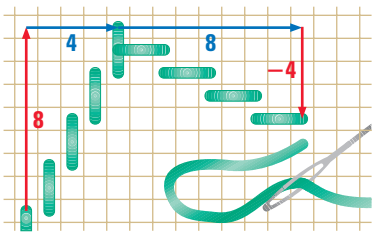
35.



36.



37. **NEEDLEPOINT** To check whether two stitched lines make a right angle, you can count the squares. For example, the lines at the right are perpendicular because one goes up 8 as it goes over 4, and the other goes over 8 as it goes down 4. Why does this mean the lines are perpendicular?



WRITING EQUATIONS **Line j is perpendicular to the line with the given equation and line j passes through P. Write an equation of line j.**

38. $y = \frac{1}{2}x - 1$, $P(0, 3)$

39. $y = \frac{5}{3}x + 2$, $P(5, 1)$

40. $y = -4x - 3$, $P(-2, 2)$

41. $3y + 4x = 12$, $P(-3, -4)$

FOCUS ON PEOPLE

HELAMAN FERGUSON'S stone drill is suspended by six cables. The computer uses the lengths of the cables to calculate the coordinates of the drill tip.

WRITING EQUATIONS **The line with the given equation is perpendicular to line *j* at point *R*. Write an equation of line *j*.**

42. $y = -\frac{3}{4}x + 6$, $R(8, 0)$

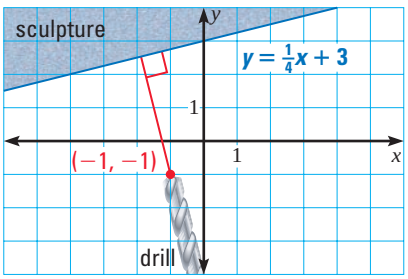
43. $y = \frac{1}{7}x - 11$, $R(7, -10)$

44. $y = 3x + 5$, $R(-3, -4)$

45. $y = -\frac{2}{5}x - 3$, $R(5, -5)$

46. **SCULPTURE** Helaman Ferguson designs sculptures on a computer. The computer is connected to his stone drill and tells how far he should drill at any given point. The distance from the drill tip to the desired surface of the sculpture is calculated along a line perpendicular to the sculpture.

Suppose the drill tip is at $(-1, -1)$ and the equation $y = \frac{1}{4}x + 3$ represents the surface of the sculpture. Write an equation of the line that passes through the drill tip and is perpendicular to the sculpture.



LINE RELATIONSHIPS **Decide whether the lines with the given equations are *parallel, perpendicular,* or *neither*.**

47. $y = -2x - 1$
$y = -2x - 3$

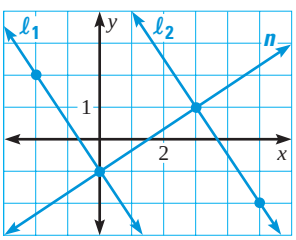
48. $y = -\frac{1}{2}x + 3$
$y = -\frac{1}{2}x + 5$

49. $y = -3x + 1$
$y = \frac{1}{3}x + 1$

50. $y = 4x + 10$
$y = -2x + 5$

Test Preparation

51. **MULTI-STEP PROBLEM** Use the diagram at the right.



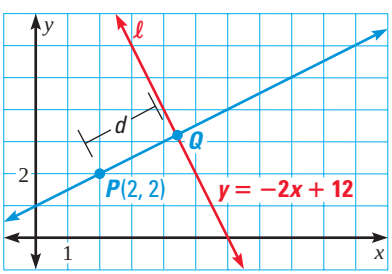
a. Is $\ell_1 \parallel \ell_2$? How do you know?

b. Is $\ell_2 \perp n$? How do you know?

c. *Writing* Describe two ways to prove that $\ell_1 \perp n$.

★ Challenge

DISTANCE TO A LINE **In Exercises 52–54, use the following information.**
The distance from a point to a line is defined to be the length of the perpendicular segment from the point to the line. In the diagram at the right, the distance d between point P and line ℓ is given by QP.



52. Find an equation of $\overleftrightarrow{QP}$.

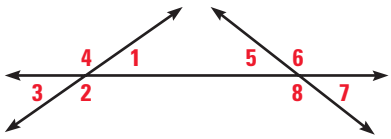
53. Solve a system of equations to find the coordinates of point Q, the intersection of the two lines.

54. Use the Distance Formula to find QP.

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

ANGLE MEASURES **Use the diagram to complete the statement.** (Review 2.6 for 4.1)



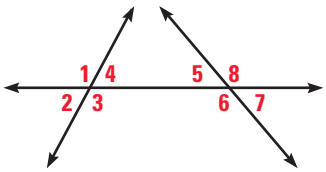
55. If $m\angle 5 = 38°$, then $m\angle 8 =$ __?__.

56. If $m\angle 3 = 36°$, then $m\angle 4 =$ __?__.

57. If $\angle 8 \cong \angle 4$ and $m\angle 2 = 145°$, then $m\angle 7 =$ __?__.

58. If $m\angle 1 = 38°$ and $\angle 3 \cong \angle 5$, then $m\angle 6 =$ __?__.

IDENTIFYING ANGLES **Use the diagram to complete the statement.** (Review 3.1 for 4.1)



59. $\angle 3$ and __?__ are consecutive interior angles.

60. $\angle 1$ and __?__ are alternate exterior angles.

61. $\angle 4$ and __?__ are alternate interior angles.

62. $\angle 1$ and __?__ are corresponding angles.

63. *Writing* Describe the three types of proofs you have learned so far. (Review 3.2)

QUIZ 3

Self-Test for Lessons 3.6 and 3.7

Find the slope of $\overleftrightarrow{AB}$. (Lesson 3.6)

1. $A(1, 2)$, $B(5, 8)$

2. $A(2, -3)$, $B(-1, 5)$

Write an equation of line j_2 that passes through point P and is parallel to line j_1. (Lesson 3.6)

3. line j_1: $y = 3x - 2$
$P(0, 2)$

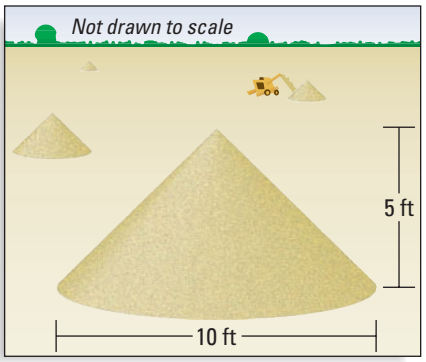
4. line j_1: $y = \frac{1}{2}x + 1$
$P(2, -4)$

Decide whether k_1 and k_2 are perpendicular. (Lesson 3.7)

5. line k_1: $y = 2x - 1$
line k_2: $y = -\frac{1}{2}x + 2$

6. line k_1: $y - 3x = -2$
line k_2: $3y - x = 12$

7. **ANGLE OF REPOSE** When a granular substance is poured into a pile, the slope of the pile depends only on the substance. For example, when barley is poured into piles, every pile has the same slope. A pile of barley that is 5 feet tall would be about 10 feet wide. What is the slope of a pile of barley? (Lesson 3.6)



CHAPTER 3

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Identify relationships between lines. (3.1)	Describe lines and planes in real-life objects, such as escalators. (p. 133)
Identify angles formed by coplanar lines intersected by a transversal. (3.1)	Lay the foundation for work with angles and proof.
Prove and use results about perpendicular lines. (3.2)	Solve real-life problems, such as deciding how many angles of a window frame to measure. (p. 141)
Write flow proofs and paragraph proofs. (3.2)	Learn to write and use different types of proof.
Prove and use results about parallel lines and transversals. (3.3)	Understand the world around you, such as how rainbows are formed. (p. 148)
Prove that lines are parallel. (3.4)	Solve real-life problems, such as predicting paths of sailboats. (p. 152)
Use properties of parallel lines. (3.4, 3.5)	Analyze light passing through glass. (p. 163)
Use slope to decide whether lines in a coordinate plane are parallel. (3.6)	Use coordinate geometry to show that two segments are parallel. (p. 170)
Write an equation of a line parallel to a given line in a coordinate plane. (3.6)	Prepare to write coordinate proofs.
Use slope to decide whether lines in a coordinate plane are perpendicular. (3.7)	Solve real-life problems, such as deciding whether two stitched lines form a right angle. (p. 176)
Write an equation of a line perpendicular to a given line. (3.7)	Find the distance from a point to a line. (p. 177)

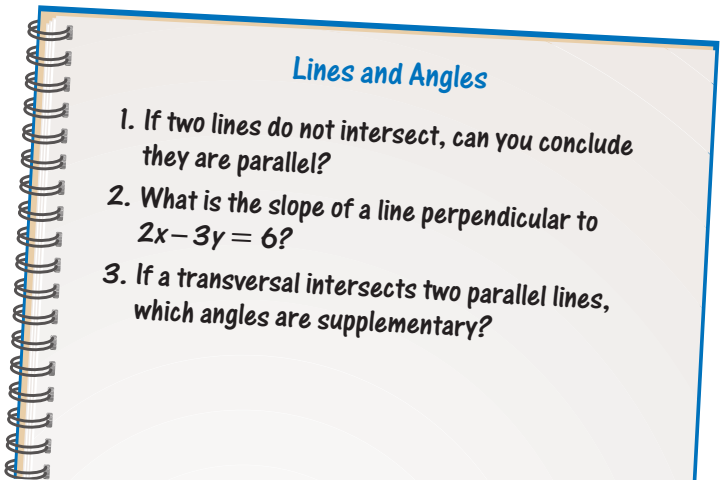
How does Chapter 3 fit into the BIGGER PICTURE of geometry?

In this chapter, you learned about properties of perpendicular and parallel lines. You also learned to write flow proofs and learned some important skills related to coordinate geometry. This work will prepare you to reach conclusions about triangles and other figures and to solve real-life problems in areas such as carpentry, engineering, and physics.

STUDY STRATEGY

How did your study questions help you learn?

The study questions you wrote, following the study strategy on page 128, may resemble this one.



CHAPTER 3

Chapter Review

VOCABULARY

- parallel lines, p. 129
- skew lines, p. 129
- parallel planes, p. 129
- transversal, p. 131
- corresponding angles, p. 131
- alternate interior angles, p. 131
- alternate exterior angles, p. 131
- consecutive interior angles, p. 131
- same side interior angles, p. 131
- flow proof, p. 136

3.1 LINES AND ANGLES

Examples on pp. 129–131

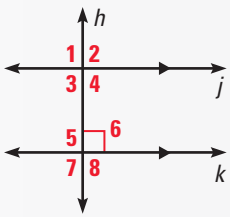
EXAMPLES In the figure, $j \parallel k$, h is a transversal, and $h \perp k$.

$\angle 1$ and $\angle 5$ are corresponding angles.

$\angle 3$ and $\angle 6$ are alternate interior angles.

$\angle 1$ and $\angle 8$ are alternate exterior angles.

$\angle 4$ and $\angle 6$ are consecutive interior angles.



Complete the statement. Use the figure above.

1. $\angle 2$ and $\angle 7$ are __?__ angles.

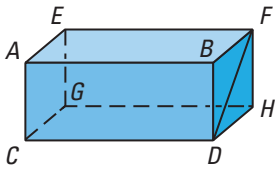
2. $\angle 4$ and $\angle 5$ are __?__ angles.

Use the figure at the right.

3. Name a line parallel to $\overleftrightarrow{DH}$.

4. Name a line perpendicular to $\overleftrightarrow{AE}$.

5. Name a line skew to $\overleftrightarrow{FD}$.

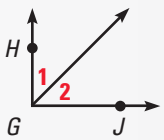


3.2 PROOF AND PERPENDICULAR LINES

Examples on pp. 136–138

EXAMPLE **GIVEN** ▶ $\angle 1$ and $\angle 2$ are complements.

PROVE ▶ $\overrightarrow{GH} \perp \overrightarrow{GJ}$



$\angle 1$ and $\angle 2$ are complements. → __?__

$m\angle 1 + m\angle 2 = 90°$ → __?__

$m\angle 1 + m\angle 2 = m\angle HGJ$ → __?__

$m\angle HGJ = 90°$ → __?__

$\angle HGJ$ is a right $\angle$. → __?__

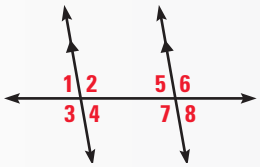
$\overrightarrow{GH} \perp \overrightarrow{GJ}$ → __?__

6. Copy the flow proof and add a reason for each statement.

3.3 PARALLEL LINES AND TRANSVERSALS

Examples on pp. 143–145

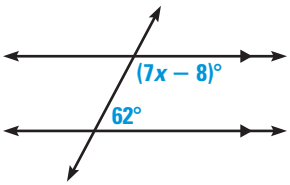
EXAMPLE In the diagram, $m\angle 1 = 75°$. By the Alternate Exterior Angles Theorem, $m\angle 8 = m\angle 1 = 75°$. Because $\angle 8$ and $\angle 7$ are a linear pair, $m\angle 8 + m\angle 7 = 180°$. So, $m\angle 7 = 180° - 75° = 105°$.



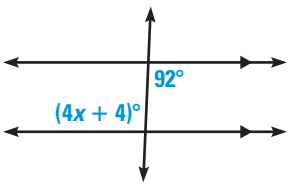
7. Find the measures of the other five angles in the diagram above.

Find the value of *x*. Explain your reasoning.

8.



9.



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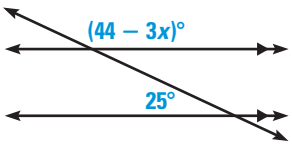


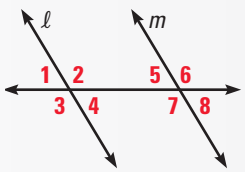
3.4 PROVING LINES ARE PARALLEL

Examples on pp. 150–152

EXAMPLE **GIVEN** ▶ $m\angle 3 = 125°$, $m\angle 6 = 125°$

PROVE ▶ $\ell \parallel m$

Plan for Proof: $m\angle 3 = 125° = m\angle 6$, so $\angle 3 \cong \angle 6$. So, $\ell \parallel m$ by the Alternate Exterior Angles Converse.



Use the diagram above to write a proof.

11. **GIVEN** ▶ $m\angle 4 = 60°$, $m\angle 7 = 120°$

PROVE ▶ $\ell \parallel m$

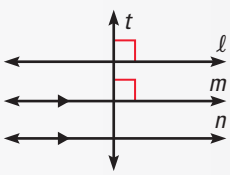
12. **GIVEN** ▶ $\angle 1$ and $\angle 7$ are supplementary.

PROVE ▶ $\ell \parallel m$

3.5 USING PROPERTIES OF PARALLEL LINES

Examples on pp. 157–159

EXAMPLE In the diagram, $\ell \perp t$, $m \perp t$, and $m \parallel n$. Because ℓ and m are coplanar and perpendicular to the same line, $\ell \parallel m$. Then, because $\ell \parallel m$ and $m \parallel n$, $\ell \parallel n$.



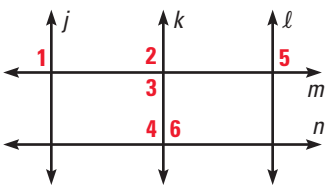
Which lines must be parallel? Explain.

13. $\angle 1$ and $\angle 2$ are right angles.

14. $\angle 3 \cong \angle 6$

15. $\angle 3$ and $\angle 4$ are supplements.

16. $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 5$



3.6 PARALLEL LINES IN THE COORDINATE PLANE

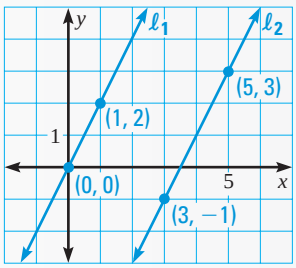
Examples on pp. 165–167

EXAMPLES slope of $\ell_1 = \frac{2-0}{1-0} = 2$

slope of $\ell_2 = \frac{3-(-1)}{5-3} = \frac{4}{2} = 2$

The slopes are the same, so $\ell_1 \parallel \ell_2$.

To write an equation for ℓ_2, substitute $(x, y) = (5, 3)$ and $m = 2$ into the slope-intercept form.

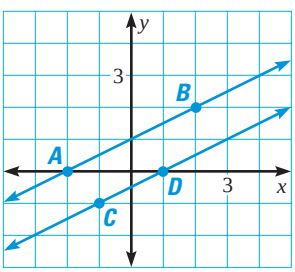


$y = mx + b$	**Slope-intercept form.**
$3 = (2)(5) + b$	**Substitute 5 for *x*, 3 for *y*, and 2 for *m*.**
$-7 = b$	**Solve for *b*.**

▶ So, an equation for ℓ_2 is $y = 2x - 7$.

Find the slope of each line. Are the lines parallel?

17.



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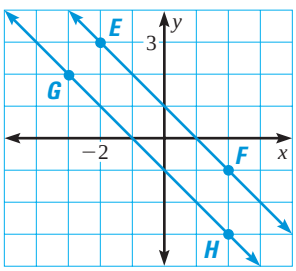

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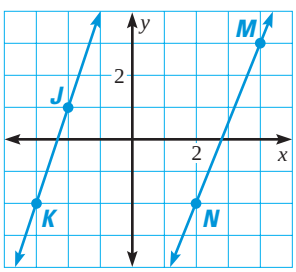


20. Find an equation of the line that is parallel to the line with equation $y = -2x + 5$ and passes through the point $(-1, -4)$.

3.7 PERPENDICULAR LINES IN THE COORDINATE PLANE

Examples on pp. 172–174

EXAMPLE The slope of line j is 3. The slope of line k is $-\frac{1}{3}$.

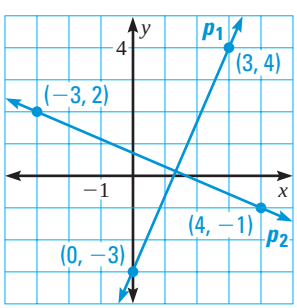
$3\left(-\frac{1}{3}\right) = -1$, so $j \perp k$.

In Exercises 21–23, decide whether lines p_1 and p_2 are perpendicular.

21. Lines p_1 and p_2 in the diagram.

22. p_1: $y = \frac{3}{5}x + 2$; p_2: $y = \frac{5}{3}x - 1$

23. p_1: $2y - x = 2$; p_2: $y + 2x = 4$

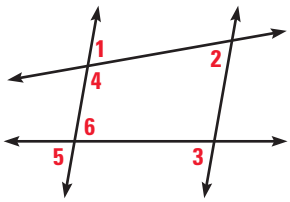


24. Line ℓ_1 has equation $y = -3x + 5$. Write an equation of line ℓ_2 which is perpendicular to ℓ_1 and passes through $(-3, 6)$.

CHAPTER 3

Chapter Test

In Exercises 1–6, identify the relationship between the angles in the diagram at the right.



1. $\angle 1$ and $\angle 2$

2. $\angle 1$ and $\angle 4$

3. $\angle 2$ and $\angle 3$

4. $\angle 1$ and $\angle 5$

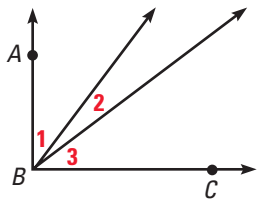
5. $\angle 4$ and $\angle 2$

6. $\angle 5$ and $\angle 6$

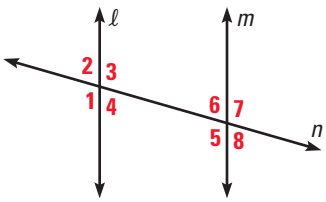
7. Write a flow proof.

GIVEN ▶ $m\angle 1 = m\angle 3 = 37°$, $\overrightarrow{BA} \perp \overrightarrow{BC}$

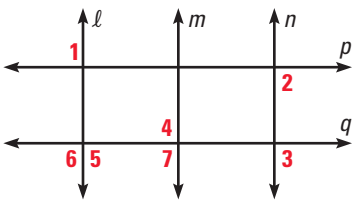
PROVE ▶ $m\angle 2 = 16°$



8. If $\ell \parallel m$, which angles are supplementary to $\angle 1$?



Use the given information and the diagram at the right to determine which lines must be parallel.



9. $\angle 1 \cong \angle 2$

10. $\angle 3$ and $\angle 4$ are right angles.

11. $\angle 1 \cong \angle 5$; $\angle 5$ and $\angle 7$ are supplementary.

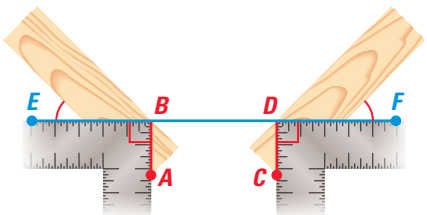
In Exercises 12 and 13, write an equation of the line described.

12. The line parallel to $y = -\frac{1}{3}x + 5$ and with a y-intercept of 1

13. The line perpendicular to $y = -2x + 4$ and that passes through the point $(-1, 2)$

14. *Writing* Describe a real-life object that has edges that are straight lines. Are any of the lines skew? If so, describe a pair.

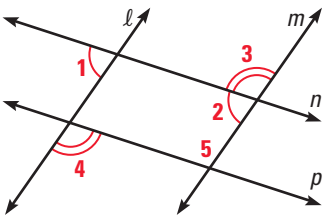
15. A carpenter wants to cut two boards to fit snugly together. The carpenter's squares are aligned along $\overline{EF}$, as shown. Are $\overline{AB}$ and $\overline{CD}$ parallel? State the theorem that justifies your answer.



16. Use the diagram to write a proof.

GIVEN ▶ $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$

PROVE ▶ $n \parallel p$



4.1 Triangles and Angles

What you should learn

GOAL 1 Classify triangles by their sides and angles, as applied in **Example 2.**

GOAL 2 Find angle measures in triangles.

Why you should learn it

▼ To solve **real-life** problems, such as finding the measures of angles in a wing deflector in **Exs. 45 and 46.**

REAL LIFE

A wing deflector is used to change the velocity of the water in a stream.

GOAL 1 CLASSIFYING TRIANGLES

A **triangle** is a figure formed by three segments joining three noncollinear points. A triangle can be classified by its sides and by its angles, as shown in the definitions below.

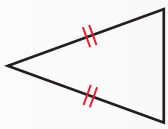
NAMES OF TRIANGLES

Classification by Sides

EQUILATERAL TRIANGLE

3 congruent sides

ISOSCELES TRIANGLE



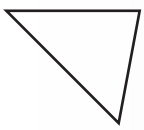
At least 2 congruent sides

SCALENE TRIANGLE

No congruent sides

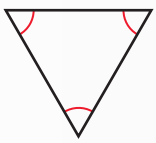
Classification by Angles

ACUTE TRIANGLE



3 acute angles

EQUIANGULAR TRIANGLE

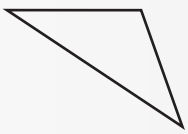


3 congruent angles

RIGHT TRIANGLE

1 right angle

OBTUSE TRIANGLE



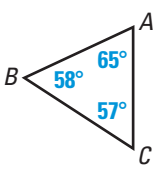
1 obtuse angle

Note: An equiangular triangle is also acute.

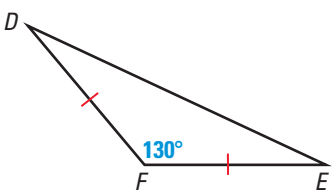
EXAMPLE 1 *Classifying Triangles*

When you classify a triangle, you need to be as specific as possible.

a. $\triangle ABC$ has three acute angles and no congruent sides. It is an acute scalene triangle. ($\triangle ABC$ is read as "triangle ABC.")

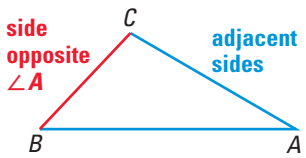


b. $\triangle DEF$ has one obtuse angle and two congruent sides. It is an obtuse isosceles triangle.



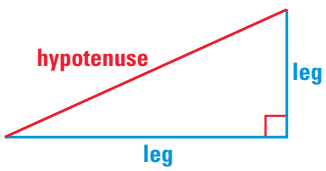
Each of the three points joining the sides of a triangle is a **vertex**. (The plural of vertex is *vertices*.) For example, in $\triangle ABC$, points A, B, and C are vertices.

In a triangle, two sides sharing a common vertex are **adjacent sides**. In $\triangle ABC$, $\overline{CA}$ and $\overline{BA}$ are adjacent sides. The third side, $\overline{BC}$, is the side *opposite* $\angle A$.

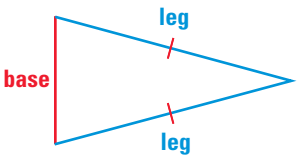


RIGHT AND ISOSCELES TRIANGLES The sides of right triangles and isosceles triangles have special names. In a right triangle, the sides that form the right angle are the **legs** of the right triangle. The side opposite the right angle is the **hypotenuse** of the triangle.

An isosceles triangle can have three congruent sides, in which case it is equilateral. When an isosceles triangle has only two congruent sides, then these two sides are the **legs** of the isosceles triangle. The third side is the **base** of the isosceles triangle.



Right triangle



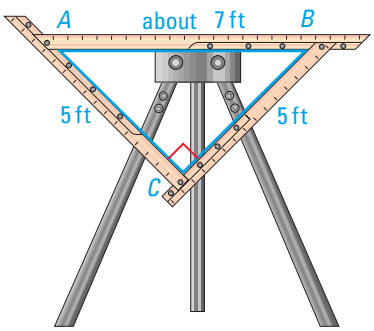
Isosceles triangle

EXAMPLE 2 *Identifying Parts of an Isosceles Right Triangle*

The diagram shows a triangular loom.

a. Explain why $\triangle ABC$ is an isosceles right triangle.

b. Identify the legs and the hypotenuse of $\triangle ABC$. Which side is the base of the triangle?



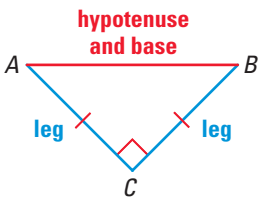
FOCUS ON APPLICATIONS

WEAVING Most looms are used to weave rectangular cloth. The loom shown in the photo is used to weave triangular pieces of cloth. A piece of cloth woven on the loom can use about 550 yards of yarn.

SOLUTION

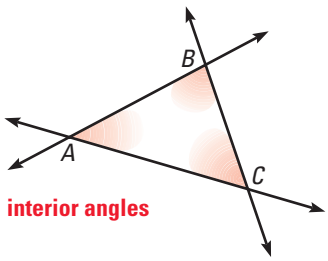
a. In the diagram, you are given that $\angle C$ is a right angle. By definition, $\triangle ABC$ is a right triangle. Because $AC = 5$ ft and $BC = 5$ ft, $\overline{AC} \cong \overline{BC}$. By definition, $\triangle ABC$ is also an isosceles triangle.

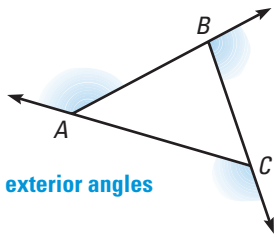
b. Sides $\overline{AC}$ and $\overline{BC}$ are adjacent to the right angle, so they are the legs. Side $\overline{AB}$ is opposite the right angle, so it is the hypotenuse. Because $\overline{AC} \cong \overline{BC}$, side $\overline{AB}$ is also the base.



GOAL 2 USING ANGLE MEASURES OF TRIANGLES

When the sides of a triangle are extended, other angles are formed. The three original angles are the **interior angles**. The angles that are adjacent to the interior angles are the **exterior angles**. Each vertex has a *pair* of congruent exterior angles. It is common to show only *one* exterior angle at each vertex.





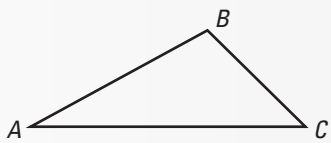
In Activity 4.1 on page 193, you may have discovered the *Triangle Sum Theorem*, shown below, and the *Exterior Angle Theorem*, shown on page 197.

THEOREM

THEOREM 4.1 ***Triangle Sum Theorem***

The sum of the measures of the interior angles of a triangle is 180°.

$$m\angle A + m\angle B + m\angle C = 180°$$

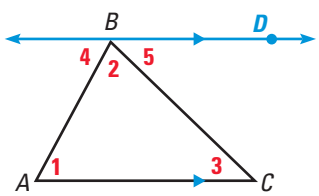


To prove some theorems, you may need to add a line, a segment, or a ray to the given diagram. Such an *auxiliary line* is used to prove the Triangle Sum Theorem.

Proof

GIVEN ▶ $\triangle ABC$

PROVE ▶ $m\angle 1 + m\angle 2 + m\angle 3 = 180°$



Plan for Proof By the Parallel Postulate, you can draw an auxiliary line through point B and parallel to $\overline{AC}$. Because $\angle 4$, $\angle 2$, and $\angle 5$ form a straight angle, the sum of their measures is 180°. You also know that $\angle 1 \cong \angle 4$ and $\angle 3 \cong \angle 5$ by the Alternate Interior Angles Theorem.

Statements	Reasons
1. Draw $\overleftrightarrow{BD}$ parallel to $\overline{AC}$.	**1.** Parallel Postulate
2. $m\angle 4 + m\angle 2 + m\angle 5 = 180°$	**2.** Angle Addition Postulate and definition of straight angle
3. $\angle 1 \cong \angle 4$ and $\angle 3 \cong \angle 5$	**3.** Alternate Interior Angles Theorem
4. $m\angle 1 = m\angle 4$ and $m\angle 3 = m\angle 5$	**4.** Definition of congruent angles
5. $m\angle 1 + m\angle 2 + m\angle 3 = 180°$	**5.** Substitution property of equality

STUDENT HELP

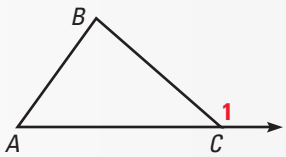
Study Tip
An auxiliary line, segment, or ray used in a proof must be justified with a reason.

THEOREM

THEOREM 4.2 ***Exterior Angle Theorem***

The measure of an exterior angle of a triangle is equal to the sum of the measures of the two nonadjacent interior angles.

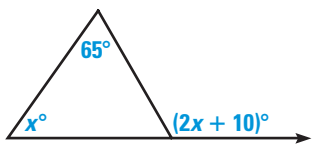
$m\angle 1 = m\angle A + m\angle B$



Using Algebra

EXAMPLE 3 *Finding an Angle Measure*

You can apply the Exterior Angle Theorem to find the measure of the exterior angle shown. First write and solve an equation to find the value of x:



$x° + 65° = (2x + 10)°$ **Apply the Exterior Angles Theorem.**

$55 = x$ **Solve for *x*.**

So, the measure of the exterior angle is $(2 \cdot 55 + 10)°$, or $120°$.

STUDENT HELP

Skills Review
For help with solving equations, see p. 790.

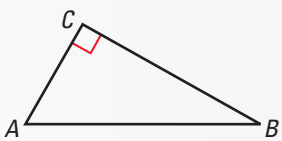
A **corollary to a theorem** is a statement that can be proved easily using the theorem. The corollary below follows from the Triangle Sum Theorem.

COROLLARY

COROLLARY TO THE TRIANGLE SUM THEOREM

The acute angles of a right triangle are complementary.

$m\angle A + m\angle B = 90°$

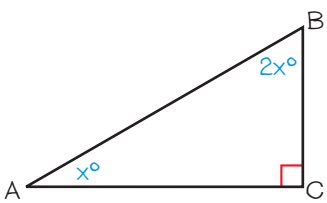


EXAMPLE 4 *Finding Angle Measures*

The measure of one acute angle of a right triangle is two times the measure of the other acute angle. Find the measure of each acute angle.

SOLUTION

Make a sketch. Let $x° = m\angle A$. Then $m\angle B = 2x°$.



$x° + 2x° = 90°$ **The acute angles of a right triangle are complementary.**

$x = 30$ **Solve for *x*.**

So, $m\angle A = 30°$ and $m\angle B = 2(30°) = 60°$.

GUIDED PRACTICE

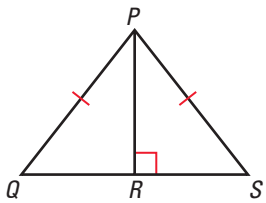
Vocabulary Check ✓

1. Sketch an obtuse scalene triangle. Label its interior angles 1, 2, and 3. Then draw its exterior angles. Shade the exterior angles.

Concept Check ✓

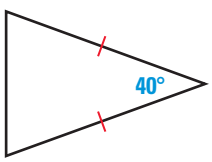
In the figure, $\overline{PQ} \cong \overline{PS}$ and $\overline{PR} \perp \overline{QS}$. Complete the sentence.

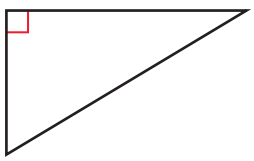
2. $\overline{PQ}$ is the __?__ of the right triangle $\triangle PQR$.
3. In $\triangle PQR$, $\overline{PQ}$ is the side opposite angle __?__.
4. $\overline{QS}$ is the __?__ of the isosceles triangle $\triangle PQS$.
5. The legs of $\triangle PRS$ are __?__ and __?__.

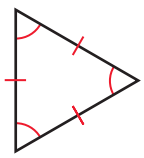


Skill Check ✓

In Exercises 6–8, classify the triangle by its angles and by its sides.

6.


7.


8.


9. The measure of one interior angle of a triangle is 25°. The other interior angles are congruent. Find the measures of the other interior angles.

PRACTICE AND APPLICATIONS

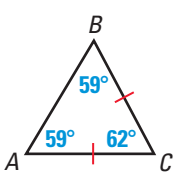
STUDENT HELP

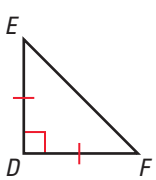
Extra Practice to help you master skills is on p. 809.

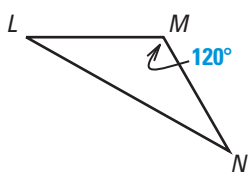
MATCHING TRIANGLES **In Exercises 10–15, match the triangle description with the most specific name.**

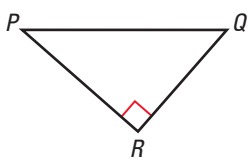
10. Side lengths: 2 cm, 3 cm, 4 cm	**A.** Equilateral
11. Side lengths: 3 cm, 2 cm, 3 cm	**B.** Scalene
12. Side lengths: 4 cm, 4 cm, 4 cm	**C.** Obtuse
13. Angle measures: 60°, 60°, 60°	**D.** Equiangular
14. Angle measures: 30°, 60°, 90°	**E.** Isosceles
15. Angle measures: 20°, 145°, 15°	**F.** Right

CLASSIFYING TRIANGLES **Classify the triangle by its angles and by its sides.**

16.


17.


18.


19.


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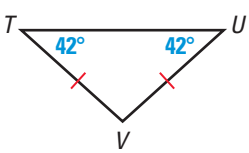

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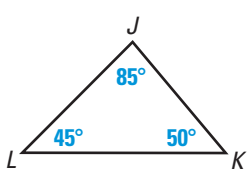


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 10–26, 34–36
Example 2: Exs. 27, 28, 45
Example 3: Exs. 31–39
Example 4: Exs. 41–44

LOGICAL REASONING **Complete the statement using *always, sometimes,* or *never*.**

22. An isosceles triangle is __?__ an equilateral triangle.

23. An obtuse triangle is __?__ an isosceles triangle.

24. An interior angle of a triangle and one of its adjacent exterior angles are __?__ supplementary.

25. The acute angles of a right triangle are __?__ complementary.

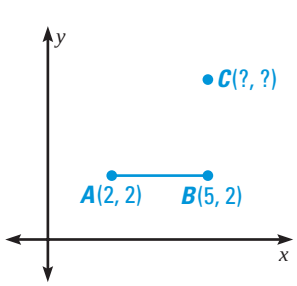
26. A triangle __?__ has a right angle and an obtuse angle.

IDENTIFYING PARTS OF TRIANGLES **Refer to the triangles in Exercises 16–21.**

27. Identify the legs and the hypotenuse of any right triangles.

28. Identify the legs and the base of any isosceles triangles. Which isosceles triangle has a base that is also the hypotenuse of a right triangle?

USING ALGEBRA **Use the graph. The segment $\overline{AB}$ is a leg of an isosceles right triangle.**



29. Find the coordinates of point C. Copy the graph and sketch $\triangle ABC$.

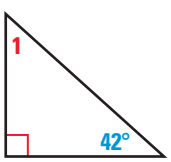
30. Find the coordinates of a point D that forms a different isosceles right triangle with leg $\overline{AB}$. Include a sketch with your answer.

STUDENT HELP

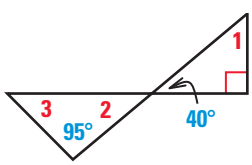
HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 31–33.

FINDING ANGLE MEASURES **Find the measure of the numbered angles.**

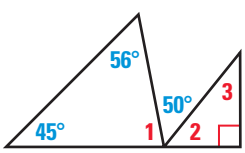
31.



32.



33.



USING ALGEBRA **The variable expressions represent the angle measures of a triangle. Find the measure of each angle. Then classify the triangle by its angles.**

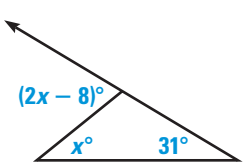
34. $m\angle A = x°$
$m\angle B = 2x°$
$m\angle C = (2x + 15)°$

35. $m\angle R = x°$
$m\angle S = 7x°$
$m\angle T = x°$

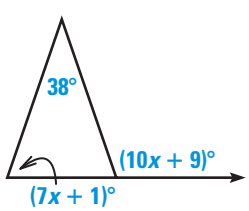
36. $m\angle W = (x - 15)°$
$m\angle Y = (2x - 165)°$
$m\angle Z = 90°$

EXTERIOR ANGLES **Find the measure of the exterior angle shown.**

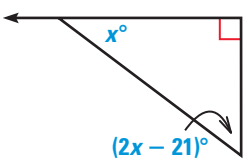
37.



38.



39.



40. **TECHNOLOGY** Use geometry software to demonstrate the Triangle Sum Theorem or the Exterior Angle Theorem. Describe your procedure.

41. **USING ALGEBRA** In $\triangle PQR$, the measure of $\angle P$ is 36°. The measure of $\angle Q$ is five times the measure of $\angle R$. Find $m\angle Q$ and $m\angle R$.

42. **USING ALGEBRA** The measure of an exterior angle of a triangle is 120°. The interior angles that are not adjacent to this exterior angle are congruent. Find the measures of the interior angles of the triangle.

43. **BILLIARD RACK** You want to make a wooden billiard rack. The rack will be an equilateral triangle whose side length is 33.5 centimeters. You have a strip of wood that is 100 centimeters long. Do you need more wood? Explain.

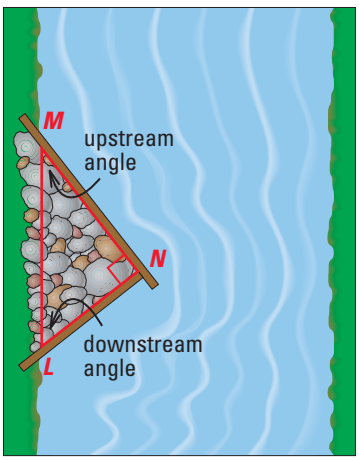
44. **COAT HANGER** You are bending a wire to make a coat hanger. The length of the wire is 88 centimeters, and 20 centimeters are needed to make the hook portion of the hanger. The triangular portion of the hanger is an isosceles triangle. The length of one leg of this triangle is $\frac{3}{5}$ the length of the base. Sketch the hanger. Give the dimensions of the triangular portion.

FOCUS ON CAREERS

HYDROLOGY A hydrologist studies how water circulates in the atmosphere, on the ground, and under the ground. A hydrologist might use a wing deflector to minimize the effects of erosion on the bank of a stream.

CAREER LINK
www.mcdougallittell.com

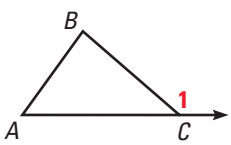
WING DEFLECTORS **In Exercises 45 and 46, use the information about wing deflectors.** A wing deflector is a structure built with rocks to redirect the flow of water in a stream and increase the rate of the water's flow. Its shape is a right triangle.



45. Identify the legs and the hypotenuse of the right triangle formed by the wing deflector.

46. It is generally recommended that the upstream angle should range from 30° to 45°. Give a range of angle measures for the downstream angle.

47. **DEVELOPING PROOF** Fill in the missing steps in the two-column proof of the Exterior Angle Theorem.



GIVEN ▶ $\angle 1$ is an exterior angle of $\triangle ABC$.

PROVE ▶ $m\angle 1 = m\angle A + m\angle B$

Statements	Reasons
1. $\angle 1$ is an exterior angle of $\triangle ABC$.	**1.** Given
2. $\angle ACB$ and $\angle 1$ are a linear pair.	**2.** Definition of exterior angle
3. $m\angle ACB + m\angle 1 = 180°$	**3.** ___?___
4. ___?___	**4.** Triangle Sum Theorem
5. $m\angle ACB + m\angle 1 = m\angle A + m\angle B + m\angle ACB$	**5.** ___?___
6. $m\angle 1 = m\angle A + m\angle B$	**6.** ___?___

48. **TWO-COLUMN PROOF** Write a two-column proof of the Corollary to the Triangle Sum Theorem on page 197.

Test Preparation

49. MULTIPLE CHOICE The lengths of the two legs of an isosceles triangle are represented by the expressions $(2x - 5)$ and $(x + 7)$. The perimeter of the triangle is 50 cm. Find the length of the base of the triangle.

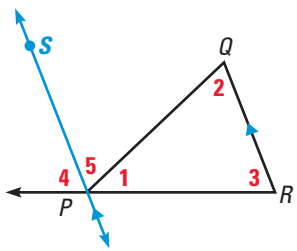
Ⓐ 11 cm Ⓑ 19 cm Ⓒ 12 cm Ⓓ 26 cm Ⓔ 32 cm

50. MULTIPLE CHOICE Which of the terms below can be used to describe a triangle with two 45° interior angles?

Ⓐ Acute Ⓑ Right Ⓒ Scalene Ⓓ Obtuse Ⓔ Equilateral

★ Challenge

51. ALTERNATIVE PROOFS There is often more than one way to prove a theorem. In the diagram, $\overline{SP}$ is constructed parallel to $\overline{QR}$. This construction is the first step of a proof of the Triangle Sum Theorem. Use the diagram to prove the Triangle Sum Theorem.



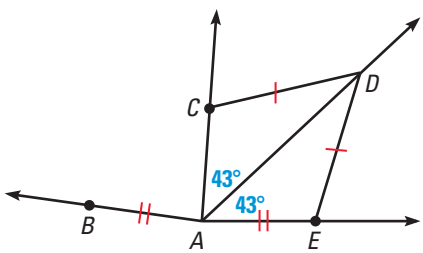
GIVEN ▶ $\triangle PQR$

PROVE ▶ $m\angle 1 + m\angle 2 + m\angle 3 = 180°$

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

EVALUATING STATEMENTS **Use the figure to determine whether the statement is *true* or *false*.** (Review 1.5 for 4.2)



52. $\overline{AE} \cong \overline{BA}$

53. $\angle CAD \cong \angle EAD$

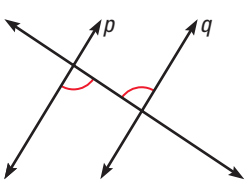
54. $m\angle CAD + m\angle EAB = 86°$

55. $\overline{CD} \cong \overline{AC}$

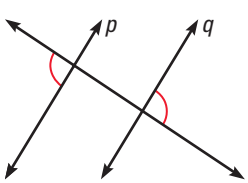
56. $\overrightarrow{AD}$ bisects $\angle CAE$.

DEVELOPING PROOF **Is it possible to prove that lines *p* and *q* are parallel? If so, state the postulate or theorem you would use.** (Review 3.4)

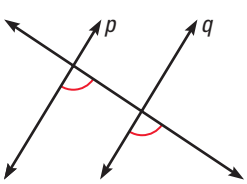
57.



58.



59.



WRITING EQUATIONS **Write an equation of the line that passes through the given point *P* and has the given slope.** (Review 3.6)

60. $P(0, -2)$, $m = 0$

61. $P(4, 7)$, $m = 1$

62. $P(-3, -5)$, $m = -1$

63. $P(9, -1)$, $m = \frac{2}{3}$

64. $P(-1, -1)$, $m = \frac{3}{4}$

65. $P(-2, -3)$, $m = -\frac{7}{2}$

66. $P(5, 2)$, $m = 0$

67. $P(8, 3)$, $m = -\frac{3}{2}$

68. $P(-6, -4)$, $m = -\frac{1}{3}$

4.2 Congruence and Triangles

What you should learn

GOAL 1 Identify congruent figures and corresponding parts.

GOAL 2 Prove that two triangles are congruent.

Why you should learn it

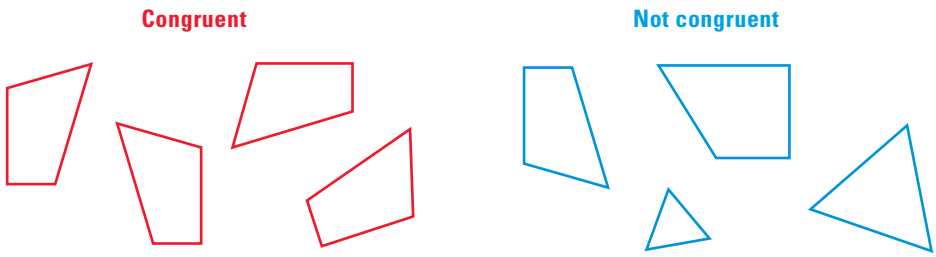
▼ To identify and describe congruent figures in **real-life** objects, such as the sculpture described in **Example 1**.

REAL LIFE

Two Open Triangles Up Gyratory II **by George Rickey**

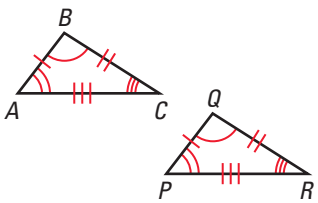
GOAL 1 IDENTIFYING CONGRUENT FIGURES

Two geometric figures are *congruent* if they have exactly the same size and shape. Each of the red figures is congruent to the other red figures. None of the blue figures is congruent to another blue figure.



When two figures are **congruent**, there is a correspondence between their angles and sides such that **corresponding angles** are congruent and **corresponding sides** are congruent. For the triangles below, you can write $\triangle ABC \cong \triangle PQR$, which is read "triangle ABC is congruent to triangle PQR." The notation shows the congruence and the correspondence.

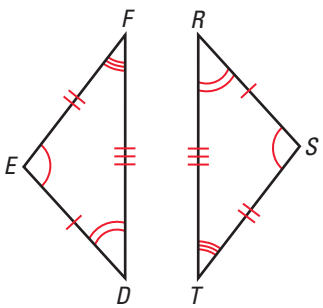
Corresponding angles	Corresponding sides
$\angle A \cong \angle P$	$\overline{AB} \cong \overline{PQ}$
$\angle B \cong \angle Q$	$\overline{BC} \cong \overline{QR}$
$\angle C \cong \angle R$	$\overline{CA} \cong \overline{RP}$



There is more than one way to write a congruence statement, but it is important to list the corresponding angles in the same order. For example, you can also write $\triangle BCA \cong \triangle QRP$.

EXAMPLE 1 *Naming Congruent Parts*

The congruent triangles represent the triangles in the photo above. Write a congruence statement. Identify all pairs of congruent corresponding parts.



SOLUTION

The diagram indicates that $\triangle DEF \cong \triangle RST$. The congruent angles and sides are as follows.

Angles: $\angle D \cong \angle R$, $\angle E \cong \angle S$, $\angle F \cong \angle T$

Sides: $\overline{DE} \cong \overline{RS}$, $\overline{EF} \cong \overline{ST}$, $\overline{FD} \cong \overline{TR}$

STUDENT HELP

Study Tip

Notice that single, double, and triple arcs are used to show congruent angles.

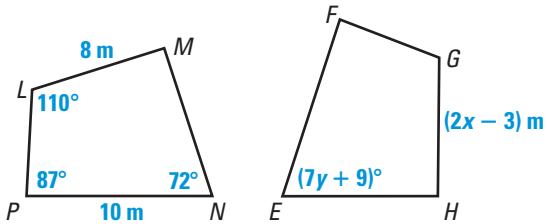
Using Algebra

EXAMPLE 2 *Using Properties of Congruent Figures*

In the diagram, $NPLM \cong EFGH$.

a. Find the value of x.

b. Find the value of y.



SOLUTION

a. You know that $\overline{LM} \cong \overline{GH}$.
So, $LM = GH$.

$$8 = 2x - 3$$
$$11 = 2x$$
$$5.5 = x$$

b. You know that $\angle N \cong \angle E$.
So, $m\angle N = m\angle E$.

$$72° = (7y + 9)°$$
$$63 = 7y$$
$$9 = y$$

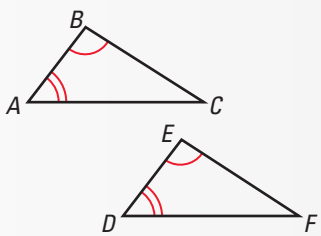
The Third Angles Theorem below follows from the Triangle Sum Theorem. You are asked to prove the Third Angles Theorem in Exercise 35.

THEOREM

THEOREM 4.3 ***Third Angles Theorem***

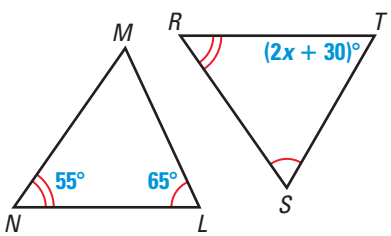
If two angles of one triangle are congruent to two angles of another triangle, then the third angles are also congruent.

If $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then $\angle C \cong \angle F$.



EXAMPLE 3 *Using the Third Angles Theorem*

Find the value of x.



SOLUTION

In the diagram, $\angle N \cong \angle R$ and $\angle L \cong \angle S$. From the Third Angles Theorem, you know that $\angle M \cong \angle T$. So, $m\angle M = m\angle T$. From the Triangle Sum Theorem, $m\angle M = 180° - 55° - 65° = 60°$.

$m\angle M = m\angle T$	**Third Angles Theorem**
$60° = (2x + 30)°$	**Substitute.**
$30 = 2x$	**Subtract 30 from each side.**
$15 = x$	**Divide each side by 2.**

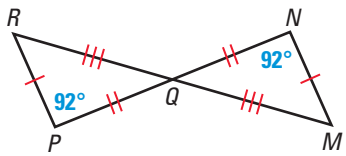
STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 PROVING TRIANGLES ARE CONGRUENT

EXAMPLE 4 *Determining Whether Triangles are Congruent*

Decide whether the triangles are congruent. Justify your reasoning.



SOLUTION

Proof

Paragraph Proof From the diagram, you are given that all three pairs of corresponding sides are congruent.

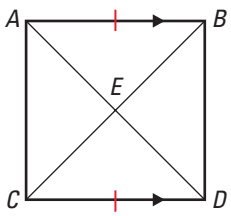
$\overline{RP} \cong \overline{MN}$, $\overline{PQ} \cong \overline{NQ}$, and $\overline{QR} \cong \overline{QM}$

Because $\angle P$ and $\angle N$ have the same measure, $\angle P \cong \angle N$. By the Vertical Angles Theorem, you know that $\angle PQR \cong \angle NQM$. By the Third Angles Theorem, $\angle R \cong \angle M$.

▶ So, all three pairs of corresponding sides and all three pairs of corresponding angles are congruent. By the definition of congruent triangles, $\triangle PQR \cong \triangle NQM$.

EXAMPLE 5 *Proving Two Triangles are Congruent*

The diagram represents the triangular stamps shown in the photo. Prove that $\triangle AEB \cong \triangle DEC$.



GIVEN ▶ $\overline{AB} \parallel \overline{DC}$, $\overline{AB} \cong \overline{DC}$, E is the midpoint of $\overline{BC}$ and $\overline{AD}$.

PROVE ▶ $\triangle AEB \cong \triangle DEC$

Plan for Proof Use the fact that $\angle AEB$ and $\angle DEC$ are vertical angles to show that those angles are congruent. Use the fact that $\overline{BC}$ intersects parallel segments $\overline{AB}$ and $\overline{DC}$ to identify other pairs of angles that are congruent.

FOCUS ON APPLICATIONS

REAL LIFE TRIANGULAR STAMP
When these stamps were issued in 1997, Postmaster General Marvin Runyon said, "Since 1847, when the first U.S. postage stamps were issued, stamps have been rectangular in shape. We want the American public to know stamps aren't 'square.'"

SOLUTION

Statements	Reasons
1. $\overline{AB} \parallel \overline{DC}$, $\overline{AB} \cong \overline{DC}$	**1.** Given
2. $\angle EAB \cong \angle EDC$, $\angle ABE \cong \angle DCE$	**2.** Alternate Interior Angles Theorem
3. $\angle AEB \cong \angle DEC$	**3.** Vertical Angles Theorem
4. E is the midpoint of $\overline{AD}$, E is the midpoint of $\overline{BC}$.	**4.** Given
5. $\overline{AE} \cong \overline{DE}$, $\overline{BE} \cong \overline{CE}$	**5.** Definition of midpoint
6. $\triangle AEB \cong \triangle DEC$	**6.** Definition of congruent triangles

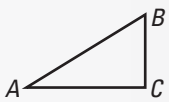
In this lesson, you have learned to prove that two triangles are congruent by the *definition of congruence*—that is, by showing that all pairs of corresponding angles and corresponding sides are congruent. In upcoming lessons, you will learn more efficient ways of proving that triangles are congruent. The properties below will be useful in such proofs.

THEOREM

THEOREM 4.4 ***Properties of Congruent Triangles***

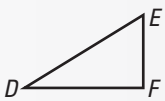
REFLEXIVE PROPERTY OF CONGRUENT TRIANGLES

Every triangle is congruent to itself.



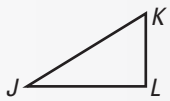
SYMMETRIC PROPERTY OF CONGRUENT TRIANGLES

If $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$.



TRANSITIVE PROPERTY OF CONGRUENT TRIANGLES

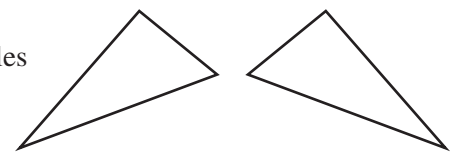
If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle JKL$, then $\triangle ABC \cong \triangle JKL$.



GUIDED PRACTICE

Vocabulary Check ✓

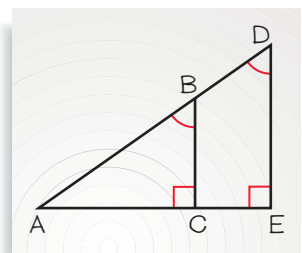
1. Copy the congruent triangles shown at the right. Then label the vertices of your triangles so that $\triangle JKL \cong \triangle RST$. Identify all pairs of congruent *corresponding angles* and *corresponding sides*.



Concept Check ✓

ERROR ANALYSIS **Use the information and the diagram below.**

On an exam, a student says that $\triangle ABC \cong \triangle ADE$ because the corresponding angles of the triangles are congruent.

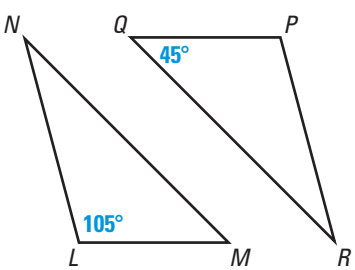


2. How does the student know that the corresponding angles are congruent?

3. Is $\triangle ABC \cong \triangle ADE$? Explain your answer.

Skill Check ✓

Use the diagram at the right, where $\triangle LMN \cong \triangle PQR$.



4. What is the measure of $\angle P$?

5. What is the measure of $\angle M$?

6. What is the measure of $\angle R$?

7. What is the measure of $\angle N$?

8. Which side is congruent to $\overline{QR}$?

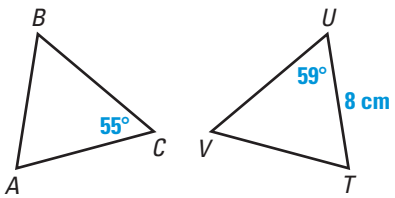
9. Which side is congruent to $\overline{LN}$?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 809.

DESCRIBING CONGRUENT TRIANGLES **In the diagram, $\triangle ABC \cong \triangle TUV$. Complete the statement.**



10. $\angle A \cong$?

11. $\overline{VT} \cong$?

12. $\triangle VTU \cong$?

13. $BC =$?

14. $m\angle A = m\angle$? $=$? °

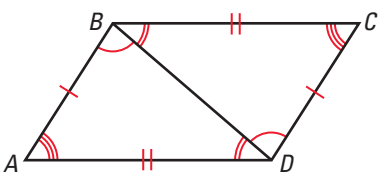
15. Which of the statements below can be used to describe the congruent triangles in Exercises 10–14? (There may be more than one answer.)

A. $\triangle CBA \cong \triangle TUV$ **B.** $\triangle CBA \cong \triangle VUT$

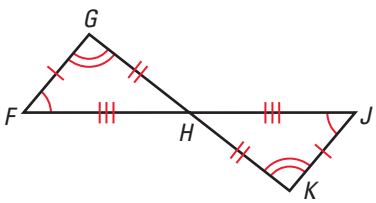
C. $\triangle UTV \cong \triangle BAC$ **D.** $\triangle TVU \cong \triangle ACB$

NAMING CONGRUENT FIGURES **Identify any figures that can be proved congruent. Explain your reasoning. For those that can be proved congruent, write a congruence statement.**

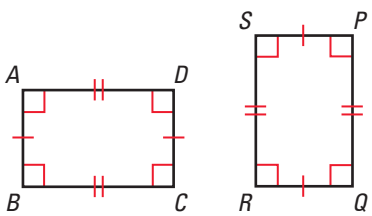
16.



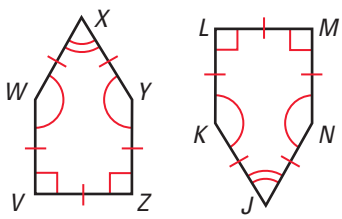
17.



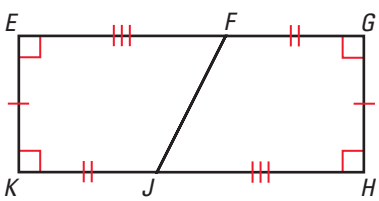
18.



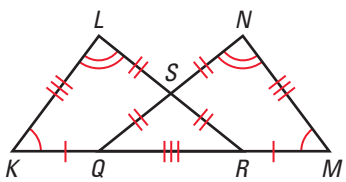
19.



20.

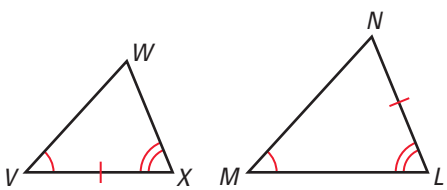


21.



22. **IDENTIFYING CORRESPONDING PARTS** Use the triangles shown in Exercise 17 above. Identify all pairs of congruent corresponding angles and corresponding sides.

23. **CRITICAL THINKING** Use the triangles shown at the right. How many pairs of angles are congruent? Are the triangles congruent? Explain your reasoning.



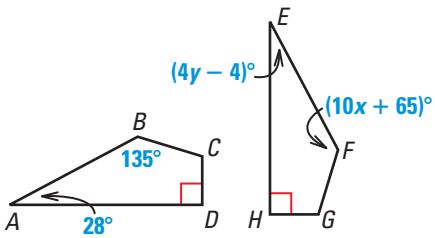
STUDENT HELP

HOMEWORK HELP

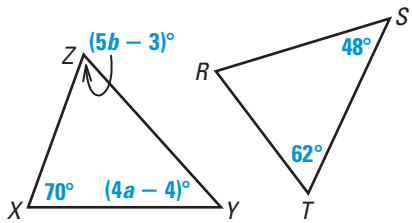
Example 1: Exs. 10–22
Example 2: Exs. 14, 24, 25
Example 3: Exs. 26–29
Example 4: Exs. 16–21, 23
Example 5: Ex. 38

USING ALGEBRA **Use the given information to find the indicated values.**

24. Given $ABCD \cong EFGH$, find the values of x and y.

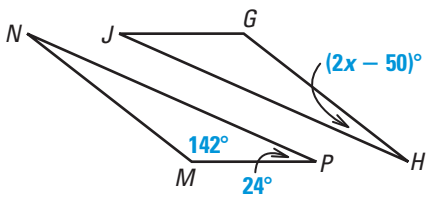


25. Given $\triangle XYZ \cong \triangle RST$, find the values of a and b.

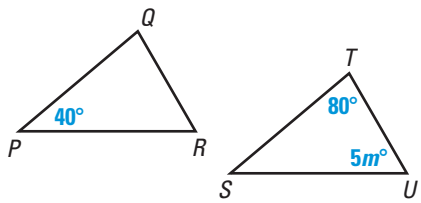


USING ALGEBRA **Use the given information to find the indicated value.**

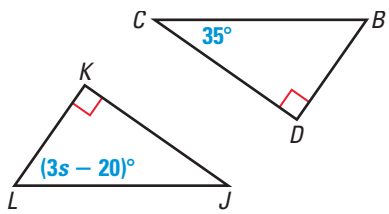
26. Given $\angle M \cong \angle G$ and $\angle N \cong \angle H$, find the value of x.



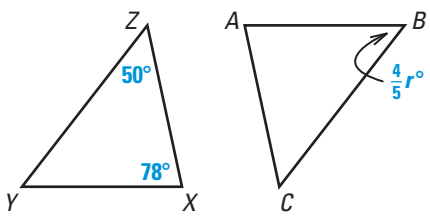
27. Given $\angle P \cong \angle S$ and $\angle Q \cong \angle T$, find the value of m.



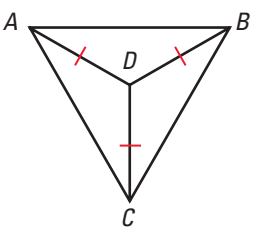
28. Given $\angle K \cong \angle D$ and $\angle J \cong \angle C$, find the value of s.



29. Given $\angle A \cong \angle X$ and $\angle C \cong \angle Z$, find the value of r.



CROP CIRCLES **Use the diagram based on the photo. The small triangles, $\triangle ADB$, $\triangle CDA$, and $\triangle CDB$, are congruent.**



This pattern was made by mowing a field in England.

30. Explain why $\triangle ABC$ is equilateral.

31. The sum of the measures of $\angle ADB$, $\angle CDA$, and $\angle CDB$ is 360°. Find $m\angle BDC$.

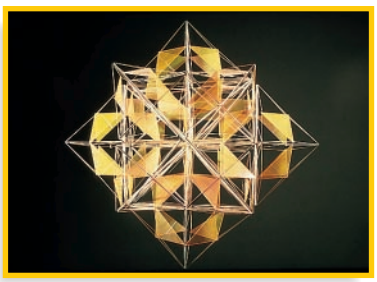
32. Each of the small isosceles triangles has two congruent acute angles. Find $m\angle DBC$ and $m\angle DCB$.

33. **LOGICAL REASONING** Explain why $\triangle ABC$ is equiangular.

FOCUS ON PEOPLE

HARRIET BRISSON is an artist who has created many works of art that rely on or express mathematical principles. The pattern used to arrange the triangles in her sculpture shown at the right can be extended indefinitely.

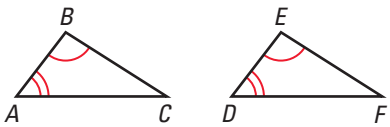
34. **SCULPTURE** The sculpture shown in the photo is made of congruent triangles cut from transparent plastic. Suppose you use one triangle as a pattern to cut all the other triangles. Which property guarantees that all the triangles are congruent to each other?



35. **DEVELOPING PROOF** Complete the proof of the Third Angles Theorem.

GIVEN ▶ $\angle A \cong \angle D$, $\angle B \cong \angle E$

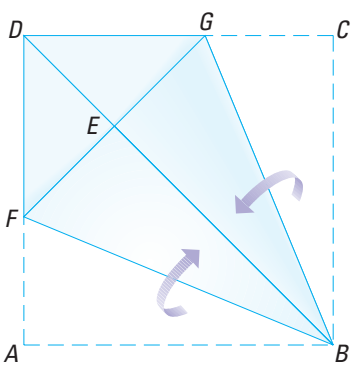
PROVE ▶ $\angle C \cong \angle F$



Statements	Reasons
1. $\angle A \cong \angle D$, $\angle B \cong \angle E$	**1.** ?
2. $m\angle$? $= m\angle$?, $m\angle$? $= m\angle$?	**2.** ?
3. $m\angle A + m\angle B + m\angle C = 180°$, $m\angle D + m\angle E + m\angle F = 180°$	**3.** ?
4. $m\angle A + m\angle B + m\angle C = m\angle D + m\angle E + m\angle F$	**4.** ?
5. $m\angle D + m\angle E + m\angle C = m\angle D + m\angle E + m\angle F$	**5.** ?
6. $m\angle C = m\angle F$	**6.** ?
7. ?	**7.** Def. of $\cong \angle$s.

ORIGAMI **Origami is the art of folding paper into interesting shapes. Follow the directions below to create a kite. Use your kite in Exercises 36–38.**

1. Fold a square piece of paper in half diagonally to create $\overline{DB}$.
2. Next fold the paper so that side $\overline{AB}$ lies directly on $\overline{DB}$.
3. Then fold the paper so that side $\overline{CB}$ lies directly on $\overline{DB}$.



36. Is $\overline{EB}$ congruent to $\overline{AB}$? Is $\overline{EF}$ congruent to $\overline{AF}$? Explain.

37. **LOGICAL REASONING** From folding, you know that $\overrightarrow{BF}$ bisects $\angle EBA$ and $\overrightarrow{FB}$ bisects $\angle AFE$. Given these facts and your answers to Exercise 36, which triangles can you conclude are congruent? Explain.

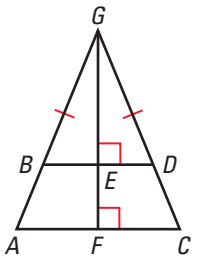
38. **PROOF** Write a proof.

GIVEN ▶ $\overline{DB} \perp \overline{FG}$, E is the midpoint of $\overline{FG}$, $\overline{BF} \cong \overline{BG}$, and $\overrightarrow{BD}$ bisects $\angle GBF$.

PROVE ▶ $\triangle FEB \cong \triangle GEB$

Test Preparation

39. MULTI-STEP PROBLEM Use the diagram, in which $ABEF \cong CDEF$.



a. Explain how you know that $\overline{BE} \cong \overline{DE}$.

b. Explain how you know that $\angle ABE \cong \angle CDE$.

c. Explain how you know that $\angle GBE \cong \angle GDE$.

d. Explain how you know that $\angle GEB \cong \angle GED$.

e. *Writing* Do you have enough information to prove that $\triangle BEG \cong \triangle DEG$? Explain.

★ Challenge

40. ORIGAMI REVISITED Look back at Exercises 36–38 on page 208. Suppose the following statements are also true about the diagram.

$\overrightarrow{BD}$ bisects $\angle ABC$ and $\overrightarrow{DB}$ bisects $\angle ADC$.
$\angle ABC$ and $\angle ADC$ are right angles.

Find all of the unknown angle measures in the figure. Use a sketch to show your answers.

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

DISTANCE FORMULA **Find the distance between each pair of points.** (Review 1.3 for 4.3)

41. $A(3, 8)$, $B(-1, -4)$

42. $C(3, -8)$, $D(-13, 7)$

43. $E(-2, -6)$, $F(3, -5)$

44. $G(0, 5)$, $H(-5, 2)$

45. $J(0, -4)$, $K(9, 2)$

46. $L(7, -2)$, $M(0, 9)$

FINDING THE MIDPOINT **Find the coordinates of the midpoint of a segment with the given endpoints.** (Review 1.5)

47. $N(-1, 5)$, $P(-3, -9)$

48. $Q(5, 7)$, $R(-1, 4)$

49. $S(-6, -2)$, $T(8, 2)$

50. $U(0, -7)$, $V(-6, 4)$

51. $W(12, 0)$, $Z(8, 6)$

52. $A(-5, -7)$, $B(0, 4)$

FINDING COMPLEMENTARY ANGLES **In Exercises 53–55, $\angle 1$ and $\angle 2$ are complementary. Find $m\angle 2$.** (Review 1.6)

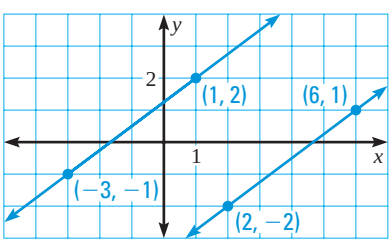
53. $m\angle 1 = 8°$, $m\angle 2 =$?

54. $m\angle 1 = 73°$, $m\angle 2 =$?

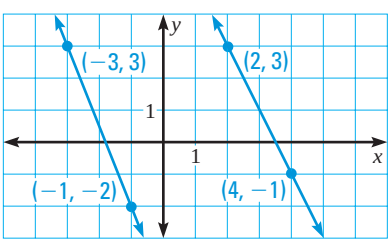
55. $m\angle 1 = 62°$, $m\angle 2 =$?

IDENTIFYING PARALLELS **Find the slope of each line. Are the lines parallel?** (Review 3.6)

56.



57.

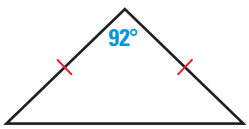


QUIZ 1

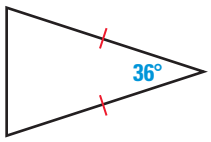
Self-Test for Lessons 4.1 and 4.2

Classify the triangle by its angles and by its sides. (Lesson 4.1)

1.



2.



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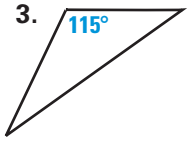
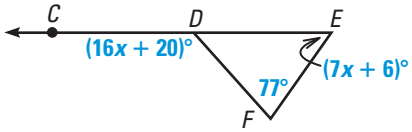
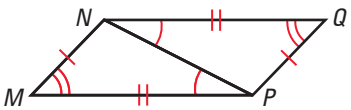


4. Find the value of x in the figure at the right. Then give the measure of each interior angle and the measure of the exterior angle shown. (Lesson 4.1)



Use the diagram at the right. (Lesson 4.2)



5. Write a congruence statement. Identify all pairs of congruent corresponding parts.

6. You are given that $m\angle NMP = 46°$ and $m\angle PNQ = 27°$. Find $m\angle MNP$.

MATH & History — Triangles In Architecture

APPLICATION LINK
www.mcdougallittell.com

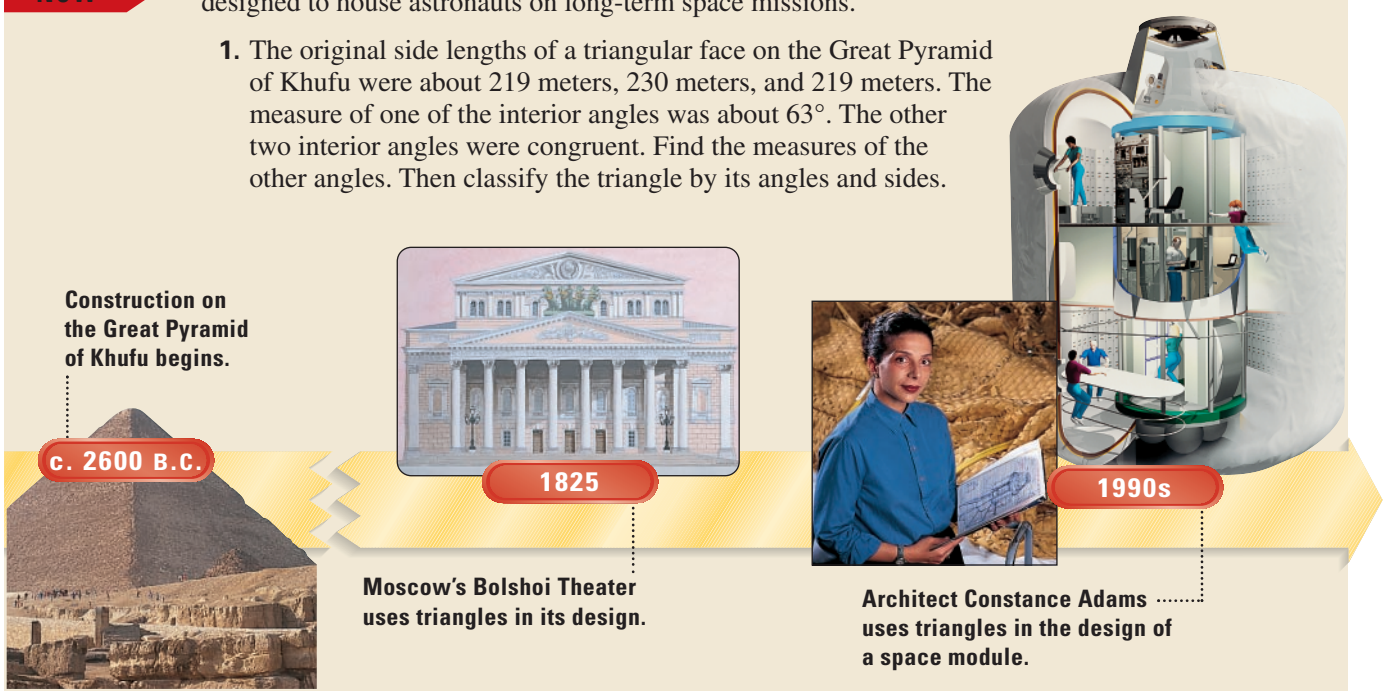
THEN

AROUND 2600 B.C., construction of the Great Pyramid of Khufu began. It took the ancient Egyptians about 30 years to transform 6.5 million tons of stone into a pyramid with a square base and four congruent triangular faces.

NOW

TODAY, triangles are still used in architecture. They are even being used in structures designed to house astronauts on long-term space missions.

1. The original side lengths of a triangular face on the Great Pyramid of Khufu were about 219 meters, 230 meters, and 219 meters. The measure of one of the interior angles was about 63°. The other two interior angles were congruent. Find the measures of the other angles. Then classify the triangle by its angles and sides.



4.3 Proving Triangles are Congruent: SSS and SAS

What you should learn

GOAL 1 Prove that triangles are congruent using the SSS and SAS Congruence Postulates.

GOAL 2 Use congruence postulates in **real-life** problems, such as bracing a structure in **Example 5**.

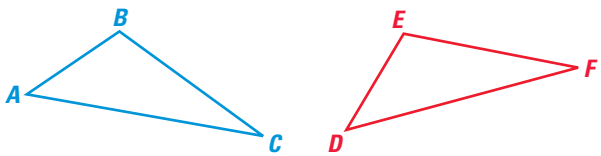
Why you should learn it

▼ Congruence postulates help you see why triangles make things stable, such as the seaplane's wing below and the objects in **Exs. 30 and 31**.

REAL LIFE

GOAL 1 SSS AND SAS CONGRUENCE POSTULATES

How much do you need to know about two triangles to prove that they are congruent? In Lesson 4.2, you learned that if all six pairs of corresponding parts (sides and angles) are congruent, then the triangles are congruent.



If	Sides are congruent	and	Angles are congruent	then	Triangles are congruent
	1. $\overline{AB} \cong \overline{DE}$		4. $\angle A \cong \angle D$		$\triangle ABC \cong \triangle DEF$
	2. $\overline{BC} \cong \overline{EF}$		5. $\angle B \cong \angle E$		
	3. $\overline{AC} \cong \overline{DF}$		6. $\angle C \cong \angle F$		

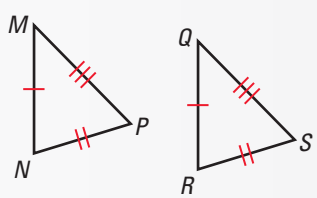
In this lesson and the next, you will learn that you do not need all six of the pieces of information above to prove that the triangles are congruent. For example, if all three pairs of corresponding sides are congruent, then the *SSS Congruence Postulate* guarantees that the triangles are congruent.

POSTULATE

POSTULATE 19 ***Side-Side-Side (SSS) Congruence Postulate***

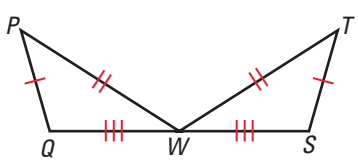
If three sides of one triangle are congruent to three sides of a second triangle, then the two triangles are congruent.

If Side $\overline{MN} \cong \overline{QR}$,
Side $\overline{NP} \cong \overline{RS}$, and
Side $\overline{PM} \cong \overline{SQ}$,
then $\triangle MNP \cong \triangle QRS$.



EXAMPLE 1 ***Using the SSS Congruence Postulate***

Prove that $\triangle PQW \cong \triangle TSW$.



Paragraph Proof The marks on the diagram show that $\overline{PQ} \cong \overline{TS}$, $\overline{PW} \cong \overline{TW}$, and $\overline{QW} \cong \overline{SW}$.

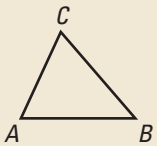
▶ So, by the SSS Congruence Postulate, you know that $\triangle PQW \cong \triangle TSW$.

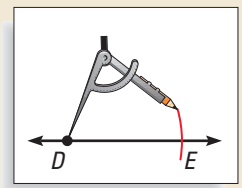
ACTIVITY

Construction

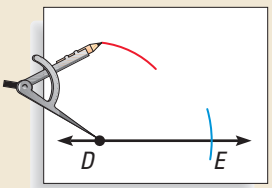
Copying a Triangle

Follow the steps below to construct a triangle that is congruent to a given $\triangle ABC$.

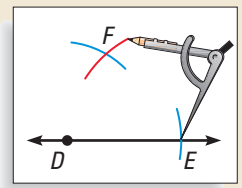




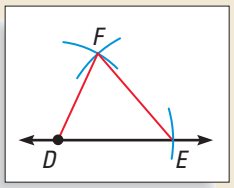
1 Construct $\overline{DE}$ so that it is congruent to $\overline{AB}$. (See page 104 for the construction.)



2 Open your compass to the length AC. Use this length to draw an arc with the compass point at D.



3 Draw an arc with radius BC and center E that intersects the arc from **Step 2**. Label the intersection point F.



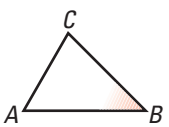
4 Draw $\triangle DEF$. By the SSS Congruence Postulate, $\triangle ABC \cong \triangle DEF$.

The SSS Congruence Postulate is a shortcut for proving two triangles are congruent without using all six pairs of corresponding parts. The postulate below is a shortcut that uses two sides and the angle that is *included* between the sides.

STUDENT HELP

Study Tip

In the triangle, $\angle B$ is the **included angle** between sides $\overline{AB}$ and $\overline{BC}$.

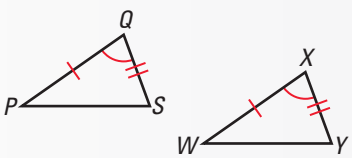


POSTULATE

POSTULATE 20 ***Side-Angle-Side (SAS) Congruence Postulate***

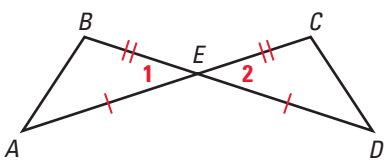
If two sides and the included angle of one triangle are congruent to two sides and the included angle of a second triangle, then the two triangles are congruent.

If Side $\overline{PQ} \cong \overline{WX}$,
Angle $\angle Q \cong \angle X$, and
Side $\overline{QS} \cong \overline{XY}$,
then $\triangle PQS \cong \triangle WXY$.



EXAMPLE 2 *Using the SAS Congruence Postulate*

Prove that $\triangle AEB \cong \triangle DEC$.



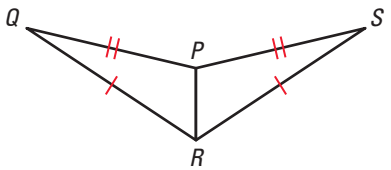
Statements	Reasons
1. $\overline{AE} \cong \overline{DE}$, $\overline{BE} \cong \overline{CE}$	**1.** Given
2. $\angle 1 \cong \angle 2$	**2.** Vertical Angles Theorem
3. $\triangle AEB \cong \triangle DEC$	**3.** SAS Congruence Postulate

GOAL 2 MODELING A REAL-LIFE SITUATION

EXAMPLE 3 *Choosing Which Congruence Postulate to Use*

Logical Reasoning

Decide whether enough information is given in the diagram to prove that $\triangle PQR \cong \triangle PSR$. If there is enough information, state the congruence postulate you would use.

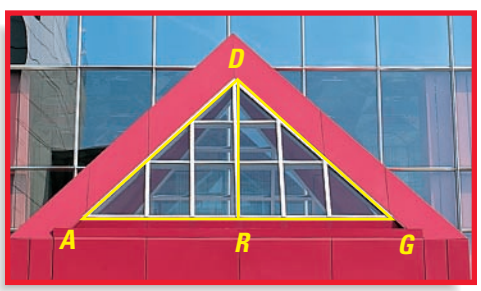


SOLUTION

Paragraph Proof The marks on the diagram show that $\overline{PQ} \cong \overline{PS}$ and $\overline{QR} \cong \overline{SR}$. By the Reflexive Property of Congruence, $\overline{RP} \cong \overline{RP}$. Because the sides of $\triangle PQR$ are congruent to the corresponding sides of $\triangle PSR$, you can use the SSS Congruence Postulate to prove that the triangles are congruent.

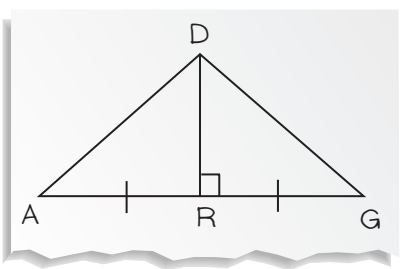
EXAMPLE 4 *Proving Triangles Congruent*

REAL LIFE **ARCHITECTURE** You are designing the window shown in the photo. You want to make $\triangle DRA$ congruent to $\triangle DRG$. You design the window so that $\overline{DR} \perp \overline{AG}$ and $\overline{RA} \cong \overline{RG}$. Can you conclude that $\triangle DRA \cong \triangle DRG$?



SOLUTION

To begin, copy the diagram and label it using the given information. Then write the given information and the statement you need to prove.



Proof

GIVEN ▶ $\overline{DR} \perp \overline{AG}$, $\overline{RA} \cong \overline{RG}$

PROVE ▶ $\triangle DRA \cong \triangle DRG$

Statements	Reasons
1. $\overline{DR} \perp \overline{AG}$	**1.** Given
2. $\angle DRA$ and $\angle DRG$ are right angles.	**2.** If 2 lines are $\perp$, then they form 4 rt. $\angle$s.
3. $\angle DRA \cong \angle DRG$	**3.** Right Angle Congruence Theorem
4. $\overline{RA} \cong \overline{RG}$	**4.** Given
5. $\overline{DR} \cong \overline{DR}$	**5.** Reflexive Property of Congruence
6. $\triangle DRA \cong \triangle DRG$	**6.** SAS Congruence Postulate

EXAMPLE 5 *Triangular Frameworks are Rigid*

STRUCTURAL SUPPORT To prevent a doorway from collapsing after an earthquake, you can reinforce it. Explain why the doorway with the diagonal brace is more stable, while the one without the brace can collapse.

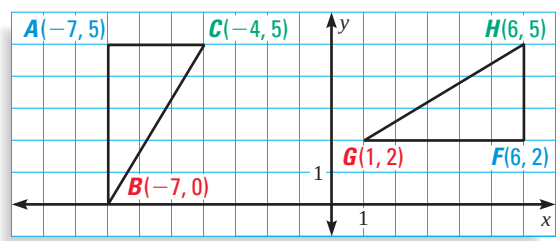
SOLUTION

In the doorway with the diagonal brace, the wood forms triangles whose sides have fixed lengths. The SSS Congruence Postulate guarantees that these triangles are rigid, because a triangle with given side lengths has only one possible size and shape. The doorway without the brace is unstable because there are many possible shapes for a four-sided figure with the given side lengths.

EXAMPLE 6 *Congruent Triangles in a Coordinate Plane*

Using Algebra

Use the SSS Congruence Postulate to show that $\triangle ABC \cong \triangle FGH$.



SOLUTION

Because $AC = 3$ and $FH = 3$, $\overline{AC} \cong \overline{FH}$. Because $AB = 5$ and $FG = 5$, $\overline{AB} \cong \overline{FG}$. Use the Distance Formula to find the lengths BC and GH.

STUDENT HELP

Look Back
For help with the Distance Formula, see page 19.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$BC = \sqrt{(-4 - (-7))^2 + (5 - 0)^2}$$

$$= \sqrt{3^2 + 5^2}$$

$$= \sqrt{34}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$GH = \sqrt{(6 - 1)^2 + (5 - 2)^2}$$

$$= \sqrt{5^2 + 3^2}$$

$$= \sqrt{34}$$

▶ Because $BC = \sqrt{34}$ and $GH = \sqrt{34}$, $\overline{BC} \cong \overline{GH}$. All three pairs of corresponding sides are congruent, so $\triangle ABC \cong \triangle FGH$ by the SSS Congruence Postulate.

GUIDED PRACTICE

Vocabulary Check ✓

1. Sketch a triangle and label its vertices. Name two sides and the included angle between the sides.

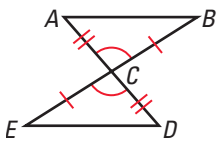
Concept Check ✓

2. **ERROR ANALYSIS** Henry believes he can use the information given in the diagram and the SAS Congruence Postulate to prove the two triangles are congruent. Explain Henry's mistake.

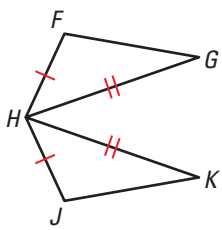
Skill Check ✓

LOGICAL REASONING Decide whether enough information is given to prove that the triangles are congruent. If there is enough information, tell which congruence postulate you would use.

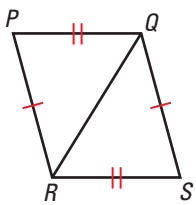
3. $\triangle ABC$, $\triangle DEC$



4. $\triangle FGH$, $\triangle JKH$



5. $\triangle PQR$, $\triangle SRQ$



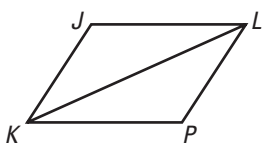
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 809.

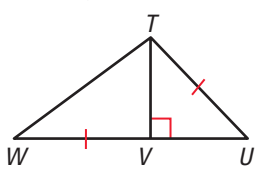
NAMING SIDES AND INCLUDED ANGLES Use the diagram. Name the included angle between the pair of sides given.

6. $\overline{JK}$ and $\overline{KL}$
7. $\overline{PK}$ and $\overline{LK}$
8. $\overline{LP}$ and $\overline{LK}$
9. $\overline{JL}$ and $\overline{JK}$
10. $\overline{KL}$ and $\overline{JL}$
11. $\overline{KP}$ and $\overline{PL}$

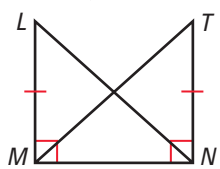


LOGICAL REASONING Decide whether enough information is given to prove that the triangles are congruent. If there is enough information, state the congruence postulate you would use.

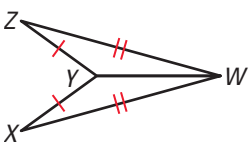
12. $\triangle UVT$, $\triangle WVT$



13. $\triangle LMN$, $\triangle TNM$



14. $\triangle YZW$, $\triangle YXW$

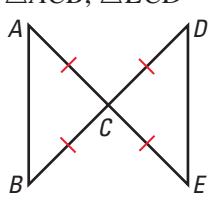


STUDENT HELP

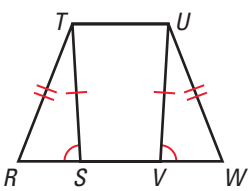
HOMEWORK HELP

Example 1: Exs. 18, 20–28
Example 2: Exs. 19–28
Example 3: Exs. 12–17
Example 4: Exs. 20–28
Example 5: Exs. 30, 31
Example 6: Exs. 33–35

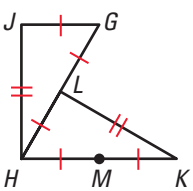
15. $\triangle ACB$, $\triangle ECD$



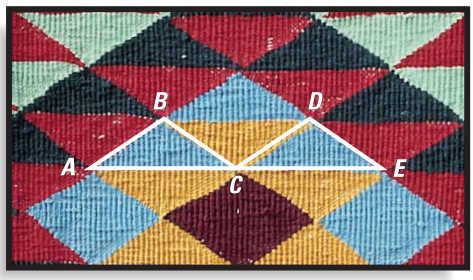
16. $\triangle RST$, $\triangle WVU$



17. $\triangle GJH$, $\triangle HLK$



DEVELOPING PROOF **In Exercises 18 and 19, use the photo of the Navajo rug. Assume that $\overline{BC} \cong \overline{DE}$ and $\overline{AC} \cong \overline{CE}$.**



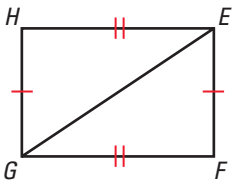
18. What other piece of information is needed to prove that $\triangle ABC \cong \triangle CDE$ using the SSS Congruence Postulate?

19. What other piece of information is needed to prove that $\triangle ABC \cong \triangle CDE$ using the SAS Congruence Postulate?

20. **DEVELOPING PROOF** Complete the proof by supplying the reasons.

GIVEN ▶ $\overline{EF} \cong \overline{GH}$, $\overline{FG} \cong \overline{HE}$

PROVE ▶ $\triangle EFG \cong \triangle GHE$

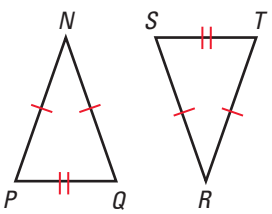


Statements	Reasons
1. $\overline{EF} \cong \overline{GH}$	1. ?
2. $\overline{FG} \cong \overline{HE}$	2. ?
3. $\overline{GE} \cong \overline{GE}$	3. ?
4. $\triangle EFG \cong \triangle GHE$	4. ?

TWO-COLUMN PROOF **Write a two-column proof.**

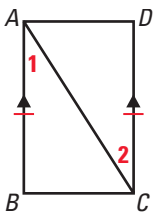
21. **GIVEN** ▶ $\overline{NP} \cong \overline{QN} \cong \overline{RS} \cong \overline{TR}$, $\overline{PQ} \cong \overline{ST}$

PROVE ▶ $\triangle NPQ \cong \triangle RST$



22. **GIVEN** ▶ $\overline{AB} \cong \overline{CD}$, $\overline{AB} \parallel \overline{CD}$

PROVE ▶ $\triangle ABC \cong \triangle CDA$



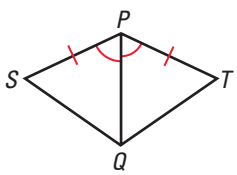
STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with paragraph proofs.

PARAGRAPH PROOF **Write a paragraph proof.**

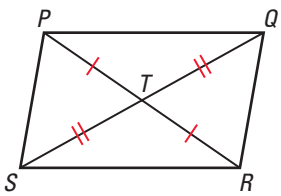
23. **GIVEN** ▶ $\overrightarrow{PQ}$ bisects $\angle SPT$, $\overline{SP} \cong \overline{TP}$

PROVE ▶ $\triangle SPQ \cong \triangle TPQ$



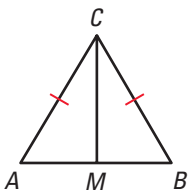
24. **GIVEN** ▶ $\overline{PT} \cong \overline{RT}$, $\overline{QT} \cong \overline{ST}$

PROVE ▶ $\triangle PQT \cong \triangle RST$

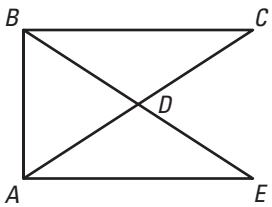


PROOF **Write a two-column proof or a paragraph proof.**

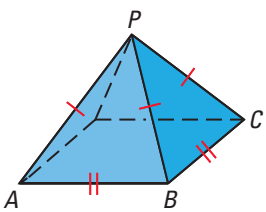
25. GIVEN ▶ $\overline{AC} \cong \overline{BC}$, M is the midpoint of $\overline{AB}$.
PROVE ▶ $\triangle ACM \cong \triangle BCM$



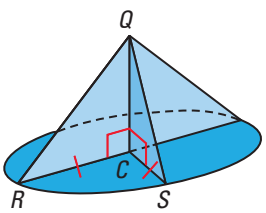
26. GIVEN ▶ $\overline{BC} \cong \overline{AE}$, $\overline{BD} \cong \overline{AD}$, $\overline{DE} \cong \overline{DC}$
PROVE ▶ $\triangle ABC \cong \triangle BAE$



27. GIVEN ▶ $\overline{PA} \cong \overline{PB} \cong \overline{PC}$, $\overline{AB} \cong \overline{BC}$
PROVE ▶ $\triangle PAB \cong \triangle PBC$



28. GIVEN ▶ $\overline{CR} \cong \overline{CS}$, $\overline{QC} \perp \overline{CR}$, $\overline{QC} \perp \overline{CS}$
PROVE ▶ $\triangle QCR \cong \triangle QCS$



STUDENT HELP

SOFTWARE HELP Visit our Web site www.mcdougallittell.com to see instructions for several software applications.

29. TECHNOLOGY Use geometry software to draw a triangle. Draw a line and reflect the triangle across the line. Measure the sides and the angles of the new triangle and tell whether it is congruent to the original one.

Writing **Explain how triangles are used in the object shown to make it more stable.**

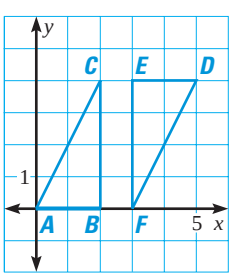
30.

31.

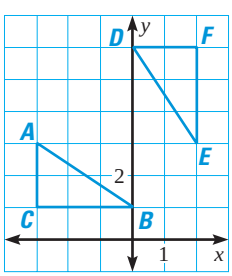
32. CONSTRUCTION Draw an isosceles triangle with vertices A, B, and C. Use a compass and straightedge to construct $\triangle DEF$ so that $\triangle DEF \cong \triangle ABC$.

USING ALGEBRA **Use the Distance Formula and the SSS Congruence Postulate to show that $\triangle ABC \cong \triangle DEF$.**

33.



34.



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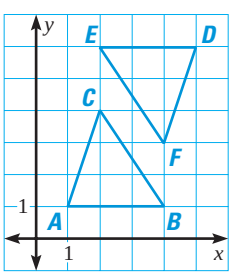


Test Preparation

36. MULTIPLE CHOICE In $\triangle RST$ and $\triangle ABC$, $\overline{RS} \cong \overline{AB}$, $\overline{ST} \cong \overline{BC}$, and $\overline{TR} \cong \overline{CA}$. Which angle is congruent to $\angle T$?

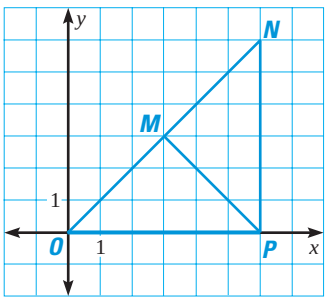
Ⓐ $\angle R$ Ⓑ $\angle A$ Ⓒ $\angle C$ Ⓓ cannot be determined

37. MULTIPLE CHOICE In equilateral $\triangle DEF$, a segment is drawn from point F to G, the midpoint of $\overline{DE}$. Which of the statements below is *not* true?

Ⓐ $\overline{DF} \cong \overline{EF}$ Ⓑ $\overline{DG} \cong \overline{DF}$ Ⓒ $\overline{DG} \cong \overline{EG}$ Ⓓ $\triangle DFG \cong \triangle EFG$

★ Challenge

38. CHOOSING A METHOD Describe how to show that $\triangle PMO \cong \triangle PMN$ using the SSS Congruence Postulate. Then find a way to show that the triangles are congruent using the SAS Congruence Postulate. You may not use a protractor to measure any angles. Compare the two methods. Which do you prefer? Why?



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

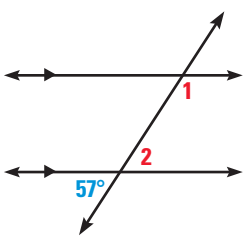
SCIENCE CONNECTION **Find an important angle in the photo. Copy the angle, extend its sides, and use a protractor to measure it to the nearest degree.** (Review 1.4)

39.

40.

USING PARALLEL LINES **Find $m\angle 1$ and $m\angle 2$. Explain your reasoning.** (Review 3.3 for 4.4)

41.



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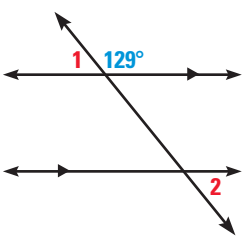

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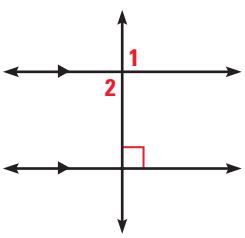
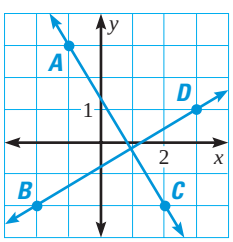


LINE RELATIONSHIPS **Find the slope of each line. Identify any parallel or perpendicular lines.** (Review 3.7)

44.



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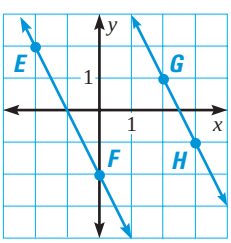

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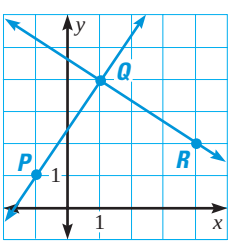


4.4 Proving Triangles are Congruent: ASA and AAS

What you should learn

GOAL 1 Prove that triangles are congruent using the ASA Congruence Postulate and the AAS Congruence Theorem.

GOAL 2 Use congruence postulates and theorems in **real-life** problems, such as taking measurements for a map in **Exs. 24 and 25**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the location of a meteorite in **Example 3**.

REAL LIFE

Lars Lindberg Christensen is an astronomer who participated in a search for a meteorite in Greenland.

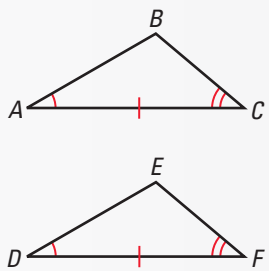
GOAL 1 USING THE ASA AND AAS CONGRUENCE METHODS

In Lesson 4.3, you studied the SSS and the SAS Congruence Postulates. Two additional ways to prove two triangles are congruent are listed below.

MORE WAYS TO PROVE TRIANGLES ARE CONGRUENT

POSTULATE 21 ***Angle-Side-Angle (ASA) Congruence Postulate***

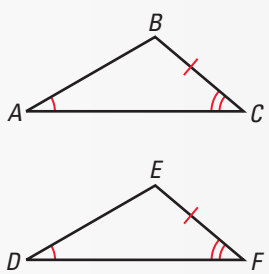
If two angles and the included side of one triangle are congruent to two angles and the included side of a second triangle, then the two triangles are congruent.



If **A**ngle $\angle A \cong \angle D$,
Side $\overline{AC} \cong \overline{DF}$, and
Angle $\angle C \cong \angle F$,
then $\triangle ABC \cong \triangle DEF$.

THEOREM 4.5 ***Angle-Angle-Side (AAS) Congruence Theorem***

If two angles and a nonincluded side of one triangle are congruent to two angles and the corresponding nonincluded side of a second triangle, then the two triangles are congruent.

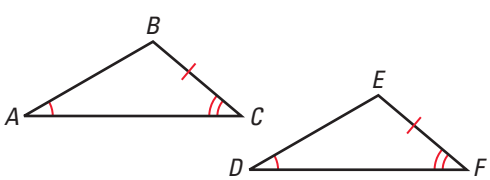


If **A**ngle $\angle A \cong \angle D$,
Angle $\angle C \cong \angle F$, and
Side $\overline{BC} \cong \overline{EF}$,
then $\triangle ABC \cong \triangle DEF$.

A proof of the Angle-Angle-Side (AAS) Congruence Theorem is given below.

GIVEN ▶ $\angle A \cong \angle D$, $\angle C \cong \angle F$, $\overline{BC} \cong \overline{EF}$

PROVE ▶ $\triangle ABC \cong \triangle DEF$



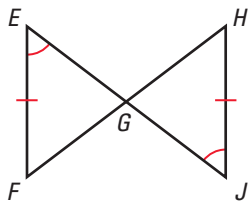
Paragraph Proof You are given that two angles of $\triangle ABC$ are congruent to two angles of $\triangle DEF$. By the Third Angles Theorem, the third angles are also congruent. That is, $\angle B \cong \angle E$. Notice that $\overline{BC}$ is the side included between $\angle B$ and $\angle C$, and $\overline{EF}$ is the side included between $\angle E$ and $\angle F$. You can apply the ASA Congruence Postulate to conclude that $\triangle ABC \cong \triangle DEF$.

Logical Reasoning

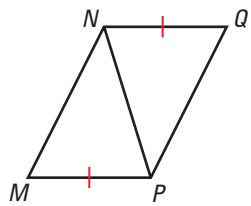
EXAMPLE 1 *Developing Proof*

Is it possible to prove that the triangles are congruent? If so, state the postulate or theorem you would use. Explain your reasoning.

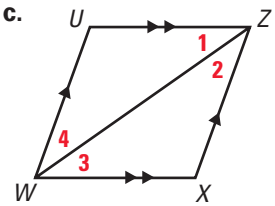
a.



b.



c.



STUDENT HELP

Study Tip
In addition to the information that is marked on a diagram, you need to consider other pairs of angles or sides that may be congruent. For instance, look for vertical angles or a side that is shared by two triangles.

SOLUTION

a. In addition to the angles and segments that are marked, $\angle EGF \cong \angle JGH$ by the Vertical Angles Theorem. Two pairs of corresponding angles and one pair of corresponding sides are congruent. You can use the AAS Congruence Theorem to prove that $\triangle EFG \cong \triangle JHG$.

b. In addition to the congruent segments that are marked, $\overline{NP} \cong \overline{NP}$. Two pairs of corresponding sides are congruent. This is not enough information to prove that the triangles are congruent.

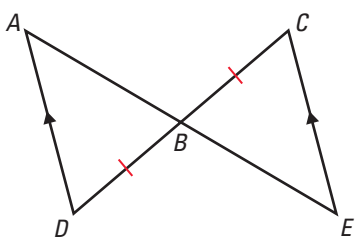
c. The two pairs of parallel sides can be used to show $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$. Because the included side $\overline{WZ}$ is congruent to itself, $\triangle WUZ \cong \triangle ZXW$ by the ASA Congruence Postulate.

Proof

EXAMPLE 2 *Proving Triangles are Congruent*

GIVEN ▶ $\overline{AD} \parallel \overline{EC}$, $\overline{BD} \cong \overline{BC}$

PROVE ▶ $\triangle ABD \cong \triangle EBC$



Plan for Proof Notice that $\angle ABD$ and $\angle EBC$ are congruent. You are given that $\overline{BD} \cong \overline{BC}$. Use the fact that $\overline{AD} \parallel \overline{EC}$ to identify a pair of congruent angles.

Statements	Reasons
1. $\overline{BD} \cong \overline{BC}$	**1.** Given
2. $\overline{AD} \parallel \overline{EC}$	**2.** Given
3. $\angle D \cong \angle C$	**3.** Alternate Interior Angles Theorem
4. $\angle ABD \cong \angle EBC$	**4.** Vertical Angles Theorem
5. $\triangle ABD \cong \triangle EBC$	**5.** ASA Congruence Postulate

. . . . . .

You can often use more than one method to prove a statement. In Example 2, you can use the parallel segments to show that $\angle D \cong \angle C$ and $\angle A \cong \angle E$. Then you can use the AAS Congruence Theorem to prove that the triangles are congruent.

GOAL 2 USING CONGRUENCE POSTULATES AND THEOREMS

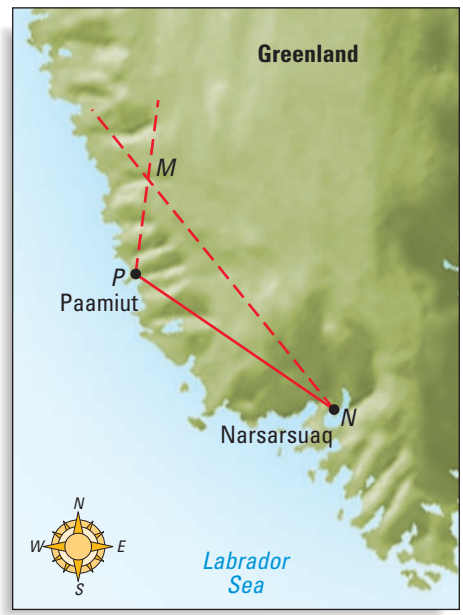
EXAMPLE 3 *Using Properties of Congruent Triangles*

FOCUS ON APPLICATIONS

REAL LIFE METEORITES When a *meteoroid* (a piece of rocky or metallic matter from space) enters Earth's atmosphere, it heats up, leaving a trail of burning gases called a *meteor*. Meteoroid fragments that reach Earth without burning up are called *meteorites*.

METEORITES On December 9, 1997, an extremely bright meteor lit up the sky above Greenland. Scientists attempted to find meteorite fragments by collecting data from eyewitnesses who had seen the meteor pass through the sky. As shown, the scientists were able to describe sightlines from observers in different towns. One sightline was from observers in Paamiut (Town P) and another was from observers in Narsarsuaq (Town N).

Assuming the sightlines were accurate, did the scientists have enough information to locate any meteorite fragments? Explain.



SOLUTION

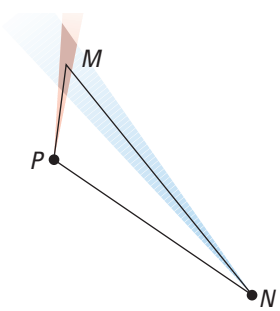
Think of Town P and Town N as two vertices of a triangle. The meteorite's position M is the other vertex. The scientists knew $m\angle P$ and $m\angle N$. They also knew the length of the included side $\overline{PN}$.

From the ASA Congruence Postulate, the scientists could conclude that any two triangles with these measurements are congruent. In other words, there is only one triangle with the given measurements and location.

▶ Assuming the sightlines were accurate, the scientists did have enough information to locate the meteorite fragments.

ACCURACY IN MEASUREMENT The conclusion in Example 3 depends on the assumption that the sightlines were accurate. If, however, the sightlines based on that information were only approximate, then the scientists could only narrow the meteorite's location to a region near point M.

For instance, if the angle measures for the sightlines were off by 2° in either direction, the meteorite's location would be known to lie within a region of about 25 square miles, which is a very large area to search.



In fact, the scientists looking for the meteorite searched over 1150 square miles of rough, icy terrain without finding any meteorite fragments.

GUIDED PRACTICE

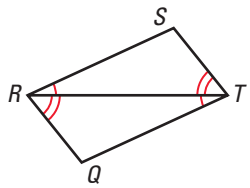
Vocabulary Check ✓

1. Name the four methods you have learned for proving triangles congruent. Only one of these is called a *theorem*. Why is it called a theorem?

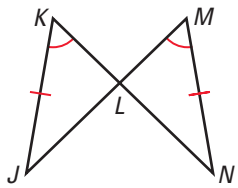
Concept Check ✓

Is it possible to prove that the triangles are congruent? If so, state the postulate or theorem you would use. Explain your reasoning.

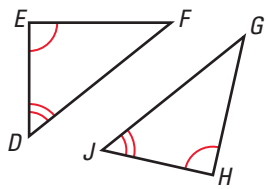
2. $\triangle RST$ and $\triangle TQR$



3. $\triangle JKL$ and $\triangle NML$



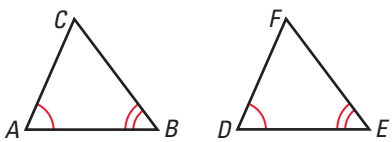
4. $\triangle DFE$ and $\triangle JGH$



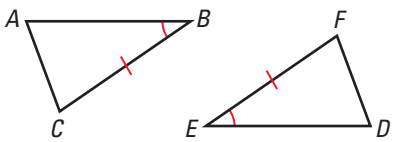
Skill Check ✓

State the third congruence that must be given to prove that $\triangle ABC \cong \triangle DEF$ using the indicated postulate or theorem.

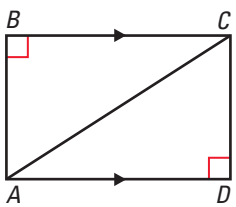
5. ASA Congruence Postulate



6. AAS Congruence Theorem



7. **RELAY RACE** A course for a relay race is marked on the gymnasium floor. Your team starts at A, goes to B, then C, then returns to A. The other team starts at C, goes to D, then A, then returns to C. Given that $\overline{AD} \parallel \overline{BC}$ and $\angle B$ and $\angle D$ are right angles, explain how you know the two courses are the same length.



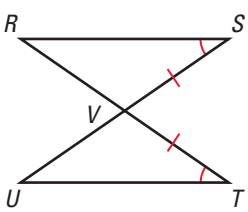
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on pp. 809 and 810.

LOGICAL REASONING **Is it possible to prove that the triangles are congruent? If so, state the postulate or theorem you would use. Explain your reasoning.**

8.



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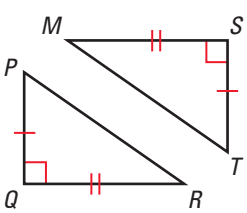

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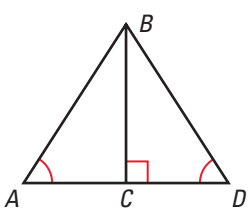
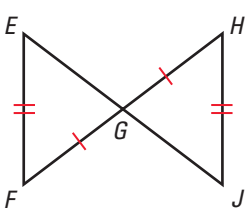


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 8–13
Example 2: Exs. 14–22
Example 3: Exs. 23–25, 28

11.



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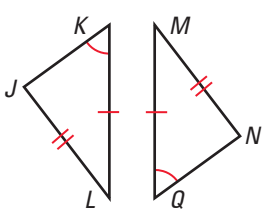

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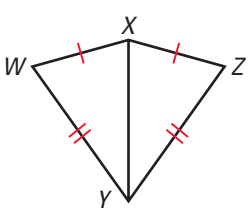


DEVELOPING PROOF **State the third congruence that must be given to prove that $\triangle PQR \cong \triangle STU$ using the indicated postulate or theorem. (*Hint:* First sketch $\triangle PQR$ and $\triangle STU$. Mark the triangles with the given information.)**

14. GIVEN ▶ $\angle Q \cong \angle T$, $\overline{PQ} \cong \overline{ST}$
Use the AAS Congruence Theorem.

15. GIVEN ▶ $\angle R \cong \angle U$, $\overline{PR} \cong \overline{SU}$
Use the ASA Congruence Postulate.

16. GIVEN ▶ $\angle R \cong \angle U$, $\angle P \cong \angle S$
Use the ASA Congruence Postulate.

17. GIVEN ▶ $\overline{PR} \cong \overline{SU}$, $\angle R \cong \angle U$
Use the SAS Congruence Postulate.

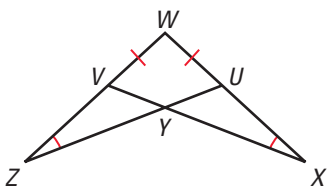
STUDENT HELP

Study Tip
When a proof involves overlapping triangles, such as the ones in Exs. 18 and 22, you may find it helpful to sketch the triangles separately.

18. DEVELOPING PROOF Complete the proof that $\triangle XWV \cong \triangle ZWU$.

GIVEN ▶ $\overline{VW} \cong \overline{UW}$
$\angle X \cong \angle Z$

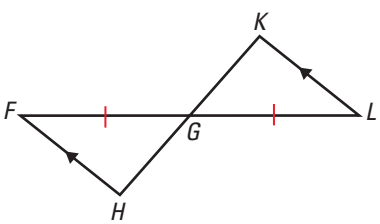
PROVE ▶ $\triangle XWV \cong \triangle ZWU$



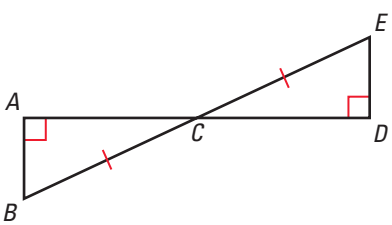
Statements	Reasons
1. $\overline{VW} \cong \overline{UW}$	1. ___?___
2. $\angle X \cong \angle Z$	2. ___?___
3. ___?___	3. Reflexive Property of Congruence
4. $\triangle XWV \cong \triangle ZWU$	4. ___?___

PROOF **Write a two-column proof or a paragraph proof.**

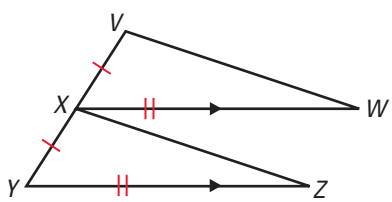
19. GIVEN ▶ $\overline{FH} \parallel \overline{LK}$, $\overline{GF} \cong \overline{GL}$
PROVE ▶ $\triangle FGH \cong \triangle LGK$



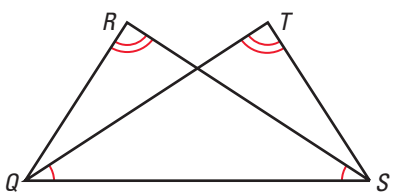
20. GIVEN ▶ $\overline{AB} \perp \overline{AD}$, $\overline{DE} \perp \overline{AD}$, $\overline{BC} \cong \overline{EC}$
PROVE ▶ $\triangle ABC \cong \triangle DEC$



21. GIVEN ▶ $\overline{VX} \cong \overline{XY}$, $\overline{XW} \cong \overline{YZ}$, $\overline{XW} \parallel \overline{YZ}$
PROVE ▶ $\triangle VXW \cong \triangle XYZ$



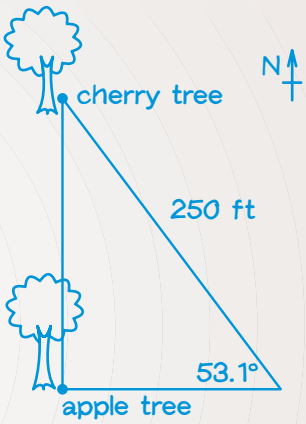
22. GIVEN ▶ $\angle TQS \cong \angle RSQ$, $\angle R \cong \angle T$
PROVE ▶ $\triangle TQS \cong \triangle RSQ$



BEARINGS **Use the information about bearings in Exercises 23–25.** In surveying and orienteering, bearings convey information about direction. For example, the bearing W 53.1° N means 53.1° to the north of west. To find this bearing, face west. Then turn 53.1° to the north.

23. You want to describe the boundary lines of a triangular piece of property to a friend. You fax the note and the sketch below to your friend. Have you provided enough information to determine the boundary lines of the property? Explain.

The southern border is a line running east from the apple tree, and the western border is the north-south line running from the cherry tree to the apple tree. The bearing from the easternmost point to the northernmost point is W 53.1° N. The distance between these points is 250 feet.



24. A surveyor wants to make a map of several streets in a village. The surveyor finds that Green Street is on an east-west line. Plain Street is at a bearing of E 55° N from its intersection with Green Street. It runs 120 yards before intersecting Ellis Avenue. Ellis Avenue runs 100 yards between Green Street and Plain Street.

Assuming these measurements are accurate, what additional measurements, if any, does the surveyor need to make to draw Ellis Avenue correctly? Explain your reasoning.

25. You are creating a map for an orienteering race. Participants start out at a large oak tree, find a boulder that is 250 yards east of the oak tree, and then find an elm tree that is W 50° N of the boulder and E 35° N of the oak tree. Use this information to sketch a map. Do you have enough information to mark the position of the elm tree? Explain.

USING ALGEBRA **Graph the equations in the same coordinate plane. Label the vertices of the two triangles formed by the lines. Show that the triangles are congruent.**

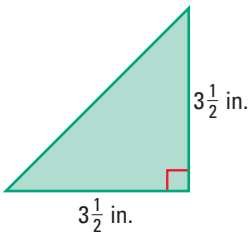
26. $y = 0$; $y = x$; $y = -x + 3$; $y = 3$

27. $y = 2$; $y = 6$; $x = 3$; $x = 5$; $y = 2x - 4$

FOCUS ON APPLICATIONS

REAL LIFE ORIENTEERING In the sport of orienteering, participants use a map and a compass to navigate a course. Along the way, they travel to various points marked on the map.

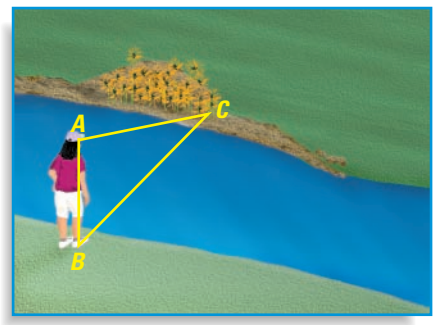
28. **QUILTING** You are making a quilt block out of congruent right triangles. Before cutting out each fabric triangle, you mark a right angle and the length of each leg, as shown. What theorem or postulate guarantees that the fabric triangles are congruent?

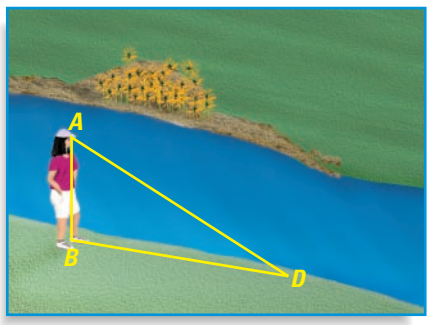


Test Preparation

29. **MULTI-STEP PROBLEM** You can use the method described below to approximate the distance across a stream without getting wet. As shown in the diagrams, you need a cap with a visor.

- Stand on the edge of the stream and look straight across to a point on the other edge of the stream. Adjust the visor of your cap so that it is in line with that point.
- Without changing the inclination of your neck and head, turn sideways until the visor is in line with a point on your side of the stream.
- Measure the distance BD between your feet and that point.





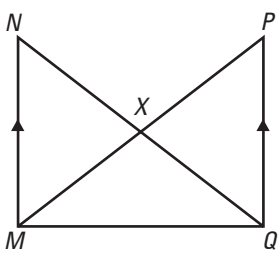
a. From the description of the measuring method, what corresponding parts of the two triangles can you assume are congruent?

b. What theorem or postulate can be used to show that the two triangles are congruent?

c. *Writing* Explain why the length of $\overline{BD}$ is also the distance across the stream.

Challenge

PROOF **Use the diagram.**



30. Alicia thinks that she can prove that $\triangle MNQ \cong \triangle QPM$ based on the information in the diagram. Explain why she cannot.

31. Suppose you are given that $\angle XMQ \cong \angle XQM$ and that $\angle N \cong \angle P$. Prove that $\triangle MNQ \cong \triangle QPM$.

EXTRA CHALLENGE
www.mcdougallittell.com

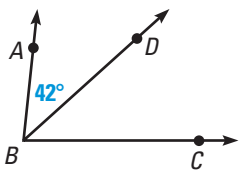
MIXED REVIEW

FINDING ENDPOINTS **Find the coordinates of the other endpoint of a segment with the given endpoint and midpoint *M*.** (Review 1.5)

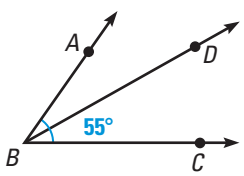
32. $B(5, 7)$, $M(-1, 0)$ **33.** $C(0, 9)$, $M(6, -2)$ **34.** $F(8, -5)$, $M(-1, -3)$

USING ANGLE BISECTORS **$\overrightarrow{BD}$ is the angle bisector of $\angle ABC$. Find the two angle measures not given in the diagram.** (Review 1.5 for 4.5)

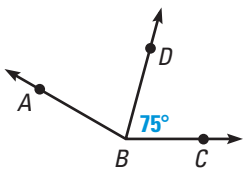
35.



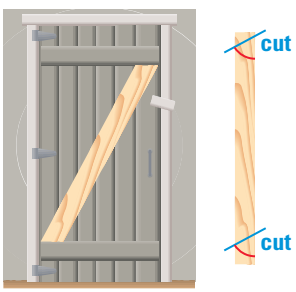
36.



37.



38. **BARN DOOR** You are making a brace for a barn door, as shown. The top and bottom pieces are parallel. To make the middle piece, you cut off the ends of a board at the same angle. What postulate or theorem guarantees that the cuts are parallel? (Review 3.4)

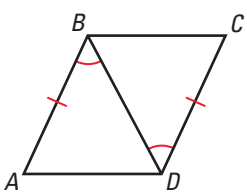


QUIZ 2

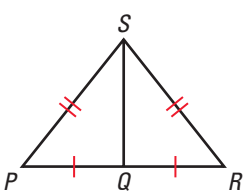
Self-Test for Lessons 4.3 and 4.4

In Exercises 1–6, decide whether it is possible to prove that the triangles are congruent. If it is possible, state the theorem or postulate you would use. Explain your reasoning. (Lessons 4.3 and 4.4)

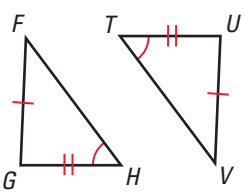
1.



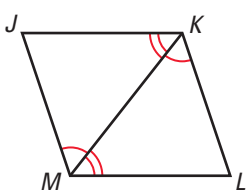
2.



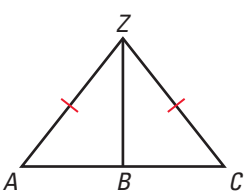
3.



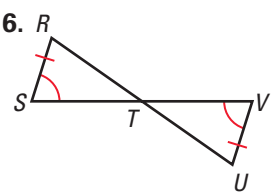
4.



5.



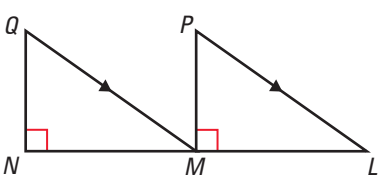
6.



7. **PROOF** Write a two-column proof. (Lesson 4.4)

GIVEN ▶ M is the midpoint of $\overline{NL}$, $\overline{NL} \perp \overline{NQ}$, $\overline{NL} \perp \overline{MP}$, $\overline{QM} \parallel \overline{PL}$

PROVE ▶ $\triangle NQM \cong \triangle MPL$



4.5 Using Congruent Triangles

What you should learn

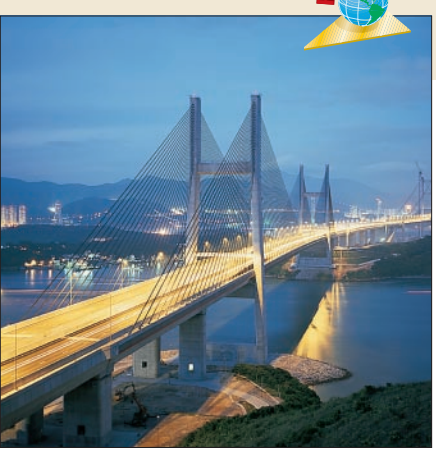
GOAL 1 Use congruent triangles to plan and write proofs.

GOAL 2 Use congruent triangles to prove constructions are valid.

Why you should learn it

▼ Congruent triangles are important in **real-life** problems, such as in designing and constructing bridges like the one in **Ex. 16**.

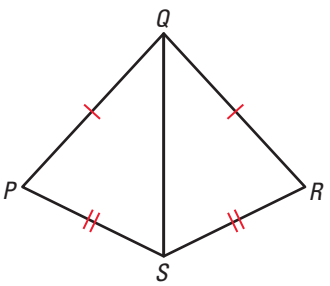
REAL LIFE



GOAL 1 PLANNING A PROOF

Knowing that all pairs of corresponding parts of congruent triangles are congruent can help you reach conclusions about congruent figures.

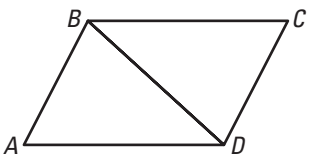
For instance, suppose you want to prove that $\angle PQS \cong \angle RQS$ in the diagram shown at the right. One way to do this is to show that $\triangle PQS \cong \triangle RQS$ by the SSS Congruence Postulate. Then you can use the fact that corresponding parts of congruent triangles are congruent to conclude that $\angle PQS \cong \angle RQS$.



EXAMPLE 1 *Planning and Writing a Proof*

GIVEN ▶ $\overline{AB} \parallel \overline{CD}$, $\overline{BC} \parallel \overline{DA}$

PROVE ▶ $\overline{AB} \cong \overline{CD}$

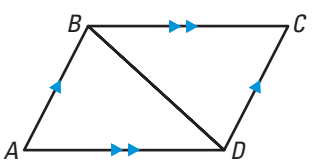


Plan for Proof Show that $\triangle ABD \cong \triangle CDB$. Then use the fact that corresponding parts of congruent triangles are congruent.

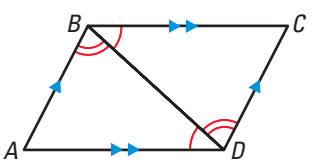
SOLUTION

First copy the diagram and mark it with the given information. Then mark any additional information that you can deduce. Because $\overline{AB}$ and $\overline{CD}$ are parallel segments intersected by a transversal, and $\overline{BC}$ and $\overline{DA}$ are parallel segments intersected by a transversal, you can deduce that two pairs of alternate interior angles are congruent.

Mark given information.



Add deduced information.



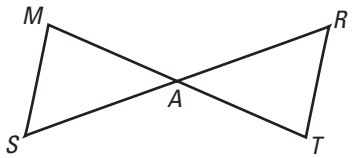
Paragraph Proof Because $\overline{AB} \parallel \overline{CD}$, it follows from the Alternate Interior Angles Theorem that $\angle ABD \cong \angle CDB$. For the same reason, $\angle ADB \cong \angle CBD$ because $\overline{BC} \parallel \overline{DA}$. By the Reflexive Property of Congruence, $\overline{BD} \cong \overline{BD}$. You can use the ASA Congruence Postulate to conclude that $\triangle ABD \cong \triangle CDB$. Finally, because corresponding parts of congruent triangles are congruent, it follows that $\overline{AB} \cong \overline{CD}$.

EXAMPLE 2 *Planning and Writing a Proof*

Proof

GIVEN ▶ A is the midpoint of $\overline{MT}$,
A is the midpoint of $\overline{SR}$.

PROVE ▶ $\overline{MS} \parallel \overline{TR}$



Plan for Proof Prove that $\triangle MAS \cong \triangle TAR$. Then use the fact that corresponding parts of congruent triangles are congruent to show that $\angle M \cong \angle T$. Because these angles are formed by two segments intersected by a transversal, you can conclude that $\overline{MS} \parallel \overline{TR}$.

STUDENT HELP

INTERNET HOMEWORK HELP

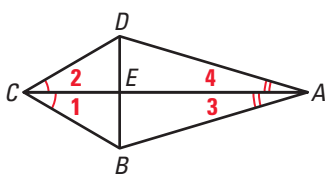
Visit our Web site at www.mcdougallittell.com for extra examples.

Statements	Reasons
1. A is the midpoint of $\overline{MT}$, A is the midpoint of $\overline{SR}$.	**1.** Given
2. $\overline{MA} \cong \overline{TA}$, $\overline{SA} \cong \overline{RA}$	**2.** Definition of midpoint
3. $\angle MAS \cong \angle TAR$	**3.** Vertical Angles Theorem
4. $\triangle MAS \cong \triangle TAR$	**4.** SAS Congruence Postulate
5. $\angle M \cong \angle T$	**5.** Corresp. parts of $\cong$ $\triangle$s are $\cong$.
6. $\overline{MS} \parallel \overline{TR}$	**6.** Alternate Interior Angles Converse

EXAMPLE 3 *Using More than One Pair of Triangles*

GIVEN ▶ $\angle 1 \cong \angle 2$
$\angle 3 \cong \angle 4$

PROVE ▶ $\triangle BCE \cong \triangle DCE$



Plan for Proof The only information you have about $\triangle BCE$ and $\triangle DCE$ is that $\angle 1 \cong \angle 2$ and that $\overline{CE} \cong \overline{CE}$. Notice, however, that sides $\overline{BC}$ and $\overline{DC}$ are also sides of $\triangle ABC$ and $\triangle ADC$. If you can prove that $\triangle ABC \cong \triangle ADC$, you can use the fact that corresponding parts of congruent triangles are congruent to get a third piece of information about $\triangle BCE$ and $\triangle DCE$.

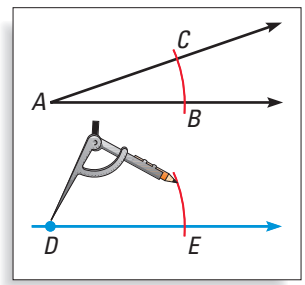
Statements	Reasons
1. $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$	**1.** Given
2. $\overline{AC} \cong \overline{AC}$	**2.** Reflexive Property of Congruence
3. $\triangle ABC \cong \triangle ADC$	**3.** ASA Congruence Postulate
4. $\overline{BC} \cong \overline{DC}$	**4.** Corresp. parts of $\cong$ $\triangle$s are $\cong$.
5. $\overline{CE} \cong \overline{CE}$	**5.** Reflexive Property of Congruence
6. $\triangle BCE \cong \triangle DCE$	**6.** SAS Congruence Postulate

GOAL 2 PROVING CONSTRUCTIONS ARE VALID

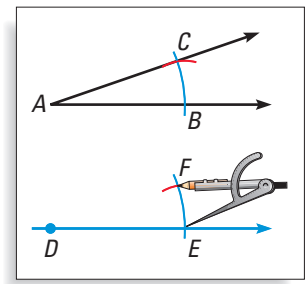
STUDENT HELP

Look Back
For help with copying an angle, see p. 159.

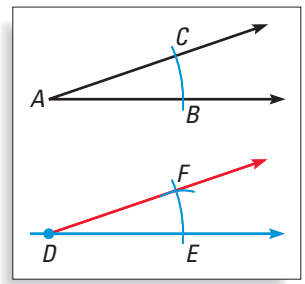
In Lesson 3.5, you learned how to copy an angle using a compass and a straightedge. The construction is summarized below. You can use congruent triangles to prove that this (and other) constructions are valid.



1 To copy $\angle A$, first draw a ray with initial point D. Then use the same compass setting to draw an arc with center A and an arc with center D. Label points B, C, and E.



2 Draw an arc with radius BC and center E. Label the intersection F.

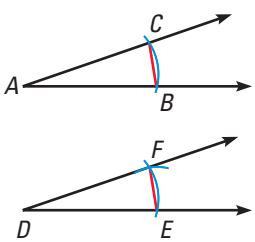


3 Draw $\overrightarrow{DF}$. $\angle FDE \cong \angle CAB$

EXAMPLE 4 *Proving a Construction*

Proof

Using the construction summarized above, you can copy $\angle CAB$ to form $\angle FDE$. Write a proof to verify that the construction is valid.



Plan for Proof Show that $\triangle CAB \cong \triangle FDE$. Then use the fact that corresponding parts of congruent triangles are congruent to conclude that $\angle CAB \cong \angle FDE$. By construction, you can assume the following statements as given.

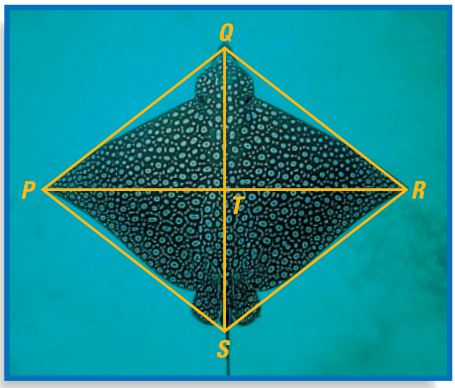
$\overline{AB} \cong \overline{DE}$ **Same compass setting is used.**
$\overline{AC} \cong \overline{DF}$ **Same compass setting is used.**
$\overline{BC} \cong \overline{EF}$ **Same compass setting is used.**

SOLUTION

Statements	Reasons
1. $\overline{AB} \cong \overline{DE}$	**1.** Given
2. $\overline{AC} \cong \overline{DF}$	**2.** Given
3. $\overline{BC} \cong \overline{EF}$	**3.** Given
4. $\triangle CAB \cong \triangle FDE$	**4.** SSS Congruence Postulate
5. $\angle CAB \cong \angle FDE$	**5.** Corresp. parts of $\cong$ $\triangle$s are $\cong$.

GUIDED PRACTICE

Concept Check ✓ **In Exercises 1–3, use the photo of the eagle ray.**



1. To prove that $\angle PQT \cong \angle RQT$, which triangles might you prove to be congruent?

2. If you know that the opposite sides of figure $PQRS$ are parallel, can you prove that $\triangle PQT \cong \triangle RST$? Explain.

Skill Check ✓

3. The statements listed below are not in order. Use the photo to order them as statements in a two-column proof. Write a reason for each statement.

GIVEN ▶ $\overline{QS} \perp \overline{RP}$, $\overline{PT} \cong \overline{RT}$

PROVE ▶ $\overline{PS} \cong \overline{RS}$

A. $\overline{QS} \perp \overline{RP}$ **B.** $\triangle PTS \cong \triangle RTS$ **C.** $\angle PTS \cong \angle RTS$

D. $\overline{PS} \cong \overline{RS}$ **E.** $\overline{PT} \cong \overline{RT}$ **F.** $\overline{TS} \cong \overline{TS}$

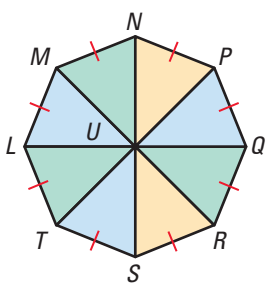
G. $\angle PTS$ and $\angle RTS$ are right angles.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 810.

STAINED GLASS WINDOW **The eight window panes in the diagram are isosceles triangles. The bases of the eight triangles are congruent.**



4. Explain how you know that $\triangle NUP \cong \triangle PUQ$.

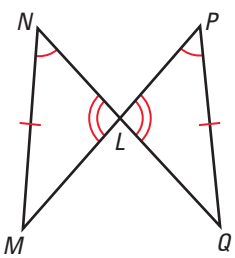
5. Explain how you know that $\triangle NUP \cong \triangle QUR$.

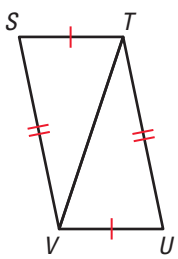
6. Do you have enough information to prove that all the triangles are congruent? Explain.

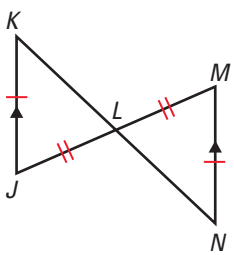
7. Explain how you know that $\angle UNP \cong \angle UPQ$.

DEVELOPING PROOF **State which postulate or theorem you can use to prove that the triangles are congruent. Then explain how proving that the triangles are congruent proves the given statement.**

8. PROVE ▶ $\overline{ML} \cong \overline{QL}$ **9. PROVE** ▶ $\angle STV \cong \angle UVT$ **10. PROVE** ▶ $KL = NL$





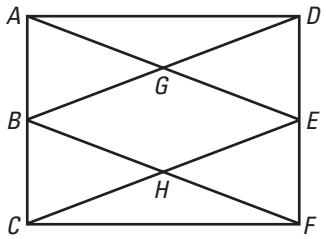


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 4–14, 17, 18
Example 2: Exs. 14, 17, 18
Example 3: Exs. 15, 16
Example 4: Exs. 19–21

CAT'S CRADLE **Use the diagram of the string game Cat's Cradle and the information given below.**



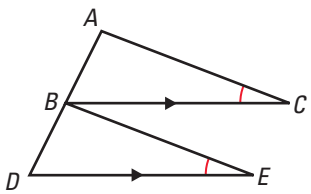
GIVEN ▶ $\triangle EDA \cong \triangle BCF$
$\triangle AGD \cong \triangle FHC$
$\triangle BFC \cong \triangle ECF$

11. PROVE ▶ $\overline{GD} \cong \overline{HC}$

12. PROVE ▶ $\angle CBH \cong \angle FEH$

13. PROVE ▶ $\overline{AE} \cong \overline{FB}$

14. **DEVELOPING PROOF** Complete the proof that $\angle BAC \cong \angle DBE$.

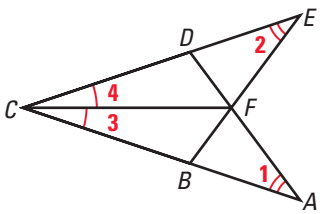


GIVEN ▶ B is the midpoint of $\overline{AD}$, $\angle C \cong \angle E$, $\overline{BC} \parallel \overline{DE}$

PROVE ▶ $\angle BAC \cong \angle DBE$

Statements	Reasons
1. B is the midpoint of $\overline{AD}$.	**1.** Given
2. $\overline{AB} \cong \overline{BD}$	**2.** ___?___
3. $\angle C \cong \angle E$	**3.** Given
4. $\overline{BC} \parallel \overline{DE}$	**4.** Given
5. $\angle EDB \cong \angle CBA$	**5.** ___?___
6. ___?___	**6.** AAS Congruence Theorem
7. $\angle BAC \cong \angle DBE$	**7.** ___?___

15. **DEVELOPING PROOF** Complete the proof that $\triangle AFB \cong \triangle EFD$.



GIVEN ▶ $\angle 1 \cong \angle 2$
$\angle 3 \cong \angle 4$

PROVE ▶ $\angle AFB \cong \angle EFD$

Statements	Reasons
1. $\angle 1 \cong \angle 2$	**1.** ___?___
2. $\angle 3 \cong \angle 4$	**2.** ___?___
3. ___?___	**3.** Reflexive Property of Congruence
4. $\triangle AFC \cong \triangle EFC$	**4.** ___?___
5. $\overline{AF} \cong \overline{EF}$	**5.** ___?___
6. ___?___	**6.** Vertical Angles Theorem
7. $\triangle AFB \cong \triangle EFD$	**7.** ___?___

FOCUS ON CAREER

CONSTRUCTION MANAGER
A construction manager plans and directs the work at a building site. Among other things, the manager reviews engineering specifications and architectural drawings to make sure that a project is proceeding according to plan.

CAREER LINK
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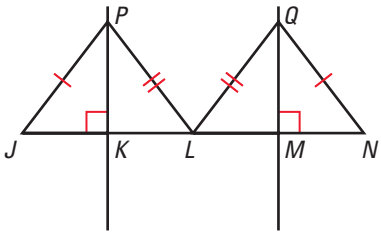
16. **BRIDGES** The diagram represents a section of the framework of the Kap Shui Mun Bridge shown in the photo on page 229. Write a two-column proof to show that $\triangle PKJ \cong \triangle QMN$.

GIVEN ▶ L is the midpoint of $\overline{JN}$, $\overline{PJ} \cong \overline{QN}$, $\overline{PL} \cong \overline{QL}$, $\angle PKJ$ and $\angle QMN$ are right angles.

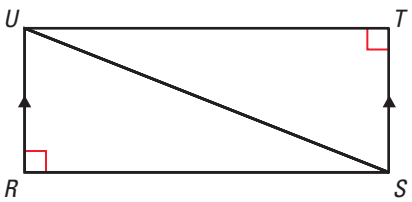
PROVE ▶ $\triangle PKJ \cong \triangle QMN$



PROOF **Write a two-column proof or a paragraph proof.**

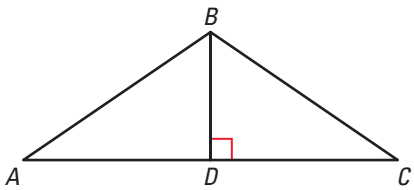
17. **GIVEN** ▶ $\overline{UR} \parallel \overline{ST}$, $\angle R$ and $\angle T$ are right angles.

PROVE ▶ $\angle RSU \cong \angle TUS$

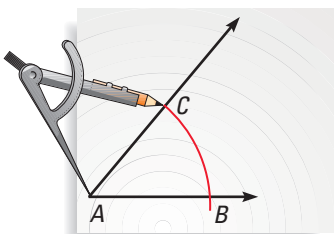


18. **GIVE**N ▶ $\overline{BD} \perp \overline{AC}$, $\overline{BD}$ bisects $\overline{AC}$.

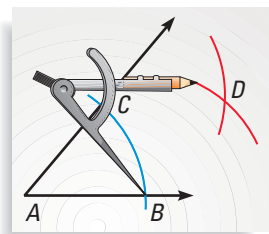
PROVE ▶ $\angle ABD$ and $\angle BCD$ are complementary angles.



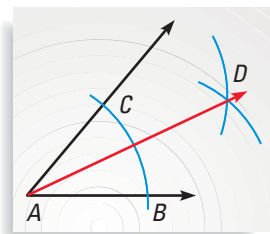
19. **PROVING A CONSTRUCTION** The diagrams below summarize the construction used to bisect $\angle A$. By construction, you can assume that $\overline{AB} \cong \overline{AC}$ and $\overline{BD} \cong \overline{CD}$. Write a proof to verify that $\overrightarrow{AD}$ bisects $\angle A$.



1 First draw an arc with center A. Label the points where the arc intersects the sides of the angle points B and C.



2 Draw an arc with center C. Using the same compass setting, draw an arc with center B. Label the intersection point D.



3 Draw $\overrightarrow{AD}$. $\angle CAD \cong \angle BAD$

PROVING A CONSTRUCTION **Use a straightedge and a compass to perform the construction. Label the important points of your construction. Then write a flow proof to verify the results.**

20. Bisect an obtuse angle.

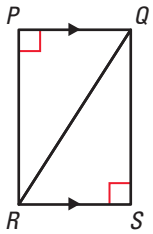
21. Copy an obtuse angle.

STUDENT HELP

Look Back
For help with bisecting an angle, see p. 36.

Test Preparation

22. MULTIPLE CHOICE Suppose $\overline{PQ} \parallel \overline{RS}$. You want to prove that $\overline{PR} \cong \overline{SQ}$. Which of the reasons below would *not* appear in your two-column proof?



Ⓐ SAS Congruence Postulate

Ⓑ Reflexive Property of Congruence

Ⓒ AAS Congruence Theorem

Ⓓ Right Angle Congruence Theorem

Ⓔ Alternate Interior Angles Theorem

23. MULTIPLE CHOICE Which statement correctly describes the congruence of the triangles in the diagram in Exercise 22?

Ⓐ $\triangle SRQ \cong \triangle RQP$ Ⓑ $\triangle PRQ \cong \triangle SRQ$

Ⓒ $\triangle QRS \cong \triangle PQR$ Ⓓ $\triangle SRQ \cong \triangle PQR$

★ Challenge

EXTRA CHALLENGE
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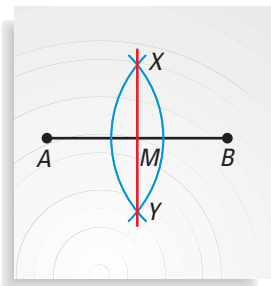
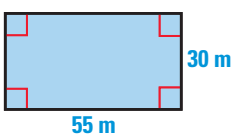
24. PROVING A CONSTRUCTION Use a straightedge and a compass to bisect a segment. (For help with this construction, look back at page 34.) Then write a proof to show that the construction is valid.



MIXED REVIEW

FINDING PERIMETER, CIRCUMFERENCE, AND AREA **Find the perimeter (or circumference) and area of the figure. (Where necessary, use $\pi \approx 3.14$.)** (Review 1.7)

25.



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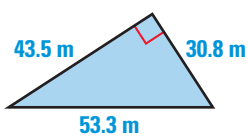
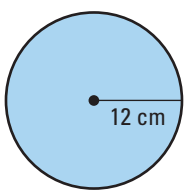


27.



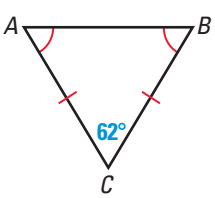
SOLVING EQUATIONS **Solve the equation and state a reason for each step.** (Review 2.4)

28. $x - 2 = 10$ **29.** $x + 11 = 21$ **30.** $9x + 2 = 29$

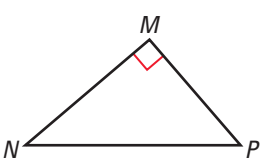
31. $8x + 13 = 3x + 38$ **32.** $3(x - 1) = 16$ **33.** $6(2x - 1) + 15 = 69$

IDENTIFYING PARTS OF TRIANGLES **Classify the triangle by its angles and by its sides. Identify the legs and the hypotenuse of any right triangles. Identify the legs and the base of any isosceles triangles.** (Review 4.1 for 4.6)

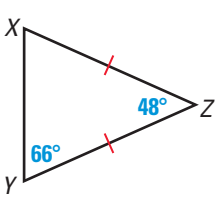
34.



35.



36.



4.6 Isosceles, Equilateral, and Right Triangles

What you should learn

GOAL 1 Use properties of isosceles and equilateral triangles.

GOAL 2 Use properties of right triangles.

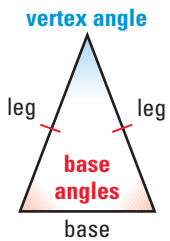
Why you should learn it

▼ Isosceles, equilateral, and right triangles are commonly used in the design of **real-life** objects, such as the exterior structure of the building in **Exs. 29–32**.

REAL LIFE

GOAL 1 USING PROPERTIES OF ISOSCELES TRIANGLES

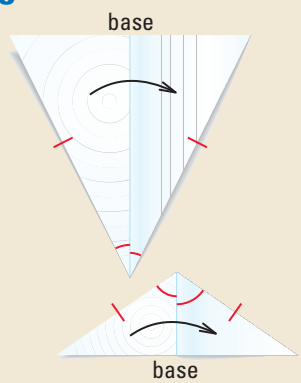
In Lesson 4.1, you learned that a triangle is isosceles if it has at least two congruent sides. If it has exactly two congruent sides, then they are the legs of the triangle and the noncongruent side is the base. The two angles adjacent to the base are the **base angles**. The angle opposite the base is the **vertex angle**.



ACTIVITY Developing Concepts

Investigating Isosceles Triangles

1. Use a straightedge and a compass to construct an acute isosceles triangle. Then fold the triangle along a line that bisects the vertex angle, as shown.
2. Repeat the procedure for an obtuse isosceles triangle.
3. What observations can you make about the base angles of an isosceles triangle? Write your observations as a conjecture.



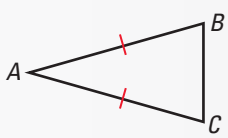
In the activity, you may have discovered the *Base Angles Theorem*, which is proved in Example 1. The converse of this theorem is also true. You are asked to prove the converse in Exercise 26.

THEOREMS

THEOREM 4.6 ***Base Angles Theorem***

If two sides of a triangle are congruent, then the angles opposite them are congruent.

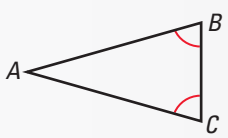
If $\overline{AB} \cong \overline{AC}$, then $\angle B \cong \angle C$.



THEOREM 4.7 ***Converse of the Base Angles Theorem***

If two angles of a triangle are congruent, then the sides opposite them are congruent.

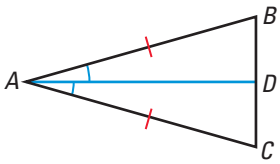
If $\angle B \cong \angle C$, then $\overline{AB} \cong \overline{AC}$.



EXAMPLE 1 *Proof of the Base Angles Theorem*

Proof

Use the diagram of $\triangle ABC$ to prove the Base Angles Theorem.



GIVEN ▶ $\triangle ABC$, $\overline{AB} \cong \overline{AC}$

PROVE ▶ $\angle B \cong \angle C$

Paragraph Proof Draw the bisector of $\angle CAB$. By construction, $\angle CAD \cong \angle BAD$. You are given that $\overline{AB} \cong \overline{AC}$. Also, $\overline{DA} \cong \overline{DA}$ by the Reflexive Property of Congruence. Use the SAS Congruence Postulate to conclude that $\triangle ADB \cong \triangle ADC$. Because corresponding parts of congruent triangles are congruent, it follows that $\angle B \cong \angle C$.

. . . . . .

Recall that an *equilateral* triangle is a special type of isosceles triangle. The corollaries below state that a triangle is equilateral if and only if it is equiangular.

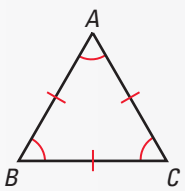
COROLLARIES

COROLLARY TO THEOREM 4.6

If a triangle is equilateral, then it is equiangular.

COROLLARY TO THEOREM 4.7

If a triangle is equiangular, then it is equilateral.

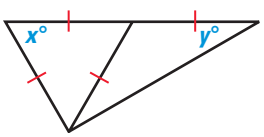


EXAMPLE 2 *Using Equilateral and Isosceles Triangles*

Using Algebra

a. Find the value of x.

b. Find the value of y.



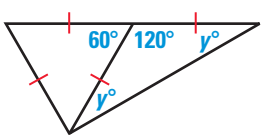
SOLUTION

a. Notice that x represents the measure of an angle of an equilateral triangle. From the corollary above, this triangle is also equiangular.

$3x° = 180°$ **Apply the Triangle Sum Theorem.**

$x = 60$ **Solve for *x*.**

b. Notice that y represents the measure of a base angle of an isosceles triangle. From the Base Angles Theorem, the other base angle has the same measure. The vertex angle forms a linear pair with a 60° angle, so its measure is 120°.



$120° + 2y° = 180°$ **Apply the Triangle Sum Theorem.**

$y = 30$ **Solve for *y*.**

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 USING PROPERTIES OF RIGHT TRIANGLES

You have learned four ways to prove that triangles are congruent.

- Side-Side-Side (SSS) Congruence Postulate (p. 212)
- Side-Angle-Side (SAS) Congruence Postulate (p. 213)
- Angle-Side-Angle (ASA) Congruence Postulate (p. 220)
- Angle-Angle-Side (AAS) Congruence Theorem (p. 220)

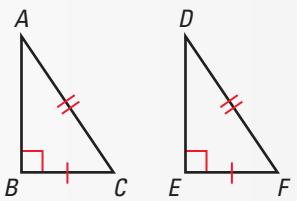
The Hypotenuse-Leg Congruence Theorem below can be used to prove that two *right* triangles are congruent. A proof of this theorem appears on page 837.

THEOREM

THEOREM 4.8 ***Hypotenuse-Leg (HL) Congruence Theorem***

If the hypotenuse and a leg of a right triangle are congruent to the hypotenuse and a leg of a second right triangle, then the two triangles are congruent.

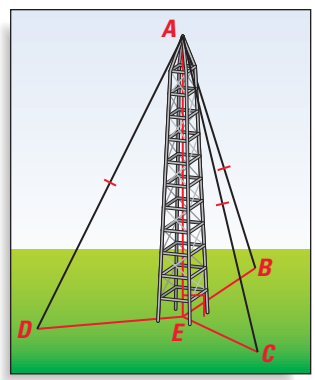
If $\overline{BC} \cong \overline{EF}$ and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.



EXAMPLE 3 *Proving Right Triangles Congruent*

Proof

The television antenna is perpendicular to the plane containing the points B, C, D, and E. Each of the stays running from the top of the antenna to B, C, and D uses the same length of cable. Prove that $\triangle AEB$, $\triangle AEC$, and $\triangle AED$ are congruent.



GIVEN ▶ $\overline{AE} \perp \overline{EB}$, $\overline{AE} \perp \overline{EC}$, $\overline{AE} \perp \overline{ED}$, $\overline{AB} \cong \overline{AC} \cong \overline{AD}$

PROVE ▶ $\triangle AEB \cong \triangle AEC \cong \triangle AED$

SOLUTION

Paragraph Proof You are given that $\overline{AE} \perp \overline{EB}$ and $\overline{AE} \perp \overline{EC}$, which implies that $\angle AEB$ and $\angle AEC$ are right angles. By definition, $\triangle AEB$ and $\triangle AEC$ are right triangles. You are given that the hypotenuses of these two triangles, $\overline{AB}$ and $\overline{AC}$, are congruent. Also, $\overline{AE}$ is a leg for both triangles, and $\overline{AE} \cong \overline{AE}$ by the Reflexive Property of Congruence. Thus, by the Hypotenuse-Leg Congruence Theorem, $\triangle AEB \cong \triangle AEC$.

▶ Similar reasoning can be used to prove that $\triangle AEC \cong \triangle AED$. So, by the Transitive Property of Congruent Triangles, $\triangle AEB \cong \triangle AEC \cong \triangle AED$.

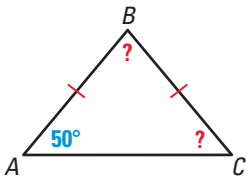
STUDENT HELP

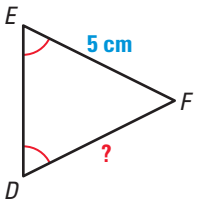
Study Tip
Before you use the HL Congruence Theorem in a proof, you need to prove that the triangles are right triangles.

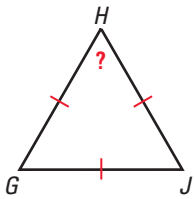
GUIDED PRACTICE

Vocabulary Check ✓ 1. Describe the meaning of *equilateral* and *equiangular*.

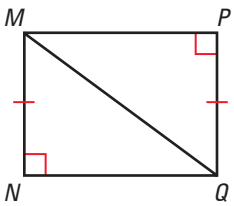
Concept Check ✓ **Find the unknown measure(s). Tell what theorems you used.**

2.


3.


4.


Skill Check ✓ **Determine whether you are given enough information to prove that the triangles are congruent. Explain your answer.**

5.


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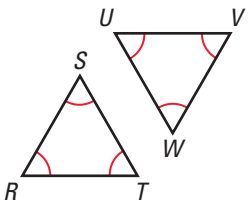

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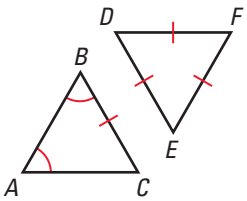
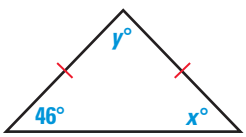


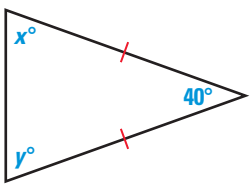
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 810.

USING ALGEBRA **Solve for *x* and *y*.**

8.


9.


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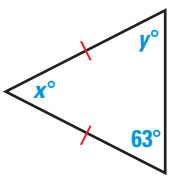
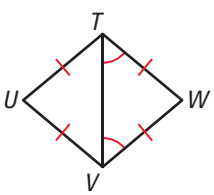


LOGICAL REASONING **Decide whether enough information is given to prove that the triangles are congruent. Explain your answer.**

11.


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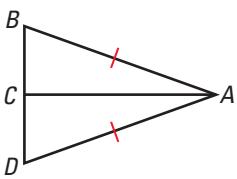
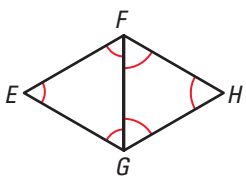
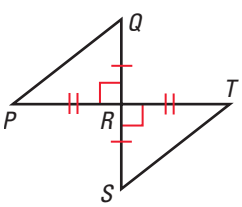


13.


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 26–28
Example 2: Exs. 8–10, 17–25
Example 3: Exs. 31, 33, 34, 39

14.


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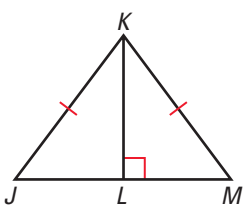

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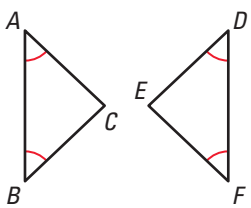
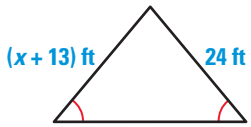


USING ALGEBRA **Find the value of *x*.**

17.



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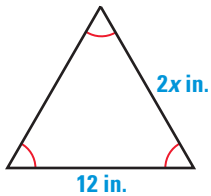

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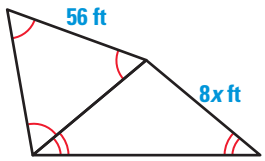
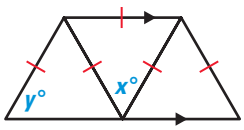


USING ALGEBRA **Find the values of *x* and *y*.**

20.



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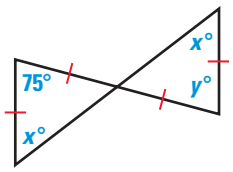

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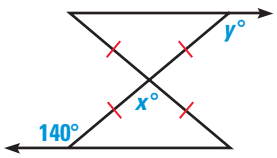

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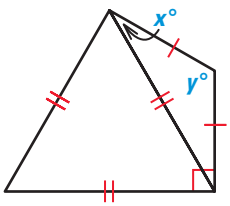

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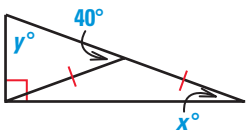

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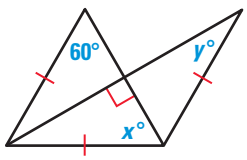


PROOF **In Exercises 26–28, use the diagrams that accompany the theorems on pages 236 and 237.**

26. The Converse of the Base Angles Theorem on page 236 states, "If two angles of a triangle are congruent, then the sides opposite them are congruent." Write a proof of this theorem.

27. The Corollary to Theorem 4.6 on page 237 states, "If a triangle is equilateral, then it is equiangular." Write a proof of this corollary.

28. The Corollary to Theorem 4.7 on page 237 states, "If a triangle is equiangular, then it is equilateral." Write a proof of this corollary.

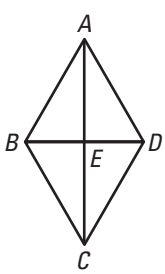
ARCHITECTURE **The diagram represents part of the exterior of the building in the photograph. In the diagram, $\triangle ABD$ and $\triangle CBD$ are congruent equilateral triangles.**

29. Explain why $\triangle ABC$ is isosceles.

30. Explain why $\angle BAE \cong \angle BCE$.

31. **PROOF** Prove that $\triangle ABE$ and $\triangle CBE$ are congruent right triangles.

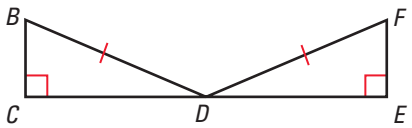
32. Find the measure of $\angle BAE$.



PROOF **Write a two-column proof or a paragraph proof.**

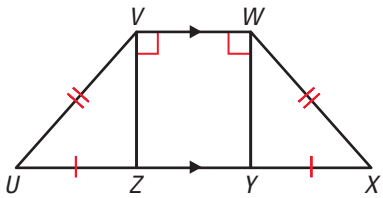
33. GIVEN ▶ D is the midpoint of $\overline{CE}$, $\angle BCD$ and $\angle FED$ are right angles, and $\overline{BD} \cong \overline{FD}$.

PROVE ▶ $\triangle BCD \cong \triangle FED$

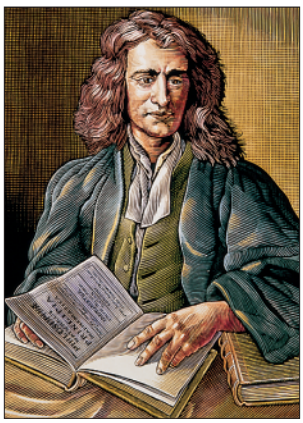


34. GIVEN ▶ $\overline{VW} \parallel \overline{ZY}$, $\overline{UV} \cong \overline{XW}$, $\overline{UZ} \cong \overline{XY}$, $\overline{VW} \perp \overline{VZ}$, $\overline{VW} \perp \overline{WY}$

PROVE ▶ $\angle U \cong \angle X$



FOCUS ON PEOPLE



ISAAC NEWTON
The English scientist Isaac Newton (1642–1727) observed that light is made up of a spectrum of colors. Newton was the first person to arrange the colors of the spectrum in a "color wheel."

APPLICATION LINK
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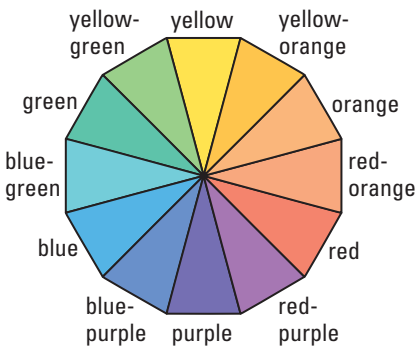
COLOR WHEEL **Artists use a color wheel to show relationships between colors. The 12 triangles in the diagram are isosceles triangles with congruent vertex angles.**



35. Complementary colors lie directly opposite each other on the color wheel. Explain how you know that the yellow triangle is congruent to the purple triangle.

36. The measure of the vertex angle of the yellow triangle is 30°. Find the measures of the base angles.

37. Trace the color wheel. Then form a triangle whose vertices are the midpoints of the bases of the red, yellow, and blue triangles. (These colors are the *primary colors*.) What type of triangle is this?

38. Form other triangles that are congruent to the triangle in Exercise 37. The colors of the vertices are called *triads*. What are the possible triads?

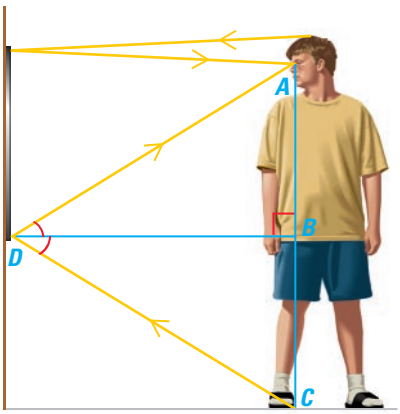
PHYSICS **Use the information below.**
When a light ray from an object meets a mirror, it is reflected back to your eye. For example, in the diagram, a light ray from point C is reflected at point D and travels back to point A. The *law of reflection* states that the angle of incidence $\angle CDB$ is equal to the angle of reflection $\angle ADB$.

39. GIVEN ▶ $\angle CDB \cong \angle ADB$
$\overline{DB} \perp \overline{AC}$

PROVE ▶ $\triangle ABD \cong \triangle CBD$

40. Verify that $\triangle ACD$ is isosceles.

41. Does moving away from the mirror have any effect on the amount of his or her reflection the person sees?



For a person to see his or her complete reflection, the mirror must be at least one half the person's height.

Test Preparation

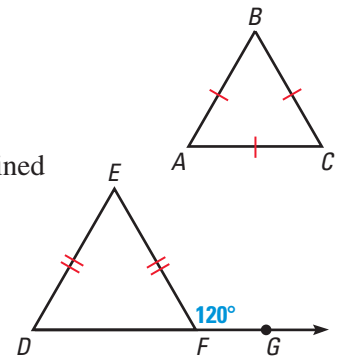
QUANTITATIVE COMPARISON **In Exercises 42 and 43, refer to the figures below. Choose the statement that is true about the given values.**

Ⓐ The value in column A is greater.

Ⓑ The value in column B is greater.

Ⓒ The two values are equal.

Ⓓ The relationship cannot be determined from the given information.

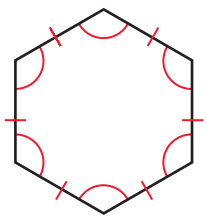


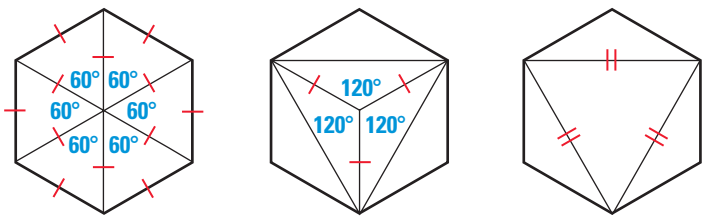
	Column A	Column B
42.	$\angle D$	$\angle EFD$
43.	$\angle B$	$\angle EFD$

★ Challenge

44. **LOGICAL REASONING** A *regular hexagon* has six congruent sides and six congruent interior angles. It can be divided into six equilateral triangles. Explain how the series of diagrams below suggests a proof that when a triangle is formed by connecting every other vertex of a regular hexagon, the result is an equilateral triangle.

Regular hexagon





EXTRA CHALLENGE

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MIXED REVIEW

CONGRUENCE **Use the Distance Formula to decide whether $\overline{AB} \cong \overline{AC}$.** (Review 1.3 for 4.7)

45. $A(0, -4)$, $B(5, 8)$, $C(-12, 1)$

46. $A(0, 0)$, $B(-6, -10)$, $C(6, 10)$

47. $A(1, -1)$, $B(-8, 7)$, $C(8, 7)$

FINDING THE MIDPOINT **Find the coordinates of the midpoint of a segment with the given endpoints.** (Review 1.5 for 4.7)

48. $C(4, 9)$, $D(10, 7)$

49. $G(0, 11)$, $H(8, -3)$

50. $L(1, 7)$, $M(-5, -5)$

51. $C(-2, 3)$, $D(5, 6)$

52. $G(0, -13)$, $H(2, -1)$

53. $L(-3, -5)$, $M(0, -20)$

WRITING EQUATIONS **Line *j* is perpendicular to the line with the given equation and line *j* passes through point *P*. Write an equation of line *j*.** (Review 3.7)

54. $y = -3x - 4$; $P(1, 1)$

55. $y = x - 7$; $P(0, 0)$

56. $y = -\frac{10}{9}x + 3$; $P(5, -12)$

57. $y = \frac{2}{3}x + 4$; $P(-3, 4)$

4.7 Triangles and Coordinate Proof

What you should learn

GOAL 1 Place geometric figures in a coordinate plane.

GOAL 2 Write a coordinate proof.

Why you should learn it

Sometimes a coordinate proof is the most efficient way to prove a statement.

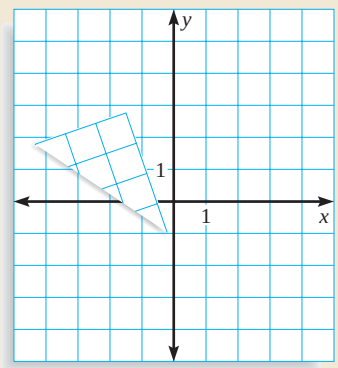
GOAL 1 PLACING FIGURES IN A COORDINATE PLANE

So far, you have studied two-column proofs, paragraph proofs, and flow proofs. A **coordinate proof** involves placing geometric figures in a coordinate plane. Then you can use the Distance Formula and the Midpoint Formula, as well as postulates and theorems, to prove statements about the figures.

ACTIVITY Developing Concepts

Placing Figures in a Coordinate Plane

1. Draw a right triangle with legs of 3 units and 4 units on a piece of grid paper. Cut out the triangle.
2. Use another piece of grid paper to draw a coordinate plane.
3. Sketch different ways that the triangle can be placed on the coordinate plane. Which of the ways that you placed the triangle is best for finding the length of the hypotenuse?

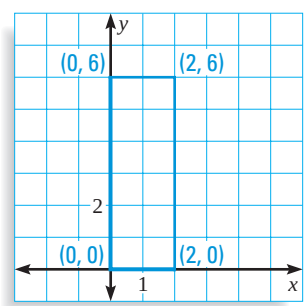


EXAMPLE 1 *Placing a Rectangle in a Coordinate Plane*

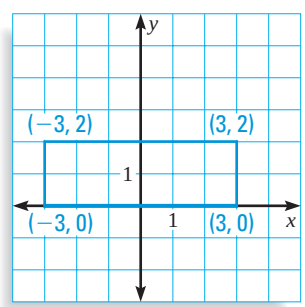
Place a 2-unit by 6-unit rectangle in a coordinate plane.

SOLUTION

Choose a placement that makes finding distances easy. Here are two possible placements.



One vertex is at the origin, and three of the vertices have at least one coordinate that is 0.



One side is centered at the origin, and the x-coordinates are opposites.

STUDENT HELP

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Once a figure has been placed in a coordinate plane, you can use the Distance Formula or the Midpoint Formula to measure distances or locate points.

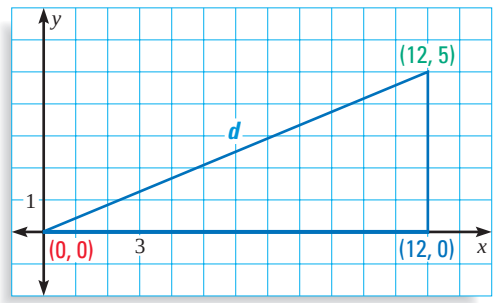
EXAMPLE 2 *Using the Distance Formula*

Using Algebra

A right triangle has legs of 5 units and 12 units. Place the triangle in a coordinate plane. Label the coordinates of the vertices and find the length of the hypotenuse.

SOLUTION

One possible placement is shown. Notice that one leg is vertical and the other leg is horizontal, which assures that the legs meet at right angles. Points on the same vertical segment have the same x-coordinate, and points on the same horizontal segment have the same y-coordinate.



You can use the Distance Formula to find the length of the hypotenuse.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \textbf{Distance Formula}$$

$$= \sqrt{(12 - 0)^2 + (5 - 0)^2} \quad \textbf{Substitute.}$$

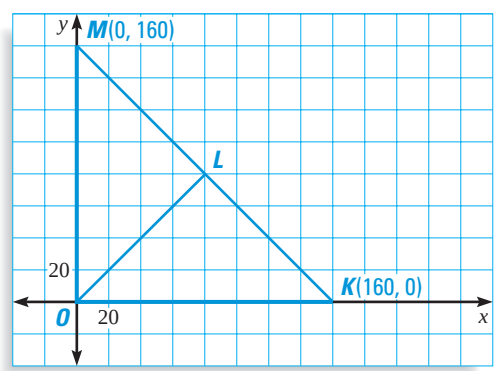
$$= \sqrt{169} \quad \textbf{Simplify.}$$

$$= 13 \quad \textbf{Evaluate square root.}$$

EXAMPLE 3 *Using the Midpoint Formula*

In the diagram, $\triangle MLO \cong \triangle KLO$.

Find the coordinates of point L.



SOLUTION

Because the triangles are congruent, it follows that $\overline{ML} \cong \overline{KL}$. So, point L must be the midpoint of $\overline{MK}$. This means you can use the Midpoint Formula to find the coordinates of point L.

$$L(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \quad \textbf{Midpoint Formula}$$

$$= \left(\frac{160 + 0}{2}, \frac{0 + 160}{2}\right) \quad \textbf{Substitute.}$$

$$= (80, 80) \quad \textbf{Simplify.}$$

▶ The coordinates of L are (80, 80).

GOAL 2 WRITING COORDINATE PROOFS

Once a figure is placed in a coordinate plane, you may be able to prove statements about the figure.

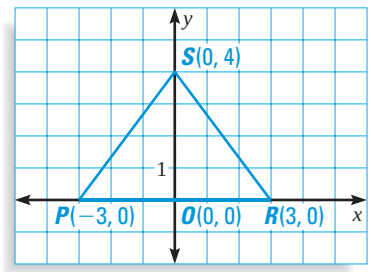
EXAMPLE 4 *Writing a Plan for a Coordinate Proof*

Proof

Write a plan to prove that $\overrightarrow{SO}$ bisects $\angle PSR$.

GIVEN ▶ Coordinates of vertices of $\triangle POS$ and $\triangle ROS$

PROVE ▶ $\overrightarrow{SO}$ bisects $\angle PSR$

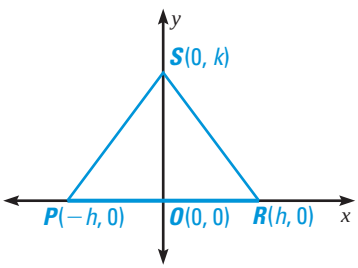


SOLUTION

Plan for Proof Use the Distance Formula to find the side lengths of $\triangle POS$ and $\triangle ROS$. Then use the SSS Congruence Postulate to show that $\triangle POS \cong \triangle ROS$. Finally, use the fact that corresponding parts of congruent triangles are congruent to conclude that $\angle PSO \cong \angle RSO$, which implies that $\overrightarrow{SO}$ bisects $\angle PSR$.

The coordinate proof in Example 4 applies to a specific triangle. When you want to prove a statement about a more general set of figures, it is helpful to use variables as coordinates.

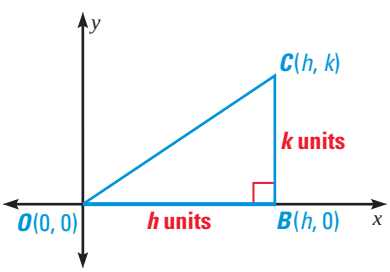
For instance, you can use variable coordinates to duplicate the proof in Example 4. Once this is done, you can conclude that $\overrightarrow{SO}$ bisects $\angle PSR$ for *any* triangle whose coordinates fit the given pattern.



EXAMPLE 5 *Using Variables as Coordinates*

Right $\triangle OBC$ has leg lengths of h units and k units. You can find the coordinates of points B and C by considering how the triangle is placed in the coordinate plane.

Point B is h units horizontally from the origin, so its coordinates are $(h, 0)$. Point C is h units horizontally from the origin and k units vertically from the origin, so its coordinates are (h, k).



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You can use the Distance Formula to find the length of the hypotenuse $\overline{OC}$.

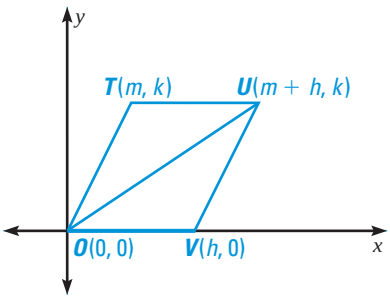
$$OC = \sqrt{(h-0)^2 + (k-0)^2} = \sqrt{h^2 + k^2}$$

Proof

EXAMPLE 6 *Writing a Coordinate Proof*

GIVEN ▶ Coordinates of figure $OTUV$

PROVE ▶ $\triangle OTU \cong \triangle UVO$



SOLUTION

COORDINATE PROOF Segments $\overline{OV}$ and $\overline{UT}$ have the same length.

$$OV = \sqrt{(h-0)^2 + (0-0)^2} = h$$

$$UT = \sqrt{(m+h-m)^2 + (k-k)^2} = h$$

Horizontal segments $\overline{UT}$ and $\overline{OV}$ each have a slope of 0, which implies that they are parallel. Segment $\overline{OU}$ intersects $\overline{UT}$ and $\overline{OV}$ to form congruent alternate interior angles $\angle TUO$ and $\angle VOU$. Because $\overline{OU} \cong \overline{OU}$, you can apply the SAS Congruence Postulate to conclude that $\triangle OTU \cong \triangle UVO$.

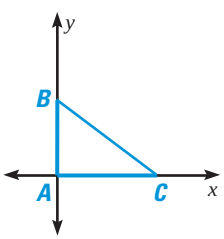
GUIDED PRACTICE

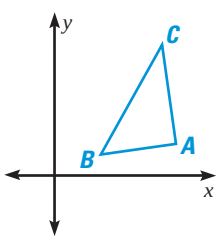
Vocabulary Check ✓

1. Prior to this section, you have studied two-column proofs, paragraph proofs, and flow proofs. How is a *coordinate proof* different from these other types of proof? How is it the same?

Concept Check ✓

2. Two different ways to place the same right triangle in a coordinate plane are shown. Which placement is more convenient for finding the side lengths? Explain your thinking. Then sketch a third placement that also makes it convenient to find the side lengths.





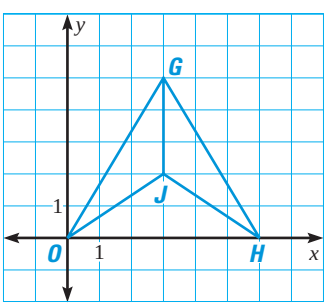
Skill Check ✓

3. A right triangle with legs of 7 units and 4 units has one vertex at (0, 0) and another at (0, 7). Give possible coordinates of the third vertex.

DEVELOPING PROOF **Describe a plan for the proof.**

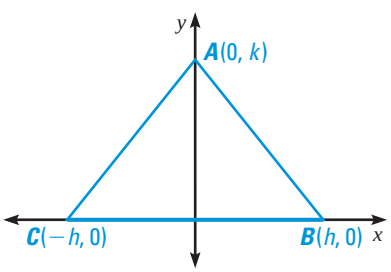
4. **GIVEN** ▶ $\overrightarrow{GJ}$ bisects $\angle OGH$.

   PROVE ▶ $\triangle GJO \cong \triangle GJH$



5. **GIVEN** ▶ Coordinates of vertices of $\triangle ABC$

   PROVE ▶ $\triangle ABC$ is isosceles.



PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 810.

PLACING FIGURES IN A COORDINATE PLANE **Place the figure in a coordinate plane. Label the vertices and give the coordinates of each vertex.**

6. A 5-unit by 8-unit rectangle with one vertex at (0, 0)

7. An 8-unit by 6-unit rectangle with one vertex at (0, −4)

8. A square with side length s and one vertex at (s, 0)

CHOOSING A GOOD PLACEMENT **Place the figure in a coordinate plane. Label the vertices and give the coordinates of each vertex. Explain the advantages of your placement.**

9. A right triangle with legs of 3 units and 8 units

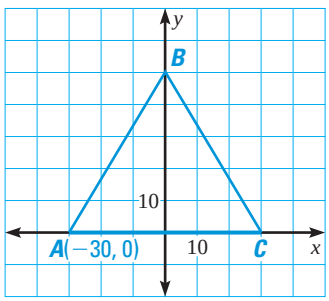
10. An isosceles right triangle with legs of 20 units

11. A rectangle with length h and width k

FINDING AND USING COORDINATES
In the diagram, $\triangle ABC$ is isosceles. Its base is 60 units and its height is 50 units.

12. Give the coordinates of points B and C.

13. Find the length of a leg of $\triangle ABC$. Round your answer to the nearest hundredth.



USING THE DISTANCE FORMULA **Place the figure in a coordinate plane and find the given information.**

14. A right triangle with legs of 7 and 9 units; find the length of the hypotenuse.

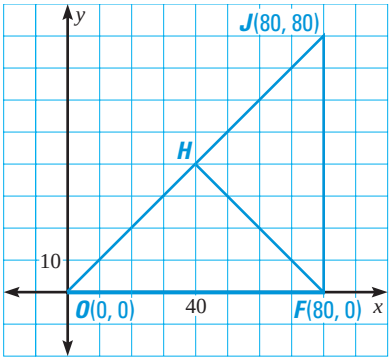
15. A rectangle with length 5 units and width 4 units; find the length of a diagonal.

16. An isosceles right triangle with legs of 3 units; find the length of the hypotenuse.

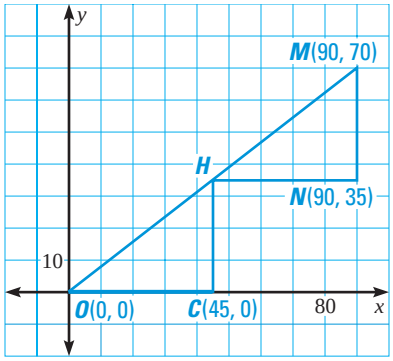
17. A 3-unit by 3-unit square; find the length of a diagonal.

USING THE MIDPOINT FORMULA **Use the given information and diagram to find the coordinates of *H*.**

18. $\triangle FOH \cong \triangle FJH$



19. $\triangle OCH \cong \triangle HNM$



STUDENT HELP

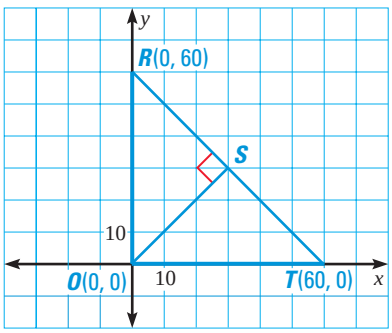
HOMEWORK HELP

Example 1: Exs. 6–11
Example 2: Exs. 12–17
Example 3: Exs. 18, 19
Example 4: Exs. 20, 21
Example 5: Exs. 22–25
Example 6: Exs. 26, 27

DEVELOPING PROOF **Write a plan for a proof.**

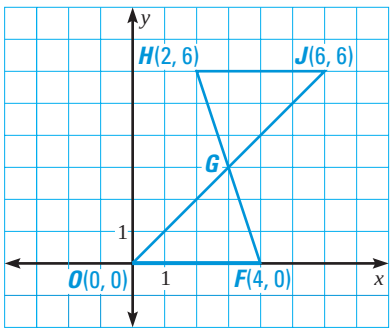
20. GIVEN ▶ $\overline{OS} \perp \overline{RT}$

PROVE ▶ $\overrightarrow{OS}$ bisects $\angle TOR$.



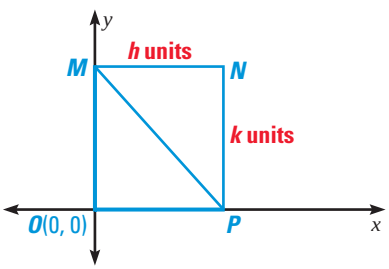
21. GIVEN ▶ G is the midpoint of $\overline{HF}$.

PROVE ▶ $\triangle GHJ \cong \triangle GFO$

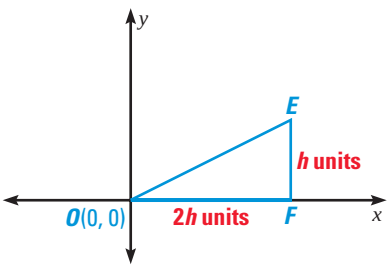


USING VARIABLES AS COORDINATES **Find the coordinates of any unlabeled points. Then find the requested information.**

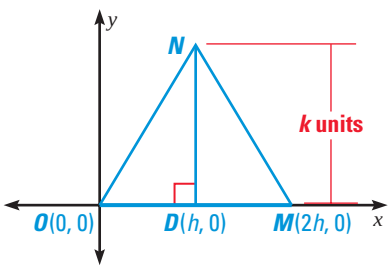
22. Find MP.



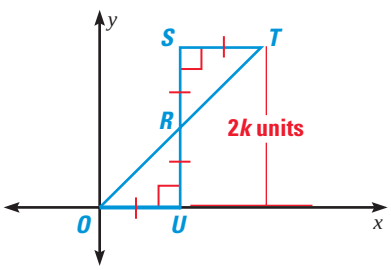
23. Find OE.



24. Find ON and MN.



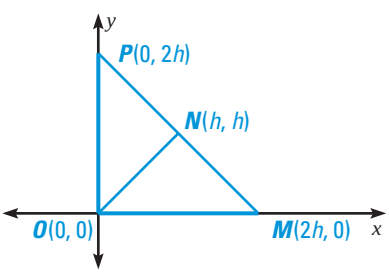
25. Find OT.



COORDINATE PROOF **Write a coordinate proof.**

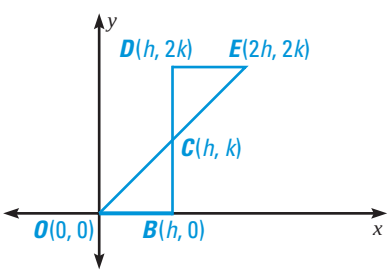
26. GIVEN ▶ Coordinates of $\triangle NPO$ and $\triangle NMO$

PROVE ▶ $\triangle NPO \cong \triangle NMO$

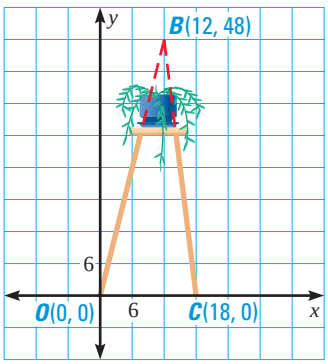


27. GIVEN ▶ Coordinates of $\triangle OBC$ and $\triangle EDC$

PROVE ▶ $\triangle OBC \cong \triangle EDC$

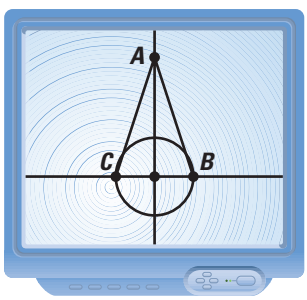


28. **PLANT STAND** You buy a tall, three-legged plant stand. When you place a plant on the stand, the stand appears to be unstable under the weight of the plant. The diagram at the right shows a coordinate plane superimposed on one pair of the plant stand's legs. The legs are extended to form $\triangle OBC$. Is $\triangle OBC$ an isosceles triangle? Explain why the plant stand may be unstable.



TECHNOLOGY Use geometry software for Exercises 29–31. Follow the steps below to construct $\triangle ABC$.

- Create a pair of axes. Construct point A on the y-axis so that the y-coordinate is positive. Construct point B on the x-axis.
- Construct a circle with a center at the origin that contains point B. Label the other point where the circle intersects the x-axis C.
- Connect points A, B, and C to form $\triangle ABC$. Find the coordinates of each vertex.



29. What type of triangle does $\triangle ABC$ appear to be? Does your answer change if you drag point A? If you drag point B?

30. Measure and compare AB and AC. What happens to these lengths as you drag point A? What happens as you drag point B?

31. Look back at the proof described in Exercise 5 on page 246. How does that proof help explain your answers to Exercises 29 and 30?

Test Preparation

32. **MULTIPLE CHOICE** A square with side length 4 has one vertex at (0, 2). Which of the points below *could* be a vertex of the square?

Ⓐ (0, −2) Ⓑ (2, −2) Ⓒ (0, 0) Ⓓ (2, 2)

33. **MULTIPLE CHOICE** A rectangle with side lengths $2h$ and k has one vertex at $(-h, k)$. Which of the points below *could not* be a vertex of the rectangle?

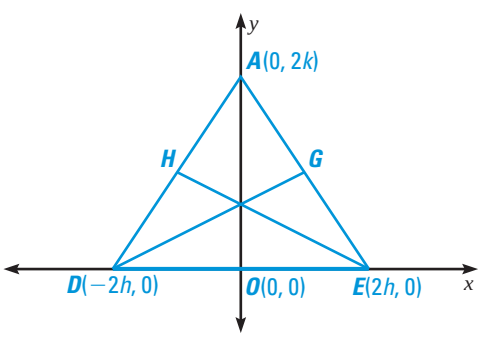
Ⓐ $(0, k)$ Ⓑ $(-h, 0)$ Ⓒ (h, k) Ⓓ $(h, 0)$

★ Challenge

34. **COORDINATE PROOF** Use the diagram and the given information to write a proof.

GIVEN ▶ Coordinates of $\triangle DEA$, H is the midpoint of $\overline{DA}$, G is the midpoint of $\overline{EA}$.

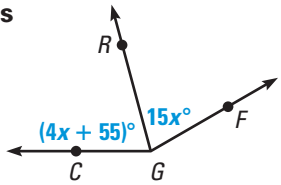
PROVE ▶ $\overline{DG} \cong \overline{EH}$



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

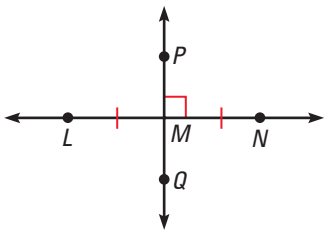
USING ALGEBRA **In the diagram, $\overrightarrow{GR}$ bisects $\angle CGF$.** (Review 1.5 for 5.1)



35. Find the value of x.

36. Find $m\angle CGF$.

PERPENDICULAR LINES AND SEGMENT BISECTORS **Use the diagram to determine whether the statement is *true* or *false*.** (Review 1.5, 2.2 for 5.1)



37. $\overleftrightarrow{PQ}$ is perpendicular to $\overleftrightarrow{LN}$.

38. Points L, Q, and N are collinear.

39. $\overleftrightarrow{PQ}$ bisects $\overline{LN}$.

40. $\angle LMQ$ and $\angle PMN$ are supplementary.

WRITING STATEMENTS **Let *p* be "two triangles are congruent" and let *q* be "the corresponding angles of the triangles are congruent." Write the symbolic statement in words. Decide whether the statement is true.** (Review 2.3)

41. $p \rightarrow q$ **42.** $q \rightarrow p$ **43.** $\sim p \rightarrow \sim q$

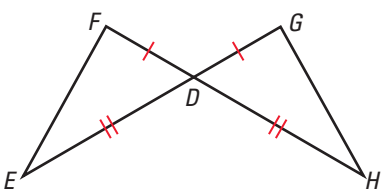
QUIZ 3

Self-Test for Lessons 4.5–4.7

PROOF **Write a two-column proof or a paragraph proof.** (Lessons 4.5 and 4.6)

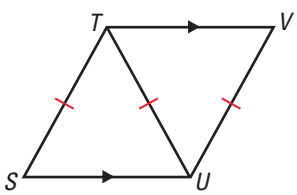
1. GIVEN ▶ $\overline{DF} \cong \overline{DG}$, $\overline{ED} \cong \overline{HD}$

PROVE ▶ $\angle EFD \cong \angle HGD$



2. GIVEN ▶ $\overline{ST} \cong \overline{UT} \cong \overline{VU}$, $\overline{SU} \parallel \overline{TV}$

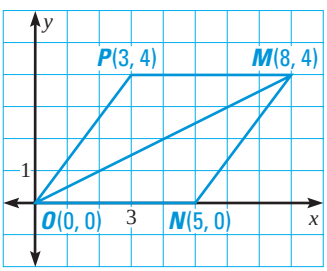
PROVE ▶ $\triangle STU \cong \triangle TUV$



3. **COORDINATE PROOF** Write a plan for a coordinate proof. (Lesson 4.7)

GIVEN ▶ Coordinates of vertices of $\triangle OPM$ and $\triangle ONM$

PROVE ▶ $\triangle OPM$ and $\triangle ONM$ are congruent isosceles triangles.



CHAPTER 4

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Classify triangles by their sides and angles. **(4.1)**	Lay the foundation for work with triangles.
Find angle measures in triangles. **(4.1)**	Find the angle measures in triangular objects, such as a wing deflector. **(p. 200)**
Identify congruent figures and corresponding parts. **(4.2)**	Analyze patterns, such as those made by the folds of an origami kite. **(p. 208)**
Prove that triangles are congruent	
• using corresponding sides and angles. **(4.2)**	Learn to work with congruent triangles.
• using the SSS and SAS Congruence Postulates. **(4.3)**	Explain why triangles are used in structural supports for buildings. **(p. 215)**
• using the ASA Congruence Postulate and the AAS Congruence Theorem. **(4.4)**	Understand how properties of triangles are applied in surveying. **(p. 225)**
• using the HL Congruence Theorem. **(4.6)**	Prove that right triangles are congruent.
• using coordinate geometry. **(4.7)**	Plan and write coordinate proofs.
Use congruent triangles to plan and write proofs. **(4.5)**	Prove that triangular parts of the framework of a bridge are congruent. **(p. 234)**
Prove that constructions are valid. **(4.5)**	Develop understanding of geometric constructions.
Use properties of isosceles, equilateral, and right triangles. **(4.6)**	Apply a law from physics, the law of reflection. **(p. 241)**

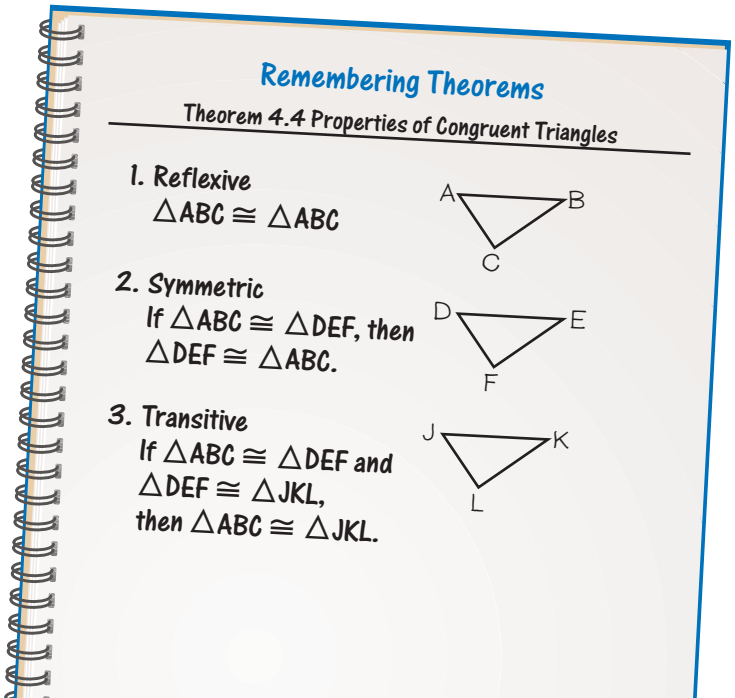
How does Chapter 4 fit into the BIGGER PICTURE of geometry?

The ways you have learned to prove triangles are congruent will be used to prove theorems about *polygons*, as well as in other topics throughout the book. Knowing the properties of triangles will help you solve real-life problems in fields such as art, architecture, and engineering.

STUDY STRATEGY

How did you use your list of theorems?

The list of theorems you made, following the **Study Strategy** on page 192, may resemble this one.



Remembering Theorems

Theorem 4.4 Properties of Congruent Triangles

1. Reflexive
   $\triangle ABC \cong \triangle ABC$

2. Symmetric
   If $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$.

3. Transitive
   If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle JKL$, then $\triangle ABC \cong \triangle JKL$.

CHAPTER 4

Chapter Review

VOCABULARY

- equilateral triangle, p. 194
- isosceles triangle, p. 194
- scalene triangle, p. 194
- acute triangle, p. 194
- equiangular triangle, p. 194
- right triangle, p. 194
- obtuse triangle, p. 194
- vertex of a triangle, p. 195
- adjacent sides of a triangle, p. 195
- legs of a right triangle, p. 195
- hypotenuse, p. 195
- legs of an isosceles triangle, p. 195
- base of an isosceles triangle, p. 195
- interior angle, p. 196
- exterior angle, p. 196
- corollary, p. 197
- congruent, p. 202
- corresponding angles, p. 202
- corresponding sides, p. 202
- base angles, p. 236
- vertex angle, p. 236
- coordinate proof, p. 243

4.1 TRIANGLES AND ANGLES

Examples on pp. 194–197

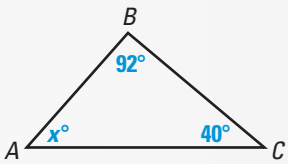
EXAMPLES You can classify triangles by their sides and by their angles.

equilateral isosceles scalene acute equiangular right obtuse

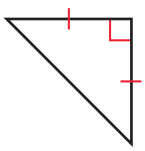
Note that an equilateral triangle is also isosceles and acute.

You can apply the Triangle Sum Theorem to find unknown angle measures in triangles.

$m\angle A + m\angle B + m\angle C = 180°$	**Triangle Sum Theorem**
$x° + 92° + 40° = 180°$	**Substitute.**
$x + 132 = 180$	**Simplify.**
$x = 48$	**Subtract 132 from each side.**
$m\angle A = 48°$	

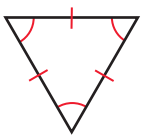


In Exercises 1–4, classify the triangle by its angles and by its sides.

1. 

2. 

3. 

4. 

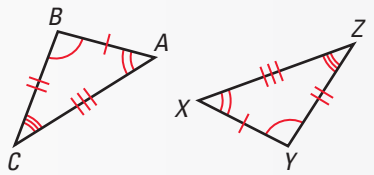
5. One acute angle of a right triangle measures 37°. Find the measure of the other acute angle.

6. In $\triangle MNP$, the measure of $\angle M$ is 24°. The measure of $\angle N$ is five times the measure of $\angle P$. Find $m\angle N$ and $m\angle P$.

4.2 CONGRUENCE AND TRIANGLES

Examples on pp. 202–205

EXAMPLE When two figures are congruent, their corresponding sides and corresponding angles are congruent. In the diagram, $\triangle ABC \cong \triangle XYZ$.



Use the diagram above of $\triangle ABC$ and $\triangle XYZ$.

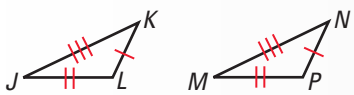
7. Identify the congruent corresponding parts of the triangles.

8. Given $m\angle A = 48°$ and $m\angle Z = 37°$, find $m\angle Y$.

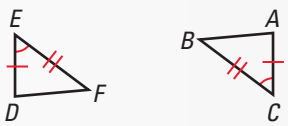
4.3 & 4.4 PROVING TRIANGLES ARE CONGRUENT: SSS, SAS, ASA, AND AAS

Examples on pp. 212–215, 220–222

EXAMPLES You can prove triangles are congruent using congruence postulates and theorems.



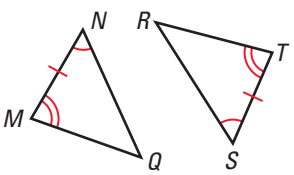
$\overline{JK} \cong \overline{MN}$, $\overline{KL} \cong \overline{NP}$, $\overline{JL} \cong \overline{MP}$, so $\triangle JKL \cong \triangle MNP$ by the SSS Congruence Postulate.



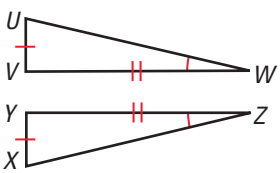
$\overline{DE} \cong \overline{AC}$, $\angle E \cong \angle C$, and $\overline{EF} \cong \overline{CB}$, so $\triangle DEF \cong \triangle ACB$ by the SAS Congruence Postulate.

Decide whether it is possible to prove that the triangles are congruent. If it is possible, tell which postulate or theorem you would use. Explain your reasoning.

9.



10.



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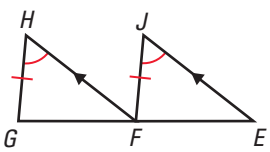
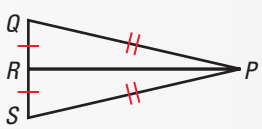


4.5 USING CONGRUENT TRIANGLES

Examples on pp. 229–231

EXAMPLE You can use congruent triangles to write proofs.



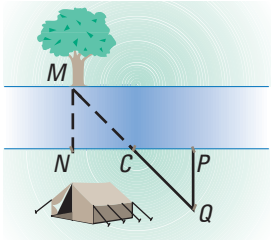
GIVEN ▶ $\overline{PQ} \cong \overline{PS}$, $\overline{RQ} \cong \overline{RS}$

PROVE ▶ $\overline{PR} \perp \overline{QS}$

Plan for Proof Use the SSS Congruence Postulate to show that $\triangle PRQ \cong \triangle PRS$. Because corresponding parts of congruent triangles are congruent, you can conclude that $\angle PRQ \cong \angle PRS$. These angles form a linear pair, so $\overline{PR} \perp \overline{QS}$.

4.5 continued

SURVEYING **You want to determine the width of a river beside a camp. You place stakes so that $\overline{MN} \perp \overline{NP}$, $\overline{PQ} \perp \overline{NP}$, and C is the midpoint of $\overline{NP}$.**



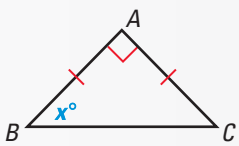
12. Are $\triangle MCN$ and $\triangle QCP$ congruent? If so, state the postulate or theorem that can be used to prove they are congruent.

13. Which segment should you measure to find the width of the river?

4.6 ISOSCELES, EQUILATERAL, AND RIGHT TRIANGLES

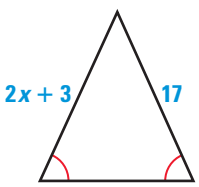
Examples on pp. 236–238

EXAMPLE To find the value of x, notice that $\triangle ABC$ is an isosceles right triangle. By the Base Angles Theorem, $\angle B \cong \angle C$. Because $\angle B$ and $\angle C$ are complementary, their sum is 90°. The measure of each must be 45°. So $x = 45°$.

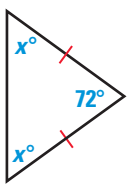


Find the value of *x*.

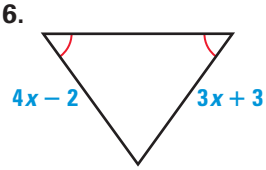
14.



15.



16.



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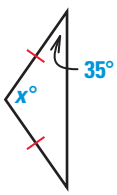


4.7 TRIANGLES AND COORDINATE PROOF

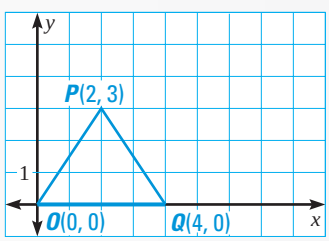
Examples on pp. 243–246

EXAMPLE You can use a coordinate proof to prove that $\triangle OPQ$ is isosceles. Use the Distance Formula to show that $\overline{OP} \cong \overline{QP}$.

$$OP = \sqrt{(2-0)^2 + (3-0)^2} = \sqrt{13}$$

$$QP = \sqrt{(2-4)^2 + (3-0)^2} = \sqrt{13}$$

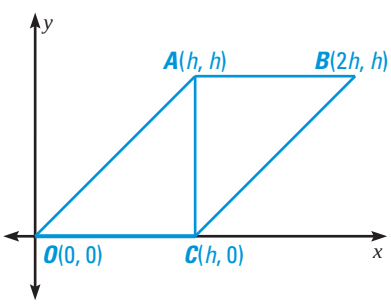
Because $\overline{OP} \cong \overline{QP}$, $\triangle OPQ$ is isosceles.



18. Write a coordinate proof.

GIVEN ▶ Coordinates of vertices of $\triangle OAC$ and $\triangle BCA$

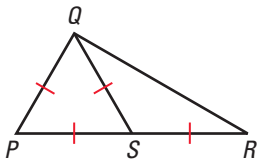
PROVE ▶ $\triangle OAC \cong \triangle BCA$



CHAPTER 4

Chapter Test

In Exercises 1–6, identify all triangles in the figure that fit the given description.



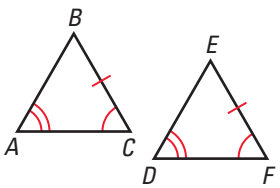
1. isosceles **2.** equilateral **3.** scalene

4. acute **5.** obtuse **6.** right

7. In $\triangle ABC$, the measure of $\angle A$ is 116°. The measure of $\angle B$ is three times the measure of $\angle C$. Find $m\angle B$ and $m\angle C$.

Decide whether it is possible to prove that the triangles are congruent. If it is possible, tell which congruence postulate or theorem you would use. Explain your reasoning.

8.



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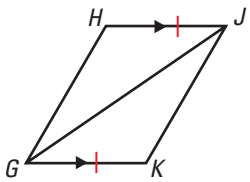

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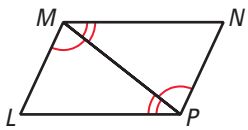

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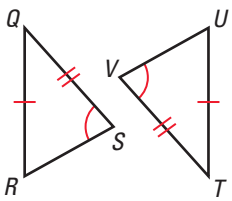

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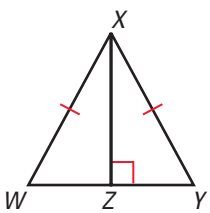

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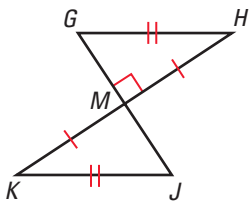
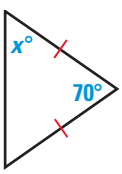


Find the value of *x*.

14.



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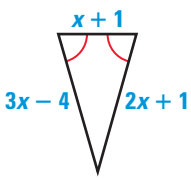

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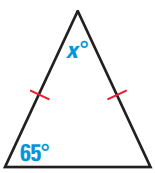
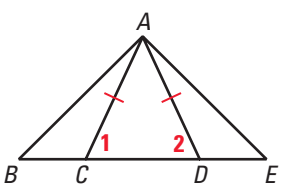
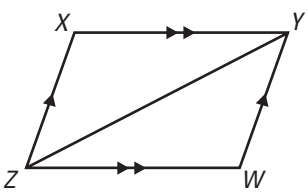


PROOF **Write a two-column proof or a paragraph proof.**

17. **GIVEN** ▶ $\overline{BD} \cong \overline{EC}$, $\overline{AC} \cong \overline{AD}$
PROVE ▶ $\overline{AB} \cong \overline{AE}$



18. **GIVEN** ▶ $\overline{XY} \parallel \overline{WZ}$, $\overline{XZ} \parallel \overline{WY}$
PROVE ▶ $\angle X \cong \angle W$



Place the figure in a coordinate plane and find the requested information.

19. A right triangle with leg lengths of 4 units and 7 units; find the length of the hypotenuse.

20. A square with side length s and vertices at (0, 0) and (s, s); find the coordinates of the midpoint of a diagonal.

5.1 Perpendiculars and Bisectors

What you should learn

GOAL 1 Use properties of perpendicular bisectors.

GOAL 2 Use properties of angle bisectors to identify equal distances, such as the lengths of beams in a roof truss in **Example 3**.

Why you should learn it

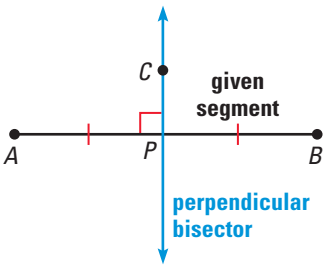
▼ To solve **real-life** problems, such as deciding where a hockey goalie should be positioned in **Exs. 33–35**.

REAL LIFE

GOAL 1 USING PROPERTIES OF PERPENDICULAR BISECTORS

In Lesson 1.5, you learned that a segment bisector intersects a segment at its midpoint. A segment, ray, line, or plane that is perpendicular to a segment at its midpoint is called a **perpendicular bisector**.

The construction below shows how to draw a line that is perpendicular to a given line or segment at a point P. You can use this method to construct a perpendicular bisector of a segment, as described below the activity.

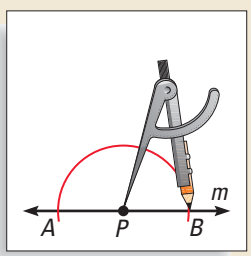


$\overleftrightarrow{CP}$ is a $\perp$ bisector of $\overline{AB}$.

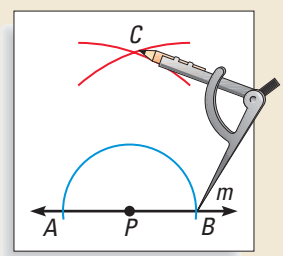
ACTIVITY
Construction

Perpendicular Through a Point on a Line

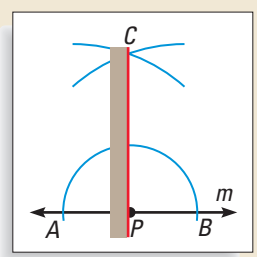
Use these steps to construct a line that is perpendicular to a given line m and that passes through a given point P on m.



1 Place the compass point at P. Draw an arc that intersects line m twice. Label the intersections as A and B.



2 Use a compass setting greater than AP. Draw an arc from A. With the same setting, draw an arc from B. Label the intersection of the arcs as C.



3 Use a straightedge to draw $\overleftrightarrow{CP}$. This line is perpendicular to line m and passes through P.

You can measure $\angle CPA$ on your construction to verify that the constructed line is perpendicular to the given line m. In the construction, $\overleftrightarrow{CP} \perp \overline{AB}$ and $PA = PB$, so $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$.

A point is **equidistant from two points** if its distance from each point is the same. In the construction above, C is equidistant from A and B because C was drawn so that $CA = CB$.

STUDENT HELP

Look Back
For a construction of a perpendicular to a line through a point not on the given line, see p. 130.

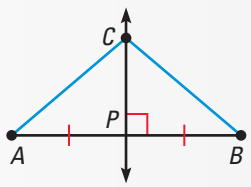
Theorem 5.1 below states that *any* point on the perpendicular bisector $\overleftrightarrow{CP}$ in the construction is equidistant from A and B, the endpoints of the segment. The converse helps you prove that a given point lies on a perpendicular bisector.

THEOREMS

THEOREM 5.1 ***Perpendicular Bisector Theorem***

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

If $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$, then $CA = CB$.

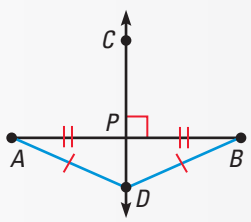


CA = CB

THEOREM 5.2 ***Converse of the Perpendicular Bisector Theorem***

If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.

If $DA = DB$, then D lies on the perpendicular bisector of $\overline{AB}$.



***D* is on $\overleftrightarrow{CP}$.**

Proof

Plan for Proof of Theorem 5.1 Refer to the diagram for Theorem 5.1 above. Suppose that you are given that $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$. Show that right triangles $\triangle APC$ and $\triangle BPC$ are congruent using the SAS Congruence Postulate. Then show that $\overline{CA} \cong \overline{CB}$.

Exercise 28 asks you to write a two-column proof of Theorem 5.1 using this plan for proof. Exercise 29 asks you to write a proof of Theorem 5.2.

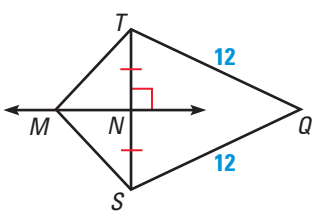
EXAMPLE 1 *Using Perpendicular Bisectors*

Logical Reasoning

In the diagram shown, $\overleftrightarrow{MN}$ is the perpendicular bisector of $\overline{ST}$.

a. What segment lengths in the diagram are equal?

b. Explain why Q is on $\overleftrightarrow{MN}$.



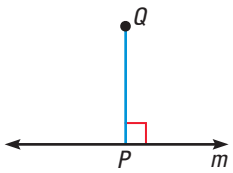
SOLUTION

a. $\overleftrightarrow{MN}$ bisects $\overline{ST}$, so $NS = NT$. Because M is on the perpendicular bisector of $\overline{ST}$, $MS = MT$ (by Theorem 5.1). The diagram shows that $QS = QT = 12$.

b. $QS = QT$, so Q is equidistant from S and T. By Theorem 5.2, Q is on the perpendicular bisector of $\overline{ST}$, which is $\overleftrightarrow{MN}$.

GOAL 2 USING PROPERTIES OF ANGLE BISECTORS

The **distance from a point to a line** is defined as the length of the perpendicular segment from the point to the line. For instance, in the diagram shown, the distance between the point Q and the line m is QP.



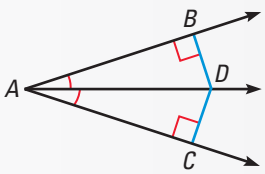
When a point is the same distance from one line as it is from another line, then the point is **equidistant from the two lines** (or rays or segments). The theorems below show that a point in the interior of an angle is equidistant from the sides of the angle if and only if the point is on the bisector of the angle.

THEOREMS

THEOREM 5.3 ***Angle Bisector Theorem***

If a point is on the bisector of an angle, then it is equidistant from the two sides of the angle.

If $m\angle BAD = m\angle CAD$, then $DB = DC$.

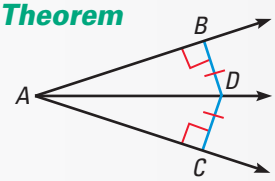


DB = DC

THEOREM 5.4 ***Converse of the Angle Bisector Theorem***

If a point is in the interior of an angle and is equidistant from the sides of the angle, then it lies on the bisector of the angle.

If $DB = DC$, then $m\angle BAD = m\angle CAD$.



m∠BAD = m∠CAD

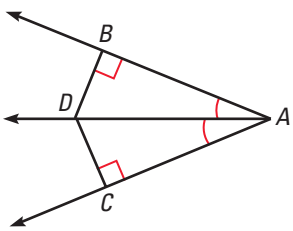
A paragraph proof of Theorem 5.3 is given in Example 2. Exercise 32 asks you to write a proof of Theorem 5.4.

EXAMPLE 2 *Proof of Theorem 5.3*

Proof

GIVEN ▶ D is on the bisector of $\angle BAC$.
$\overline{DB} \perp \overrightarrow{AB}$, $\overline{DC} \perp \overrightarrow{AC}$

PROVE ▶ $DB = DC$



Plan for Proof Prove that $\triangle ADB \cong \triangle ADC$. Then conclude that $\overline{DB} \cong \overline{DC}$, so $DB = DC$.

SOLUTION

Paragraph Proof By the definition of an angle bisector, $\angle BAD \cong \angle CAD$. Because $\angle ABD$ and $\angle ACD$ are right angles, $\angle ABD \cong \angle ACD$. By the Reflexive Property of Congruence, $\overline{AD} \cong \overline{AD}$. Then $\triangle ADB \cong \triangle ADC$ by the AAS Congruence Theorem. Because corresponding parts of congruent triangles are congruent, $\overline{DB} \cong \overline{DC}$. By the definition of congruent segments, $DB = DC$.

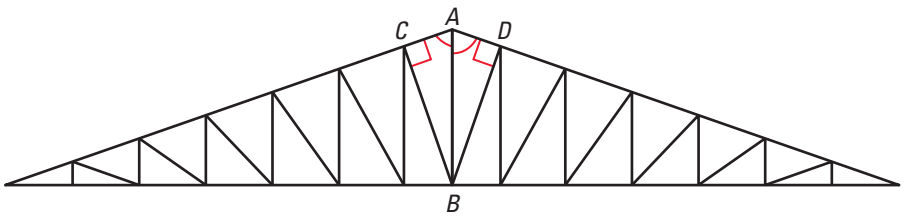
FOCUS ON CAREERS

ENGINEERING TECHNICIAN
In manufacturing, engineering technicians prepare specifications for products such as roof trusses, and devise and run tests for quality control.

CAREER LINK
www.mcdougallittell.com

EXAMPLE 3 *Using Angle Bisectors*

ROOF TRUSSES Some roofs are built with wooden trusses that are assembled in a factory and shipped to the building site. In the diagram of the roof truss shown below, you are given that $\overrightarrow{AB}$ bisects $\angle CAD$ and that $\angle ACB$ and $\angle ADB$ are right angles. What can you say about $\overline{BC}$ and $\overline{BD}$?



SOLUTION

Because $\overline{BC}$ and $\overline{BD}$ meet $\overline{AC}$ and $\overline{AD}$ at right angles, they are perpendicular segments to the sides of $\angle CAD$. This implies that their lengths represent the distances from the point B to $\overrightarrow{AC}$ and $\overrightarrow{AD}$. Because point B is on the bisector of $\angle CAD$, it is equidistant from the sides of the angle.

▶ So, $BC = BD$, and you can conclude that $\overline{BC} \cong \overline{BD}$.

GUIDED PRACTICE

Vocabulary Check ✓

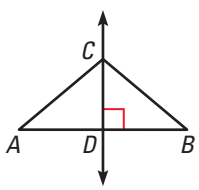
1. If D is on the __?__ of $\overline{AB}$, then D is *equidistant* from A and B.

Concept Check ✓

2. Point G is in the interior of $\angle HJK$ and is equidistant from the sides of the angle, $\overrightarrow{JH}$ and $\overrightarrow{JK}$. What can you conclude about G? Use a sketch to support your answer.

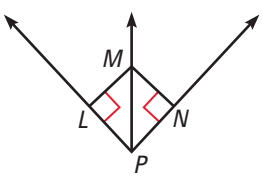
Skill Check ✓

In the diagram, $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$.



3. What is the relationship between $\overline{AD}$ and $\overline{BD}$?
4. What is the relationship between $\angle ADC$ and $\angle BDC$?
5. What is the relationship between $\overline{AC}$ and $\overline{BC}$? Explain your answer.

In the diagram, $\overrightarrow{PM}$ is the bisector of $\angle LPN$.



6. What is the relationship between $\angle LPM$ and $\angle NPM$?
7. How is the distance between point M and $\overrightarrow{PL}$ related to the distance between point M and $\overrightarrow{PN}$?

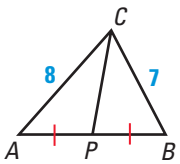
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 811.

LOGICAL REASONING **Tell whether the information in the diagram allows you to conclude that *C* is on the perpendicular bisector of $\overline{AB}$. Explain your reasoning.**

8.



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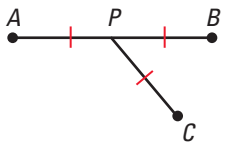

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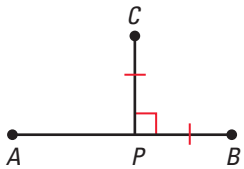
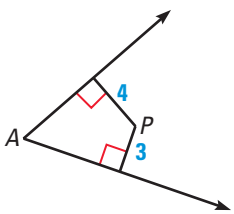


LOGICAL REASONING **In Exercises 11–13, tell whether the information in the diagram allows you to conclude that *P* is on the bisector of $\angle A$. Explain.**

11.



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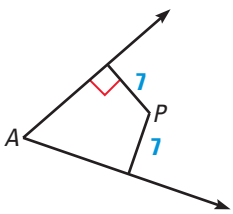

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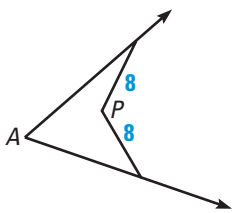
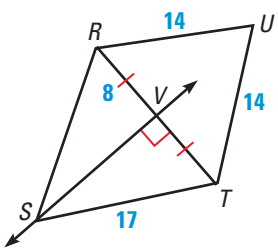


14. **CONSTRUCTION** Draw $\overline{AB}$ with a length of 8 centimeters. Construct a perpendicular bisector and draw a point D on the bisector so that the distance between D and $\overline{AB}$ is 3 centimeters. Measure $\overline{AD}$ and $\overline{BD}$.

15. **CONSTRUCTION** Draw a large $\angle A$ with a measure of 60°. Construct the angle bisector and draw a point D on the bisector so that $AD = 3$ inches. Draw perpendicular segments from D to the sides of $\angle A$. Measure these segments to find the distance between D and the sides of $\angle A$.

USING PERPENDICULAR BISECTORS **Use the diagram shown.**

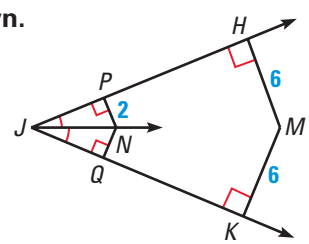


16. In the diagram, $\overleftrightarrow{SV} \perp \overline{RT}$ and $\overline{VR} \cong \overline{VT}$. Find VT.

17. In the diagram, $\overleftrightarrow{SV} \perp \overline{RT}$ and $\overline{VR} \cong \overline{VT}$. Find SR.

18. In the diagram, $\overleftrightarrow{SV}$ is the perpendicular bisector of $\overline{RT}$. Because $UR = UT = 14$, what can you conclude about point U?

USING ANGLE BISECTORS **Use the diagram shown.**



19. In the diagram, $\overrightarrow{JN}$ bisects $\angle HJK$, $\overline{NP} \perp \overrightarrow{JP}$, $\overline{NQ} \perp \overrightarrow{JQ}$, and $NP = 2$. Find NQ.

20. In the diagram, $\overrightarrow{JN}$ bisects $\angle HJK$, $\overline{MH} \perp \overrightarrow{JH}$, $\overline{MK} \perp \overrightarrow{JK}$, and $MH = MK = 6$. What can you conclude about point M?

STUDENT HELP

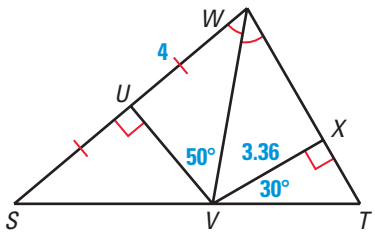
HOMEWORK HELP

Example 1: Exs. 8–10, 14, 16–18, 21–26
Example 2: Exs. 11–13, 15, 19, 20, 21–26
Example 3: Exs. 31, 33–35

USING BISECTOR THEOREMS **In Exercises 21–26, match the angle measure or segment length described with its correct value.**

A. 60°
B. 8
C. 40°
D. 4
E. 50°
F. 3.36

21. SW
22. $m\angle XTV$
23. $m\angle VWX$
24. VU
25. WX
26. $m\angle WVX$



27. **PROVING A CONSTRUCTION** Write a proof to verify that $\overleftrightarrow{CP} \perp \overline{AB}$ in the construction on page 264.

28. **PROVING THEOREM 5.1** Write a proof of Theorem 5.1, the Perpendicular Bisector Theorem. You may want to use the plan for proof given on page 265.

GIVEN ▶ $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$.

PROVE ▶ C is equidistant from A and B.

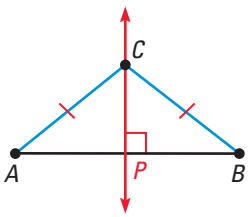
STUDENT HELP

Look Back
For help with proving that constructions are valid, see p. 231.

29. **PROVING THEOREM 5.2** Use the diagram shown to write a two-column proof of Theorem 5.2, the Converse of the Perpendicular Bisector Theorem.

GIVEN ▶ C is equidistant from A and B.

PROVE ▶ C is on the perpendicular bisector of $\overline{AB}$.

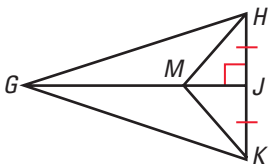


Plan for Proof Use the Perpendicular Postulate to draw $\overleftrightarrow{CP} \perp \overleftrightarrow{AB}$. Show that $\triangle APC \cong \triangle BPC$ by the HL Congruence Theorem. Then $\overline{AP} \cong \overline{BP}$, so $AP = BP$.

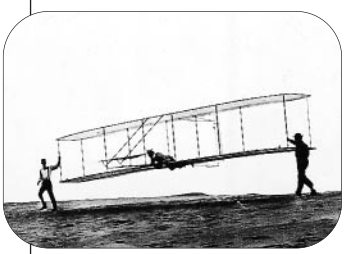
30. **PROOF** Use the diagram shown.

GIVEN ▶ $\overline{GJ}$ is the perpendicular bisector of $\overline{HK}$.

PROVE ▶ $\triangle GHM \cong \triangle GKM$

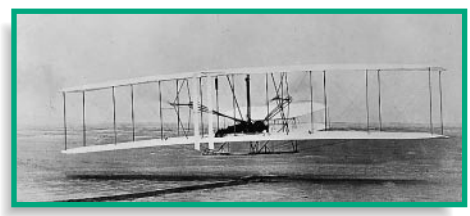


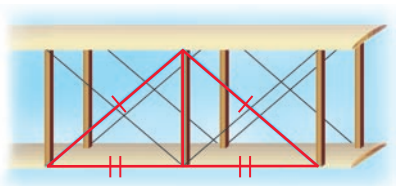
FOCUS ON PEOPLE



THE WRIGHT BROTHERS
In Kitty Hawk, North Carolina, on December 17, 1903, Orville and Wilbur Wright became the first people to successfully fly an engine-driven, heavier-than-air machine.

31. **EARLY AIRCRAFT** On many of the earliest airplanes, wires connected vertical posts to the edges of the wings, which were wooden frames covered with cloth. Suppose the lengths of the wires from the top of a post to the edges of the frame are the same and the distances from the bottom of the post to the ends of the two wires are the same. What does that tell you about the post and the section of frame between the ends of the wires?

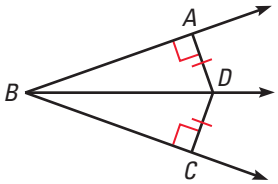




32. **DEVELOPING PROOF** Use the diagram to complete the proof of Theorem 5.4, the Converse of the Angle Bisector Theorem.

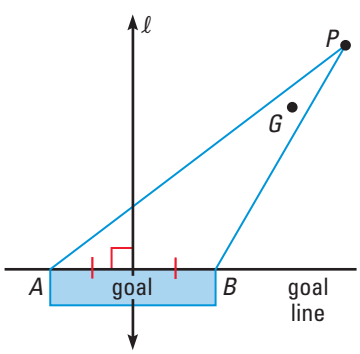
GIVEN ▶ D is in the interior of $\angle ABC$ and is equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

PROVE ▶ D lies on the angle bisector of $\angle ABC$.



Statements	Reasons
1. D is in the interior of $\angle ABC$.	**1.** ___?___
2. D is ___?___ from $\overrightarrow{BA}$ and $\overrightarrow{BC}$.	**2.** Given
3. ___?___ = ___?___	**3.** Definition of equidistant
4. $\overline{DA} \perp$ ___?___, ___?___ $\perp \overrightarrow{BC}$	**4.** Definition of distance from a point to a line
5. ___?___	**5.** If 2 lines are $\perp$, then they form 4 rt. $\angle$s.
6. ___?___	**6.** Definition of right triangle
7. $\overline{BD} \cong \overline{BD}$	**7.** ___?___
8. ___?___	**8.** HL Congruence Thm.
9. $\angle ABD \cong \angle CBD$	**9.** ___?___
10. $\overrightarrow{BD}$ bisects $\angle ABC$ and point D is on the bisector of $\angle ABC$.	**10.** ___?___

ICE HOCKEY **In Exercises 33–35, use the following information.**
In the diagram, the goalie is at point G and the puck is at point P. The goalie's job is to prevent the puck from entering the goal.

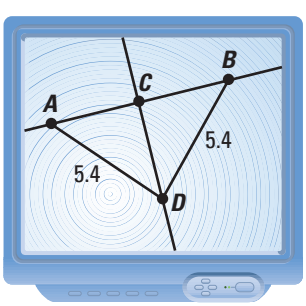


33. When the puck is at the other end of the rink, the goalie is likely to be standing on line ℓ. How is ℓ related to $\overline{AB}$?

34. As an opposing player with the puck skates toward the goal, the goalie is likely to move from line ℓ to other places on the ice. What should be the relationship between $\overrightarrow{PG}$ and $\angle APB$?

35. How does $m\angle APB$ change as the puck gets closer to the goal? Does this change make it easier or more difficult for the goalie to defend the goal? Explain.

36. **TECHNOLOGY** Use geometry software to construct $\overline{AB}$. Find the midpoint C. Draw the perpendicular bisector of $\overline{AB}$ through C. Construct a point D along the perpendicular bisector and measure $\overline{DA}$ and $\overline{DB}$. Move D along the perpendicular bisector. What theorem does this construction demonstrate?



Test Preparation

37. MULTI-STEP PROBLEM Use the map shown and the following information. A town planner is trying to decide whether a new household X should be covered by fire station A, B, or C.

a. Trace the map and draw the segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

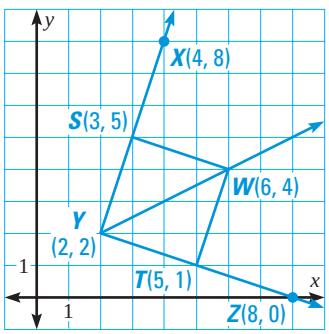
b. Construct the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. Do the perpendicular bisectors meet at a point?

c. The perpendicular bisectors divide the town into regions. Shade the region closest to fire station A red. Shade the region closest to fire station B blue. Shade the region closest to fire station C gray.

d. *Writing* In an emergency at household X, which fire station should respond? Explain your choice.

★ Challenge

USING ALGEBRA Use the graph at the right.



38. Use slopes to show that $\overline{WS} \perp \overrightarrow{YX}$ and that $\overline{WT} \perp \overrightarrow{YZ}$.

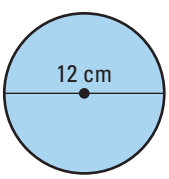
39. Find WS and WT.

40. Explain how you know that $\overrightarrow{YW}$ bisects $\angle XYZ$.

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

CIRCLES Find the missing measurement for the circle shown. Use 3.14 as an approximation for π. (Review 1.7 for 5.2)



41. radius **42.** circumference **43.** area

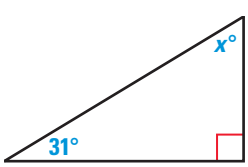
CALCULATING SLOPE Find the slope of the line that passes through the given points. (Review 3.6)

44. $A(-1, 5)$, $B(-2, 10)$ **45.** $C(4, -3)$, $D(-6, 5)$ **46.** $E(4, 5)$, $F(9, 5)$

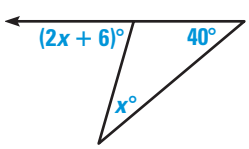
47. $G(0, 8)$, $H(-7, 0)$ **48.** $J(3, 11)$, $K(-10, 12)$ **49.** $L(-3, -8)$, $M(8, -8)$

USING ALGEBRA Find the value of x. (Review 4.1)

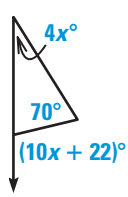
50.



51.



52.



5.2 Bisectors of a Triangle

What you should learn

GOAL 1 Use properties of perpendicular bisectors of a triangle, as applied in **Example 1**.

GOAL 2 Use properties of angle bisectors of a triangle.

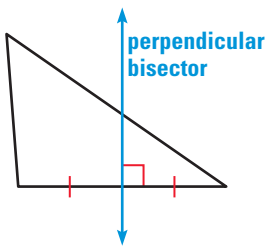
Why you should learn it

▼ To solve **real-life** problems, such as finding the center of a mushroom ring in **Exs. 24–26**.

REAL LIFE

GOAL 1 USING PERPENDICULAR BISECTORS OF A TRIANGLE

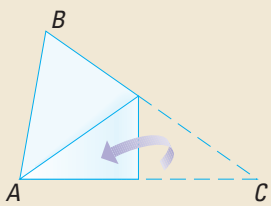
In Lesson 5.1, you studied properties of perpendicular bisectors of segments and angle bisectors. In this lesson, you will study the special cases in which the segments and angles being bisected are parts of a triangle.



A **perpendicular bisector of a triangle** is a line (or ray or segment) that is perpendicular to a side of the triangle at the midpoint of the side.

ACTIVITY Developing Concepts

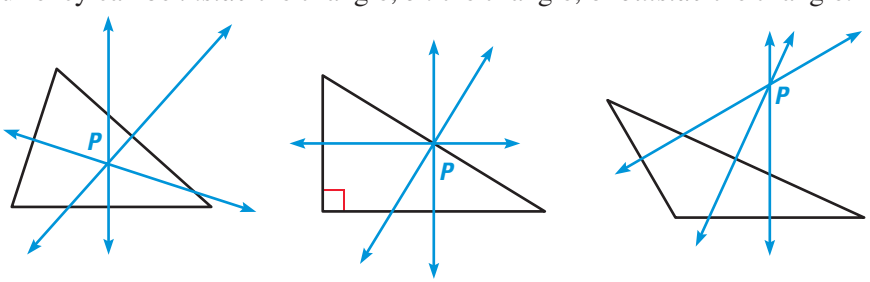
Perpendicular Bisectors of a Triangle



1. Cut four large acute scalene triangles out of paper. Make each one different.
2. Choose one triangle. Fold the triangle to form the perpendicular bisectors of the sides. Do the three bisectors intersect at the same point?
3. Repeat the process for the other three triangles. What do you observe? Write your observation in the form of a conjecture.
4. Choose one triangle. Label the vertices A, B, and C. Label the point of intersection of the perpendicular bisectors as P. Measure $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$. What do you observe?

When three or more lines (or rays or segments) intersect in the same point, they are called **concurrent lines** (or rays or segments). The point of intersection of the lines is called the **point of concurrency**.

The three perpendicular bisectors of a triangle are concurrent. The point of concurrency can be *inside* the triangle, *on* the triangle, or *outside* the triangle.



acute triangle — right triangle — obtuse triangle

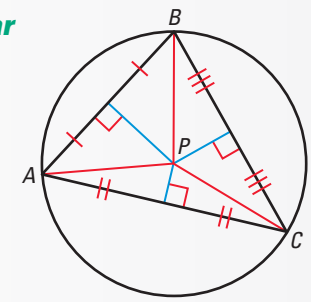
The point of concurrency of the perpendicular bisectors of a triangle is called the **circumcenter of the triangle.** In each triangle at the bottom of page 272, the circumcenter is at P. The circumcenter of a triangle has a special property, as described in Theorem 5.5. You will use coordinate geometry to illustrate this theorem in Exercises 29–31. A proof appears on page 835.

THEOREM

THEOREM 5.5 ***Concurrency of Perpendicular Bisectors of a Triangle***

The perpendicular bisectors of a triangle intersect at a point that is equidistant from the vertices of the triangle.

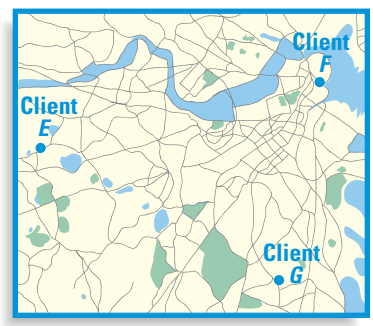
$$PA = PB = PC$$



The diagram for Theorem 5.5 shows that the circumcenter is the center of the circle that passes through the vertices of the triangle. The circle is *circumscribed* about $\triangle ABC$. Thus, the radius of this circle is the distance from the center to any of the vertices.

EXAMPLE 1 *Using Perpendicular Bisectors*

REAL LIFE **FACILITIES PLANNING** A company plans to build a distribution center that is convenient to three of its major clients. The planners start by roughly locating the three clients on a sketch and finding the circumcenter of the triangle formed.



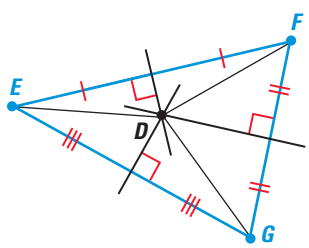
a. Explain why using the circumcenter as the location of a distribution center would be convenient for all the clients.

b. Make a sketch of the triangle formed by the clients. Locate the circumcenter of the triangle. Tell what segments are congruent.

SOLUTION

a. Because the circumcenter is equidistant from the three vertices, each client would be equally close to the distribution center.

b. Label the vertices of the triangle as E, F, and G. Draw the perpendicular bisectors. Label their intersection as D.



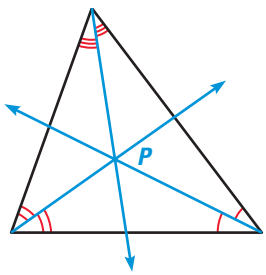
▶ By Theorem 5.5, $DE = DF = DG$.

STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 USING ANGLE BISECTORS OF A TRIANGLE

An **angle bisector of a triangle** is a bisector of an angle of the triangle. The three angle bisectors are concurrent. The point of concurrency of the angle bisectors is called the **incenter of the triangle**, and it always lies inside the triangle. The incenter has a special property that is described below in Theorem 5.6. Exercise 22 asks you to write a proof of this theorem.

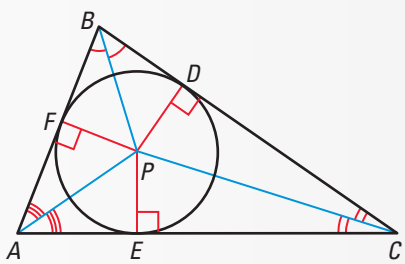


THEOREM

THEOREM 5.6 ***Concurrency of Angle Bisectors of a Triangle***

The angle bisectors of a triangle intersect at a point that is equidistant from the sides of the triangle.

$$PD = PE = PF$$



The diagram for Theorem 5.6 shows that the incenter is the center of the circle that touches each side of the triangle once. The circle is *inscribed* within $\triangle ABC$. Thus, the radius of this circle is the distance from the center to any of the sides.

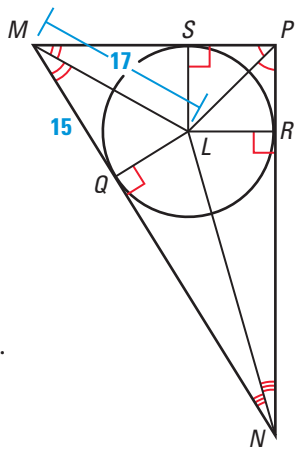
EXAMPLE 2 *Using Angle Bisectors*

Logical Reasoning

The angle bisectors of $\triangle MNP$ meet at point L.

a. What segments are congruent?

b. Find LQ and LR.



SOLUTION

a. By Theorem 5.6, the three angle bisectors of a triangle intersect at a point that is equidistant from the sides of the triangle. So, $\overline{LR} \cong \overline{LQ} \cong \overline{LS}$.

b. Use the Pythagorean Theorem to find LQ in $\triangle LQM$.

$$(LQ)^2 + (MQ)^2 = (LM)^2$$

$$(LQ)^2 + 15^2 = 17^2 \quad \textbf{Substitute.}$$

$$(LQ)^2 + 225 = 289 \quad \textbf{Multiply.}$$

$$(LQ)^2 = 64 \quad \textbf{Subtract 225 from each side.}$$

$$LQ = 8 \quad \textbf{Find the positive square root.}$$

▶ So, $LQ = 8$ units. Because $\overline{LR} \cong \overline{LQ}$, $LR = 8$ units.

STUDENT HELP

Look Back
For help with the Pythagorean Theorem, see p. 20.

GUIDED PRACTICE

Vocabulary Check ✓

1. If three or more lines intersect at the same point, the lines are __?__.

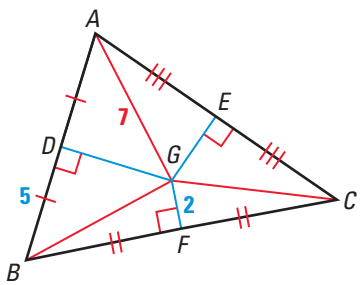
Concept Check ✓

2. Think of something about the words *incenter* and *circumcenter* that you can use to remember which special parts of a triangle meet at each point.

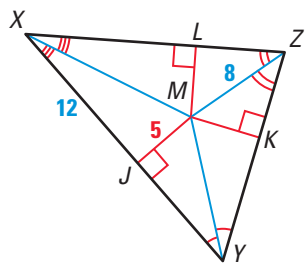
Skill Check ✓

Use the diagram and the given information to find the indicated measure.

3. The perpendicular bisectors of $\triangle ABC$ meet at point G. Find GC.



4. The angle bisectors of $\triangle XYZ$ meet at point M. Find MK.



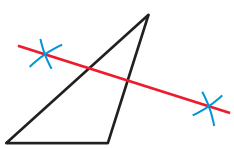
PRACTICE AND APPLICATIONS

STUDENT HELP

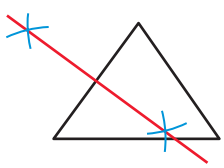
Extra Practice to help you master skills is on p. 811.

CONSTRUCTION **Draw a large example of the given type of triangle. Construct perpendicular bisectors of the sides. (See page 264.) For the type of triangle, do the bisectors intersect *inside, on,* or *outside* the triangle?**

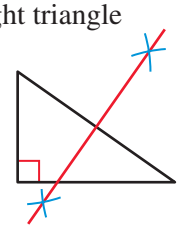
5. obtuse triangle



6. acute triangle



7. right triangle



DRAWING CONCLUSIONS **Draw a large $\triangle ABC$.**

8. Construct the angle bisectors of $\triangle ABC$. Label the point where the angle bisectors meet as D.

9. Construct perpendicular segments from D to each of the sides of the triangle. Measure each segment. What do you notice? Which theorem have you just confirmed?

LOGICAL REASONING **Use the results of Exercises 5–9 to complete the statement using *always, sometimes,* or *never*.**

10. A perpendicular bisector of a triangle __?__ passes through the midpoint of a side of the triangle.

11. The angle bisectors of a triangle __?__ intersect at a single point.

12. The angle bisectors of a triangle __?__ meet at a point outside the triangle.

13. The circumcenter of a triangle __?__ lies outside the triangle.

STUDENT HELP

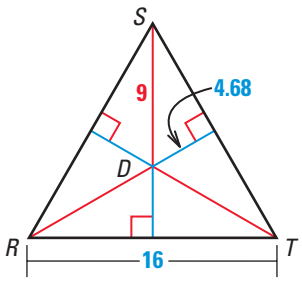
HOMEWORK HELP

Example 1: Exs. 5–7, 10–13, 14, 17, 20, 21

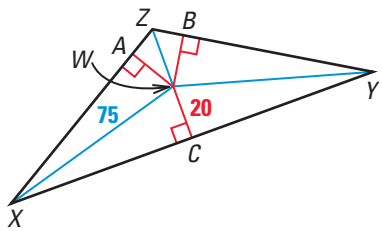
Example 2: Exs. 8, 9, 10–13, 15, 16, 22

BISECTORS **In each case, find the indicated measure.**

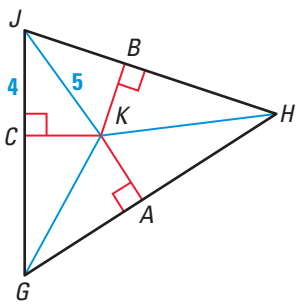
14. The perpendicular bisectors of $\triangle RST$ meet at point D. Find DR.



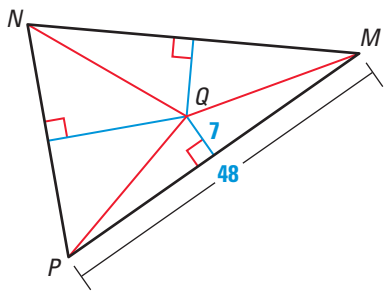
15. The angle bisectors of $\triangle XYZ$ meet at point W. Find WB.



16. The angle bisectors of $\triangle GHJ$ meet at point K. Find KB.

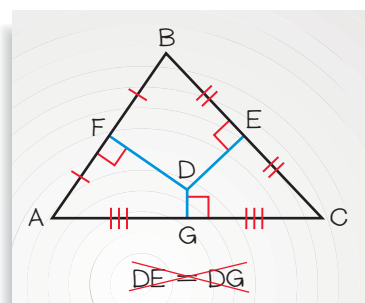


17. The perpendicular bisectors of $\triangle MNP$ meet at point Q. Find QN.



ERROR ANALYSIS **Explain why the student's conclusion is *false*. Then state a correct conclusion that can be deduced from the diagram.**

18.



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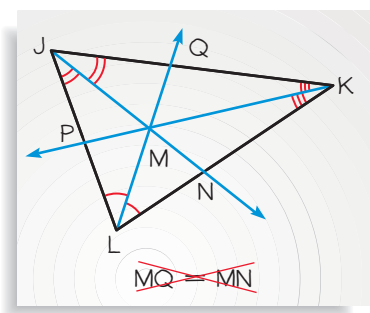
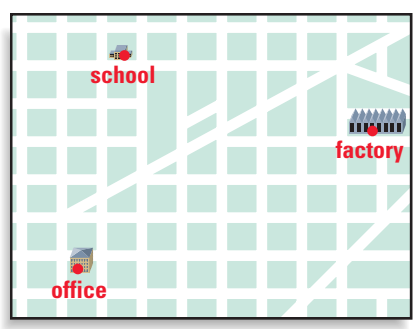


LOGICAL REASONING **In Exercises 20 and 21, use the following information and map.**

Your family is considering moving to a new home. The diagram shows the locations of where your parents work and where you go to school. The locations form a triangle.



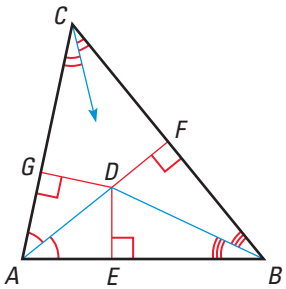
20. In the diagram, how could you find a point that is equidistant from each location? Explain your answer.

21. Make a sketch of the situation. Find the best location for the new home.

22. **DEVELOPING PROOF** Complete the proof of Theorem 5.6, the Concurrency of Angle Bisectors.

GIVEN ▶ $\triangle ABC$, the bisectors of $\angle A$, $\angle B$, and $\angle C$, $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{BC}$, $\overline{DG} \perp \overline{CA}$

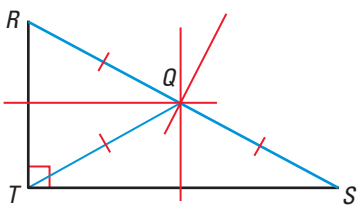
PROVE ▶ The angle bisectors intersect at a point that is equidistant from $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.



Plan for Proof Show that D, the point of intersection of the bisectors of $\angle A$ and $\angle B$, also lies on the bisector of $\angle C$. Then show that D is equidistant from the sides of the triangle.

Statements	Reasons
1. $\triangle ABC$, the bisectors of $\angle A$, $\angle B$, and $\angle C$, $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{BC}$, $\overline{DG} \perp \overline{CA}$	**1.** Given
2. __?__ $= DG$	**2.** $\overrightarrow{AD}$ bisects $\angle BAC$, so D is __?__ from the sides of $\angle BAC$.
3. $DE = DF$	**3.** __?__
4. $DF = DG$	**4.** __?__
5. D is on the __?__ of $\angle C$.	**5.** Converse of the Angle Bisector Theorem
6. __?__	**6.** Givens and Steps __?__

23. ***Writing*** Joannie thinks that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle. Explain how she could use perpendicular bisectors to verify her conjecture.

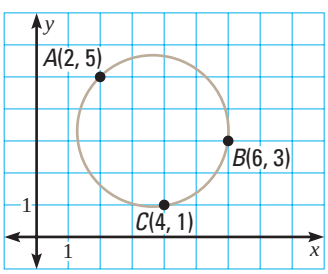


FOCUS ON APPLICATIONS

REAL LIFE **MUSHROOMS** live for only a few days. As the mycelium spreads outward, new mushroom rings are formed. A mushroom ring in France is almost half a mile in diameter and is about 700 years old.

SCIENCE CONNECTION **In Exercises 24–26, use the following information.** A *mycelium* fungus grows underground in all directions from a central point. Under certain conditions, mushrooms sprout up in a ring at the edge. The radius of the mushroom ring is an indication of the mycelium's age.

24. Suppose three mushrooms in a mushroom ring are located as shown. Make a large copy of the diagram and draw $\triangle ABC$. Each unit on your coordinate grid should represent 1 foot.



25. Draw perpendicular bisectors on your diagram to find the center of the mushroom ring. Estimate the radius of the ring.

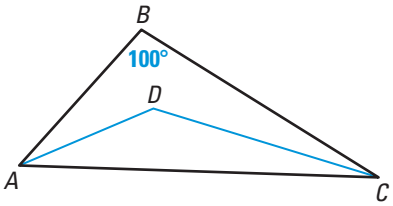
26. Suppose the radius of the mycelium increases at a rate of about 8 inches per year. Estimate its age.

Test Preparation

MULTIPLE CHOICE **Choose the correct answer from the list given.**

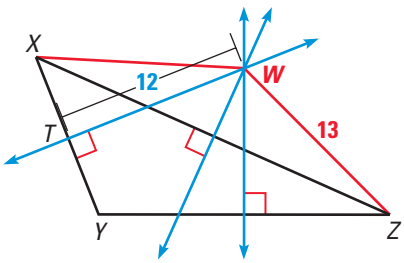
27. $\overline{AD}$ and $\overline{CD}$ are angle bisectors of $\triangle ABC$ and $m\angle ABC = 100°$. Find $m\angle ADC$.

Ⓐ 80° Ⓑ 90° Ⓒ 100°
Ⓓ 120° Ⓔ 140°



28. The perpendicular bisectors of $\triangle XYZ$ intersect at point W, $WT = 12$, and $WZ = 13$. Find XY.

Ⓐ 5 Ⓑ 8 Ⓒ 10
Ⓓ 12 Ⓔ 13



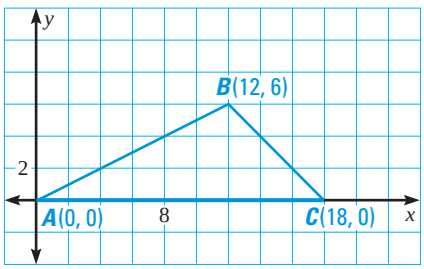
★ Challenge

USING ALGEBRA **Use the graph of $\triangle ABC$ to illustrate Theorem 5.5, the Concurrency of Perpendicular Bisectors.**

29. Find the midpoint of each side of $\triangle ABC$. Use the midpoints to find the equations of the perpendicular bisectors of $\triangle ABC$.

30. Using your equations from Exercise 29, find the intersection of two of the lines. Show that the point is on the third line.

31. Show that the point in Exercise 30 is equidistant from the vertices of $\triangle ABC$.



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

FINDING AREAS **Find the area of the triangle described.** (Review 1.7 for 5.3)

32. base = 9, height = 5

33. base = 22, height = 7

WRITING EQUATIONS **The line with the given equation is perpendicular to line *j* at point *P*. Write an equation of line *j*.** (Review 3.7)

34. $y = 3x - 2$, $P(1, 4)$

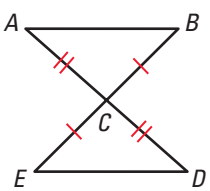
35. $y = -2x + 5$, $P(7, 6)$

36. $y = -\frac{2}{3}x - 1$, $P(2, 8)$

37. $y = \frac{10}{11}x + 3$, $P(-2, -9)$

LOGICAL REASONING **Decide whether enough information is given to prove that the triangles are congruent. If there is enough information, tell which congruence postulate or theorem you would use.** (Review 4.3, 4.4, and 4.6)

38.



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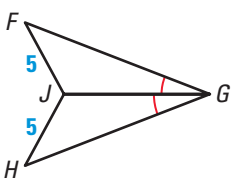

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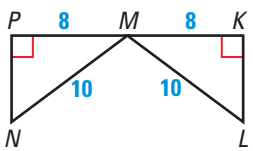


5.3 Medians and Altitudes of a Triangle

What you should learn

GOAL 1 Use properties of medians of a triangle.

GOAL 2 Use properties of altitudes of a triangle.

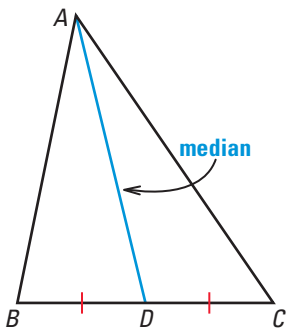
Why you should learn it

▼ To solve **real-life** problems, such as locating points in a triangle used to measure a person's heart fitness as in **Exs. 30–33**.

REAL LIFE

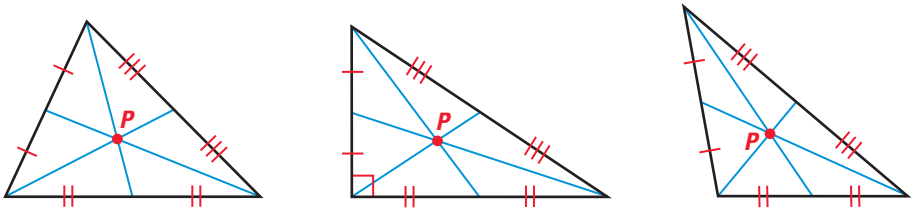
GOAL 1 USING MEDIANS OF A TRIANGLE

In Lesson 5.2, you studied two special types of segments of a triangle: perpendicular bisectors of the sides and angle bisectors. In this lesson, you will study two other special types of segments of a triangle: *medians* and *altitudes*.



A **median of a triangle** is a segment whose endpoints are a vertex of the triangle and the midpoint of the opposite side. For instance, in $\triangle ABC$ shown at the right, D is the midpoint of side $\overline{BC}$. So, $\overline{AD}$ is a median of the triangle.

The three medians of a triangle are concurrent. The point of concurrency is called the **centroid of the triangle**. The centroid, labeled P in the diagrams below, is always inside the triangle.



acute triangle **right triangle** **obtuse triangle**

The medians of a triangle have a special concurrency property, as described in Theorem 5.7. Exercises 13–16 ask you to use paper folding to demonstrate the relationships in this theorem. A proof appears on pages 836–837.

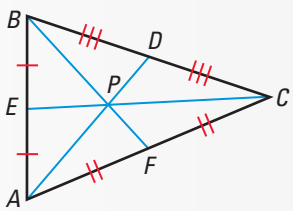
THEOREM

THEOREM 5.7 ***Concurrency of Medians of a Triangle***

The medians of a triangle intersect at a point that is two thirds of the distance from each vertex to the midpoint of the opposite side.

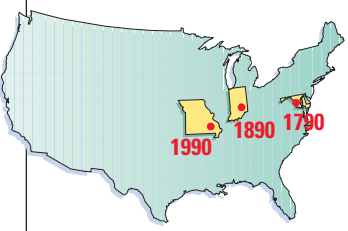
If P is the centroid of $\triangle ABC$, then

$AP = \frac{2}{3}AD$, $BP = \frac{2}{3}BF$, and $CP = \frac{2}{3}CE$.



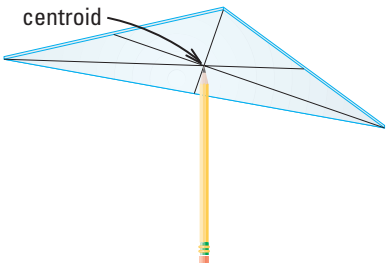
The centroid of a triangle can be used as its balancing point, as shown on the next page.

FOCUS ON APPLICATIONS



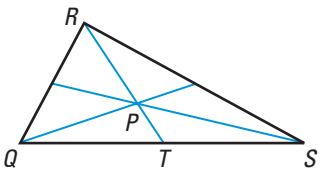
REAL LIFE — CENTER OF POPULATION
Suppose the location of each person counted in a census is identified by a weight placed on a flat, weightless map of the United States. The map would balance at a point that is the center of the population. This center has been moving westward over time.

A triangular model of uniform thickness and density will balance at the centroid of the triangle. For instance, in the diagram shown at the right, the triangular model will balance if the tip of a pencil is placed at its centroid.



EXAMPLE 1 *Using the Centroid of a Triangle*

P is the centroid of $\triangle QRS$ shown below and $PT = 5$. Find RT and RP.



SOLUTION

Because P is the centroid, $RP = \frac{2}{3}RT$.

Then $PT = RT - RP = \frac{1}{3}RT$.

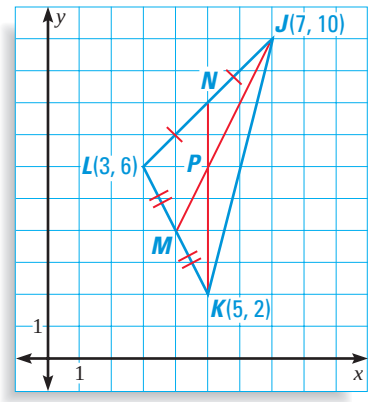
Substituting 5 for PT, $5 = \frac{1}{3}RT$, so $RT = 15$.

Then $RP = \frac{2}{3}RT = \frac{2}{3}(15) = 10$.

▶ So, $RP = 10$ and $RT = 15$.

EXAMPLE 2 *Finding the Centroid of a Triangle*

Find the coordinates of the centroid of $\triangle JKL$.



SOLUTION

You know that the centroid is two thirds of the distance from each vertex to the midpoint of the opposite side.

Choose the median $\overline{KN}$. Find the coordinates of N, the midpoint of $\overline{JL}$. The coordinates of N are

$$\left(\frac{3+7}{2}, \frac{6+10}{2}\right) = \left(\frac{10}{2}, \frac{16}{2}\right) = (5, 8).$$

Find the distance from vertex K to midpoint N. The distance from $K(5, 2)$ to $N(5, 8)$ is $8 - 2$, or 6 units.

Determine the coordinates of the centroid, which is $\frac{2}{3} \cdot 6$, or **4** units up from vertex K along the median $\overline{KN}$.

▶ The coordinates of centroid P are $(5, 2 + \mathbf{4})$, or $(5, 6)$.

STUDENT HELP

INTERNET — HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

Exercises 21–23 ask you to use the Distance Formula to confirm that the distance from vertex J to the centroid P in Example 2 is two thirds of the distance from J to M, the midpoint of the opposite side.

GOAL 2 USING ALTITUDES OF A TRIANGLE

An **altitude of a triangle** is the perpendicular segment from a vertex to the opposite side or to the line that contains the opposite side. An altitude can lie inside, on, or outside the triangle.

Every triangle has three altitudes. The lines containing the altitudes are concurrent and intersect at a point called the **orthocenter of the triangle**.

EXAMPLE 3 *Drawing Altitudes and Orthocenters*

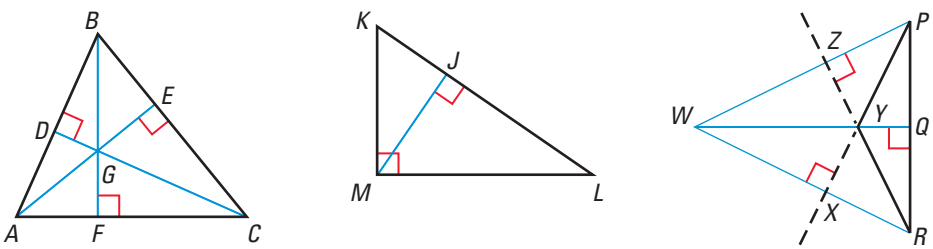
Logical Reasoning

Where is the orthocenter located in each type of triangle?

a. Acute triangle **b.** Right triangle **c.** Obtuse triangle

SOLUTION

Draw an example of each type of triangle and locate its orthocenter.



a. $\triangle ABC$ is an acute triangle. The three altitudes intersect at G, a point *inside* the triangle.

b. $\triangle KLM$ is a right triangle. The two legs, $\overline{LM}$ and $\overline{KM}$, are also altitudes. They intersect at the triangle's right angle. This implies that the orthocenter is *on* the triangle at M, the vertex of the right angle of the triangle.

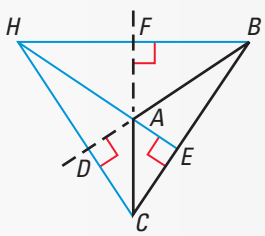
c. $\triangle YPR$ is an obtuse triangle. The three lines that contain the altitudes intersect at W, a point that is *outside* the triangle.

THEOREM

THEOREM 5.8 ***Concurrency of Altitudes of a Triangle***

The lines containing the altitudes of a triangle are concurrent.

If $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ are the altitudes of $\triangle ABC$, then the lines $\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$, and $\overleftrightarrow{CD}$ intersect at some point H.



Exercises 24–26 ask you to use construction to verify Theorem 5.8. A proof appears on page 838.

GUIDED PRACTICE

Vocabulary Check ✓

1. The *centroid* of a triangle is the point where the three __?__ intersect.

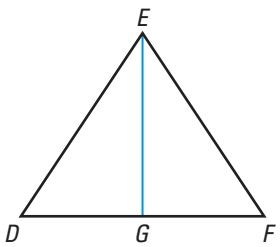
Concept Check ✓

2. In Example 3 on page 281, explain why the two legs of the right triangle in part (b) are also altitudes of the triangle.

Skill Check ✓

Use the diagram shown and the given information to decide in each case whether $\overline{EG}$ is a *perpendicular bisector*, an *angle bisector*, a *median*, or an *altitude* of $\triangle DEF$.

3. $\overline{DG} \cong \overline{FG}$
4. $\overline{EG} \perp \overline{DF}$
5. $\angle DEG \cong \angle FEG$
6. $\overline{EG} \perp \overline{DF}$ and $\overline{DG} \cong \overline{FG}$
7. $\triangle DGE \cong \triangle FGE$



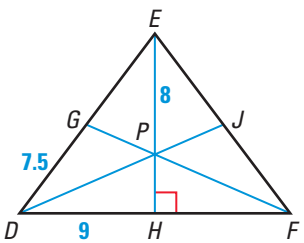
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 811.

USING MEDIANS OF A TRIANGLE **In Exercises 8–12, use the figure below and the given information.**

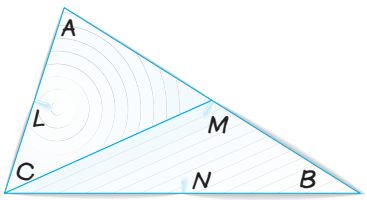
P is the centroid of $\triangle DEF$, $\overline{EH} \perp \overline{DF}$, $DH = 9$, $DG = 7.5$, $EP = 8$, and $DE = FE$.



8. Find the length of $\overline{FH}$.
9. Find the length of $\overline{EH}$.
10. Find the length of $\overline{PH}$.
11. Find the perimeter of $\triangle DEF$.
12. **LOGICAL REASONING** In the diagram of $\triangle DEF$ above, $\frac{EP}{EH} = \frac{2}{3}$. Find $\frac{PH}{EH}$ and $\frac{PH}{EP}$.

PAPER FOLDING **Cut out a large acute, right, or obtuse triangle. Label the vertices. Follow the steps in Exercises 13–16 to verify Theorem 5.7.**

13. Fold the sides to locate the midpoint of each side. Label the midpoints.
14. Fold to form the median from each vertex to the midpoint of the opposite side.
15. Did your medians meet at about the same point? If so, label this centroid point.
16. Verify that the distance from the centroid to a vertex is two thirds of the distance from that vertex to the midpoint of the opposite side.

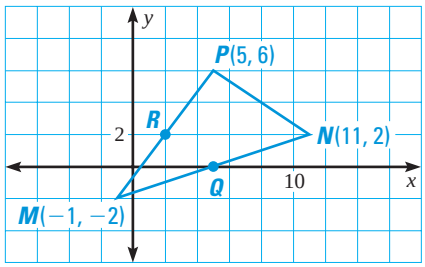


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 8–11, 13–16
Example 2: Exs. 17–23
Example 3: Exs. 24–26

USING ALGEBRA **Use the graph shown.**



17. Find the coordinates of Q, the midpoint of $\overline{MN}$.

18. Find the length of the median $\overline{PQ}$.

19. Find the coordinates of the centroid. Label this point as T.

20. Find the coordinates of R, the midpoint of $\overline{MP}$. Show that the quotient $\frac{NT}{NR}$ is $\frac{2}{3}$.

USING ALGEBRA **Refer back to Example 2 on page 280.**

21. Find the coordinates of M, the midpoint of $\overline{KL}$.

22. Use the Distance Formula to find the lengths of $\overline{JP}$ and $\overline{JM}$.

23. Verify that $JP = \frac{2}{3}JM$.

STUDENT HELP

Look Back
To construct an altitude, use the construction of a perpendicular to a line through a point not on the line, as shown on p. 130.

CONSTRUCTION **Draw and label a large scalene triangle of the given type and construct the altitudes. Verify Theorem 5.8 by showing that the lines containing the altitudes are concurrent, and label the orthocenter.**

24. an acute $\triangle ABC$

25. a right $\triangle EFG$ with right angle at G

26. an obtuse $\triangle KLM$

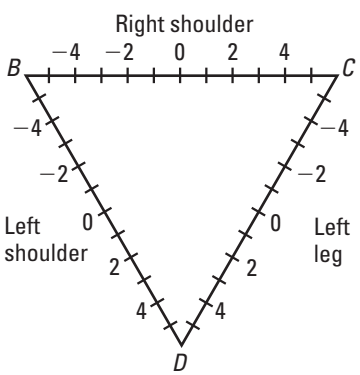
TECHNOLOGY **Use geometry software to draw a triangle. Label the vertices as *A*, *B*, and *C*.**

27. Construct the altitudes of $\triangle ABC$ by drawing perpendicular lines through each side to the opposite vertex. Label them $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.

28. Find and label G and H, the intersections of $\overline{AD}$ and $\overline{BE}$ and of $\overline{BE}$ and $\overline{CF}$.

29. Prove that the altitudes are concurrent by showing that $GH = 0$.

ELECTROCARDIOGRAPH **In Exercises 30–33, use the following information about electrocardiographs.**
The equilateral triangle $\triangle BCD$ is used to plot electrocardiograph readings. Consider a person who has a left shoulder reading (S) of -1, a right shoulder reading (R) of 2, and a left leg reading (L) of 3.



30. On a large copy of $\triangle BCD$, plot the reading to form the vertices of $\triangle SRL$. (This triangle is an *Einthoven's Triangle*, named for the inventor of the electrocardiograph.)

31. Construct the circumcenter M of $\triangle SRL$.

32. Construct the centroid P of $\triangle SRL$. Draw line r through P parallel to $\overline{BC}$.

33. Estimate the measure of the acute angle between line r and $\overline{MP}$. Cardiologists call this the angle of a person's heart.

FOCUS ON CAREERS

REAL LIFE CARDIOLOGY TECHNICIAN
Technicians use equipment like electrocardiographs to test, monitor, and evaluate heart function.

INTERNET CAREER LINK
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Test Preparation

34. MULTI-STEP PROBLEM Recall the formula for the area of a triangle, $A = \frac{1}{2}bh$, where b is the length of the base and h is the height. The height of a triangle is the length of an altitude.

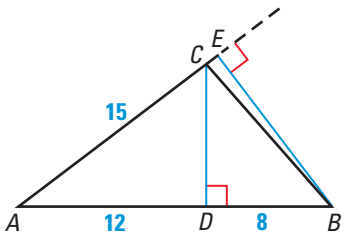
a. Make a sketch of $\triangle ABC$. Find CD, the height of the triangle (the length of the altitude to side $\overline{AB}$).

b. Use CD and AB to find the area of $\triangle ABC$.

c. Draw $\overline{BE}$, the altitude to the line containing side $\overline{AC}$.

d. Use the results of part (b) to find the length of $\overline{BE}$.

e. *Writing* Write a formula for the length of an altitude in terms of the base and the area of the triangle. Explain.

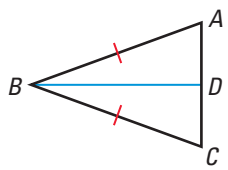


★ Challenge

SPECIAL TRIANGLES **Use the diagram at the right.**

35. GIVEN ▶ $\triangle ABC$ is isosceles. $\overline{BD}$ is a median to base $\overline{AC}$.

PROVE ▶ $\overline{BD}$ is also an altitude.



36. Are the medians to the *legs* of an isosceles triangle also altitudes? Explain your reasoning.

37. Are the medians of an *equilateral* triangle also altitudes? Are they contained in the angle bisectors? Are they contained in the perpendicular bisectors?

EXTRA CHALLENGE
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38. LOGICAL REASONING In a proof, if you are given a median of an equilateral triangle, what else can you conclude about the segment?

MIXED REVIEW

USING ALGEBRA **Write an equation of the line that passes through point *P* and is parallel to the line with the given equation.** (Review 3.6 for 5.4)

39. $P(1, 7)$, $y = -x + 3$

40. $P(-3, -8)$, $y = -2x - 3$

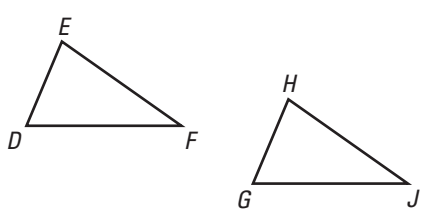
41. $P(4, -9)$, $y = 3x + 5$

42. $P(4, -2)$, $y = -\frac{1}{2}x - 1$

DEVELOPING PROOF **In Exercises 43 and 44, state the third congruence that must be given to prove that $\triangle DEF \cong \triangle GHJ$ using the indicated postulate or theorem.** (Review 4.4)

43. GIVEN ▶ $\angle D \cong \angle G$, $\overline{DF} \cong \overline{GJ}$
AAS Congruence Theorem

44. GIVEN ▶ $\angle E \cong \angle H$, $\overline{EF} \cong \overline{HJ}$
ASA Congruence Postulate

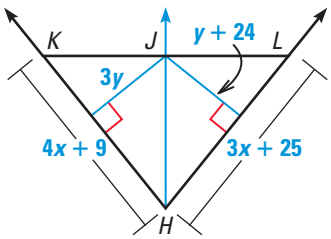


45. USING THE DISTANCE FORMULA Place a right triangle with legs of length 9 units and 13 units in a coordinate plane and use the Distance Formula to find the length of the hypotenuse. (Review 4.7)

QUIZ 1

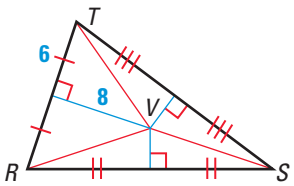
Self-Test for Lessons 5.1–5.3

Use the diagram shown and the given information. (Lesson 5.1)
$\overline{HJ}$ is the perpendicular bisector of $\overline{KL}$.
$\overrightarrow{HJ}$ bisects $\angle KHL$.

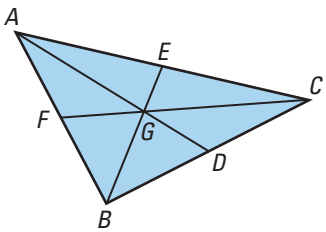


1. Find the value of x.
2. Find the value of y.

In the diagram shown, the perpendicular bisectors of $\triangle RST$ meet at V. (Lesson 5.2)



3. Find the length of $\overline{VT}$.
4. What is the length of $\overline{VS}$? Explain.
5. **BUILDING A MOBILE** Suppose you want to attach the items in a mobile so that they hang horizontally. You would want to find the balancing point of each item. For the triangular metal plate shown, describe where the balancing point would be located. (Lesson 5.3)



$\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are medians. $CF = 12$ in.

MATH & History — Optimization

APPLICATION LINK
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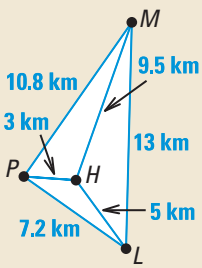
THEN

THROUGHOUT HISTORY, people have faced problems involving minimizing resources or maximizing output, a process called optimization. The use of mathematics in solving these types of problems has increased greatly since World War II, when mathematicians found the optimal shape for naval convoys to avoid enemy fire.

NOW

TODAY, with the help of computers, optimization techniques are used in many industries, including manufacturing, economics, and architecture.



1. Your house is located at point H in the diagram. You need to do errands at the post office (P), the market (M), and the library (L). In what order should you do your errands to minimize the distance traveled?
2. Look back at Exercise 34 on page 270. Explain why the goalie's position on the angle bisector optimizes the chances of blocking a scoring shot.

1611

Johannes Kepler proposes the optimal way to stack cannonballs.

WWII naval convoy

1942

1972

This Olympic stadium roof uses a minimum of materials.

Thomas Hales proves Kepler's cannonball conjecture.

1997

5.4 Midsegment Theorem

What you should learn

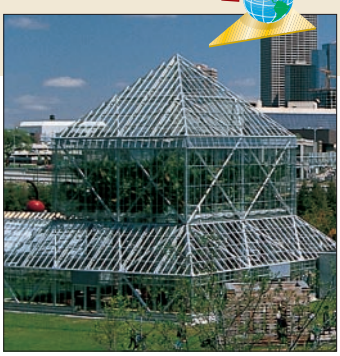
GOAL 1 Identify the midsegments of a triangle.

GOAL 2 Use properties of midsegments of a triangle.

Why you should learn it

▼ To solve **real-life** problems involving midsegments, as applied in **Exs. 32 and 35.**

REAL LIFE

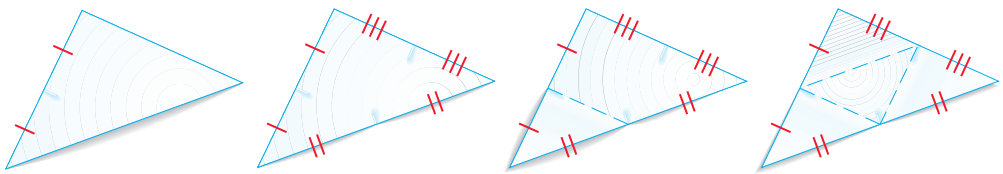


The roof of the Cowles Conservatory in Minneapolis, Minnesota, shows the midsegments of a triangle.

GOAL 1 USING MIDSEGMENTS OF A TRIANGLE

In Lessons 5.2 and 5.3, you studied four special types of segments of a triangle: perpendicular bisectors, angle bisectors, medians, and altitudes. Another special type of segment is called a *midsegment*. A **midsegment of a triangle** is a segment that connects the midpoints of two sides of a triangle.

You can form the three midsegments of a triangle by tracing the triangle on paper, cutting it out, and folding it, as shown below.

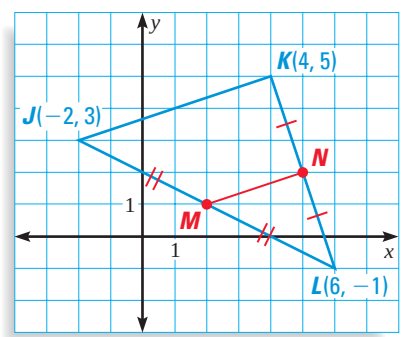


1. Fold one vertex onto another to find one midpoint.
2. Repeat the process to find the other two midpoints.
3. Fold a segment that contains two of the midpoints.
4. Fold the remaining two midsegments of the triangle.

The midsegments and sides of a triangle have a special relationship, as shown in Example 1 and Theorem 5.9 on the next page.

EXAMPLE 1 *Using Midsegments*

Show that the midsegment $\overline{MN}$ is parallel to side $\overline{JK}$ and is half as long.



SOLUTION

Use the Midpoint Formula to find the coordinates of M and N.

$$M = \left(\frac{-2+6}{2}, \frac{3+(-1)}{2}\right) = (2, 1)$$

$$N = \left(\frac{4+6}{2}, \frac{5+(-1)}{2}\right) = (5, 2)$$

Next, find the slopes of $\overline{JK}$ and $\overline{MN}$.

$$\text{Slope of } \overline{JK} = \frac{5-3}{4-(-2)} = \frac{2}{6} = \frac{1}{3} \qquad \text{Slope of } \overline{MN} = \frac{2-1}{5-2} = \frac{1}{3}$$

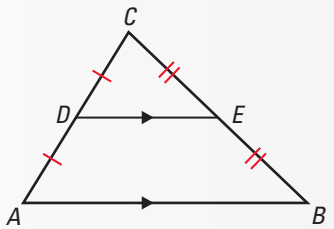
▶ Because their slopes are equal, $\overline{JK}$ and $\overline{MN}$ are parallel. You can use the Distance Formula to show that $MN = \sqrt{10}$ and $JK = \sqrt{40} = 2\sqrt{10}$. So, $\overline{MN}$ is half as long as $\overline{JK}$.

THEOREM

THEOREM 5.9 ***Midsegment Theorem***

The segment connecting the midpoints of two sides of a triangle is parallel to the third side and is half as long.

$\overline{DE} \parallel \overline{AB}$ and $DE = \frac{1}{2}AB$



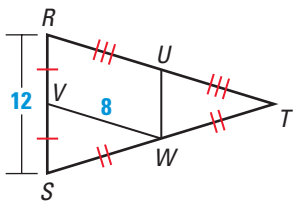
EXAMPLE 2 *Using the Midsegment Theorem*

$\overline{UW}$ and $\overline{VW}$ are midsegments of $\triangle RST$. Find UW and RT.

SOLUTION

$$UW = \frac{1}{2}(RS) = \frac{1}{2}(12) = 6$$

$$RT = 2(VW) = 2(8) = 16$$

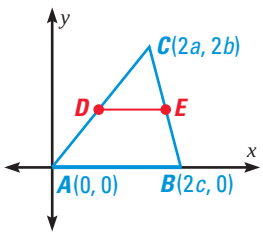


A coordinate proof of Theorem 5.9 for one midsegment of a triangle is given below. Exercises 23–25 ask for proofs about the other two midsegments. To set up a coordinate proof, remember to place the figure in a convenient location.

EXAMPLE 3 *Proving Theorem 5.9*

Proof

Write a coordinate proof of the Midsegment Theorem.



SOLUTION

Place points A, B, and C in convenient locations in a coordinate plane, as shown. Use the Midpoint Formula to find the coordinates of the midpoints D and E.

$$D = \left(\frac{2a + 0}{2}, \frac{2b + 0}{2}\right) = (a, b) \qquad E = \left(\frac{2a + 2c}{2}, \frac{2b + 0}{2}\right) = (a + c, b)$$

Find the slope of midsegment $\overline{DE}$. Points D and E have the same y-coordinates, so the slope of $\overline{DE}$ is zero.

▶ $\overline{AB}$ also has a slope of zero, so the slopes are equal and $\overline{DE}$ and $\overline{AB}$ are parallel.

Calculate the lengths of $\overline{DE}$ and $\overline{AB}$. The segments are both horizontal, so their lengths are given by the absolute values of the differences of their x-coordinates.

$$AB = |2c - 0| = 2c \qquad DE = |a + c - a| = c$$

▶ The length of $\overline{DE}$ is half the length of $\overline{AB}$.

STUDENT HELP

Study Tip

In Example 3, it is convenient to locate a vertex at (0, 0) and it also helps to make one side horizontal. To use the Midpoint Formula, it is helpful for the coordinates to be multiples of 2.

GOAL 2 USING PROPERTIES OF MIDSEGMENTS

Suppose you are given only the three midpoints of the sides of a triangle. Is it possible to draw the original triangle? Example 4 shows one method.

EXAMPLE 4 *Using Midpoints to Draw a Triangle*

Using Algebra

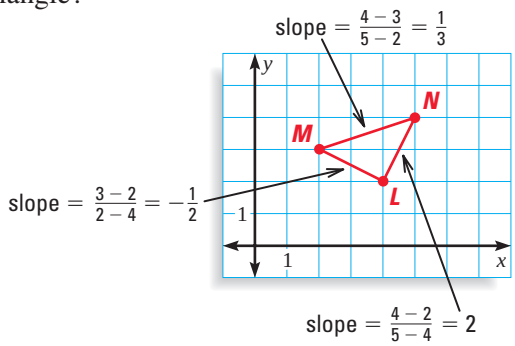
The midpoints of the sides of a triangle are $L(4, 2)$, $M(2, 3)$, and $N(5, 4)$. What are the coordinates of the vertices of the triangle?

SOLUTION

Plot the midpoints in a coordinate plane.

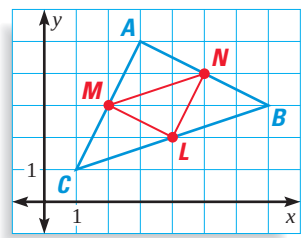
Connect these midpoints to form the midsegments $\overline{LN}$, $\overline{MN}$, and $\overline{ML}$.

Find the slopes of the midsegments. Use the slope formula as shown.



Each midsegment contains two of the unknown triangle's midpoints and is parallel to the side that contains the third midpoint. So, you know a point on each side of the triangle and the slope of each side.

Draw the lines that contain the three sides.



▶ The lines intersect at $A(3, 5)$, $B(7, 3)$, and $C(1, 1)$, which are the vertices of the triangle.

The perimeter of the triangle formed by the three midsegments of a triangle is *half* the perimeter of the original triangle, as shown in Example 5.

EXAMPLE 5 *Perimeter of Midsegment Triangle*

FOCUS ON APPLICATIONS

REAL LIFE **ORIGAMI** is an ancient method of paper folding. The pattern of folds for a number of objects, such as the flower shown, involve midsegments.

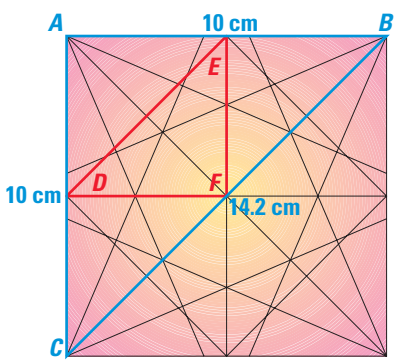
ORIGAMI $\overline{DE}$, $\overline{EF}$, and $\overline{DF}$ are midsegments in $\triangle ABC$. Find the perimeter of $\triangle DEF$.

SOLUTION The lengths of the midsegments are half the lengths of the sides of $\triangle ABC$.

$$DF = \frac{1}{2}AB = \frac{1}{2}(10) = 5$$

$$EF = \frac{1}{2}AC = \frac{1}{2}(10) = 5$$

$$ED = \frac{1}{2}BC = \frac{1}{2}(14.2) = 7.1$$



Crease pattern of origami flower

▶ The perimeter of $\triangle DEF$ is $5 + 5 + 7.1$, or 17.1. The perimeter of $\triangle ABC$ is $10 + 10 + 14.2$, or 34.2, so the perimeter of the triangle formed by the midsegments is half the perimeter of the original triangle.

GUIDED PRACTICE

Vocabulary Check ✓

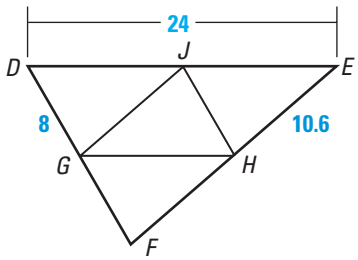
1. In $\triangle ABC$, if M is the midpoint of $\overline{AB}$, N is the midpoint of $\overline{AC}$, and P is the midpoint of $\overline{BC}$, then $\overline{MN}$, $\overline{NP}$, and $\overline{PN}$ are __?__ of $\triangle ABC$.

Concept Check ✓

2. In Example 3 on page 288, why was it convenient to position one of the sides of the triangle along the x-axis?

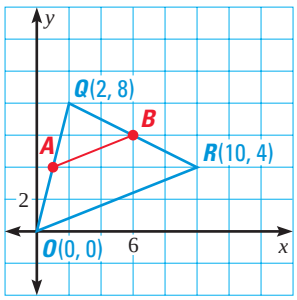
Skill Check ✓

In Exercises 3–9, $\overline{GH}$, $\overline{HJ}$, and $\overline{JG}$ are midsegments of $\triangle DEF$.



3. $\overline{JH} \parallel$ __?__
4. __?__ $\parallel \overline{DE}$
5. $EF =$ __?__
6. $GH =$ __?__
7. $DF =$ __?__
8. $JH =$ __?__
9. Find the perimeter of $\triangle GHJ$.

WALKWAYS The triangle below shows a section of walkways on a college campus.



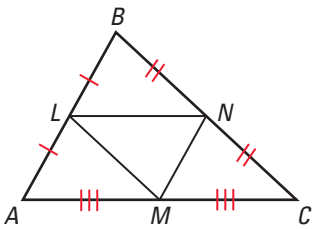
10. The midsegment $\overline{AB}$ represents a new walkway that is to be constructed on the campus. What are the coordinates of points A and B?
11. Each unit in the coordinate plane represents 10 yards. Use the Distance Formula to find the length of the new walkway.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 812.

COMPLETE THE STATEMENT In Exercises 12–19, use $\triangle ABC$, where L, M, and N are midpoints of the sides.



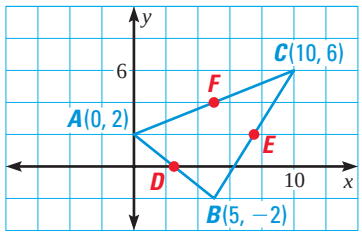
12. $\overline{LM} \parallel$ __?__
13. $\overline{AB} \parallel$ __?__
14. If $AC = 20$, then $LN =$ __?__.
15. If $MN = 7$, then $AB =$ __?__.
16. If $NC = 9$, then $LM =$ __?__.
17. **USING ALGEBRA** If $LM = 3x + 7$ and $BC = 7x + 6$, then $LM =$ __?__.
18. **USING ALGEBRA** If $MN = x - 1$ and $AB = 6x - 18$, then $AB =$ __?__.
19. **LOGICAL REASONING** Which angles in the diagram are congruent? Explain your reasoning.
20. **CONSTRUCTION** Use a straightedge to draw a triangle. Then use the straightedge and a compass to construct the three midsegments of the triangle.

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 21, 22
Example 2: Exs. 12–16
Example 3: Exs. 23–25
Example 4: Exs. 26, 27
Example 5: Exs. 28, 29

USING ALGEBRA **Use the diagram.**



21. Find the coordinates of the endpoints of each midsegment of $\triangle ABC$.

22. Use slope and the Distance Formula to verify that the Midsegment Theorem is true for $\overline{DF}$.

USING ALGEBRA **Copy the diagram in Example 3 on page 288 to complete the proof of Theorem 5.9, the Midsegment Theorem.**

23. Locate the midpoint of $\overline{AB}$ and label it F. What are the coordinates of F? Draw midsegments $\overline{DF}$ and $\overline{EF}$.

24. Use slopes to show that $\overline{DF} \parallel \overline{CB}$ and $\overline{EF} \parallel \overline{CA}$.

25. Use the Distance Formula to find DF, EF, CB, and CA. Verify that $DF = \frac{1}{2}CB$ and $EF = \frac{1}{2}CA$.

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 26 and 27.

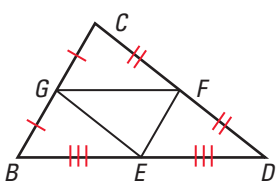
USING ALGEBRA **In Exercises 26 and 27, you are given the midpoints of the sides of a triangle. Find the coordinates of the vertices of the triangle.**

26. $L(1, 3)$, $M(5, 9)$, $N(4, 4)$

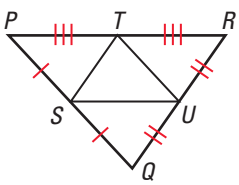
27. $L(7, 1)$, $M(9, 6)$, $N(5, 4)$

FINDING PERIMETER **In Exercises 28 and 29, use the diagram shown.**

28. Given $CD = 14$, $GF = 8$, and $GC = 5$, find the perimeter of $\triangle BCD$.



29. Given $PQ = 20$, $SU = 12$, and $QU = 9$, find the perimeter of $\triangle STU$.



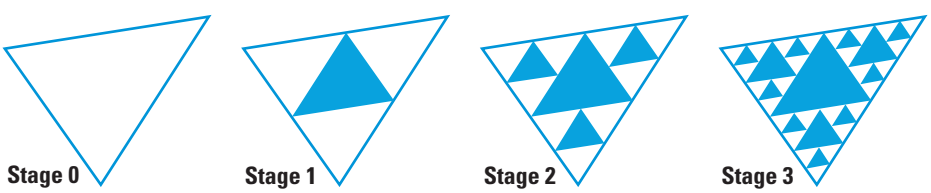
FOCUS ON APPLICATIONS

FRACTALS are shapes that look the same at many levels of magnification. Take a small part of the image above and you will see that it looks about the same as the whole image.

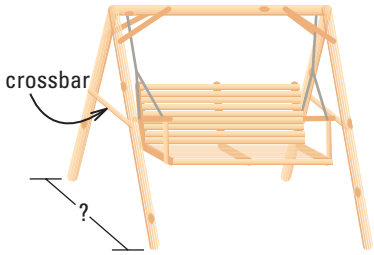
APPLICATION LINK www.mcdougallittell.com

30. **TECHNOLOGY** Use geometry software to draw any $\triangle ABC$. Construct the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. Label them as D, E, and F. Construct the midpoints of $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$. Label them as G, H, and I. What is the relationship between the perimeters of $\triangle ABC$ and $\triangle GHI$?

31. **FRACTALS** The design below, which approximates a *fractal*, is created with midsegments. Beginning with any triangle, shade the triangle formed by the three midsegments. Continue the process for each unshaded triangle. Suppose the perimeter of the original triangle is 1. What is the perimeter of the triangle that is shaded in Stage 1? What is the total perimeter of all the triangles that are shaded in Stage 2? in Stage 3?



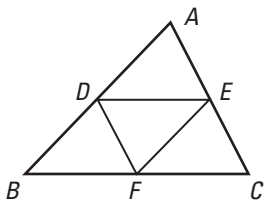
32. **PORCH SWING** You are assembling the frame for a porch swing. The horizontal crossbars in the kit you purchased are each 30 inches long. You attach the crossbars at the midpoints of the legs. At each end of the frame, how far apart will the bottoms of the legs be when the frame is assembled? Explain.



33. **WRITING A PROOF** Write a paragraph proof using the diagram shown and the given information.

GIVEN ▶ $\triangle ABC$ with midsegments $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$

PROVE ▶ $\triangle ADE \cong \triangle DBF$



Plan for Proof Use the SAS Congruence Postulate. Show that $\overline{AD} \cong \overline{DB}$. Show that because $DE = BF = \frac{1}{2}BC$, then $\overline{DE} \cong \overline{BF}$.

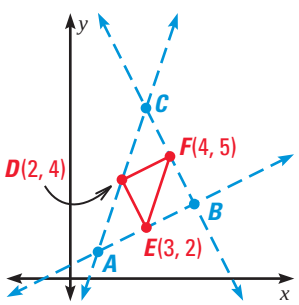
Use parallel lines to show that $\angle ADE \cong \angle ABC$.

34. **WRITING A PLAN** Using the information from Exercise 33, write a plan for a proof showing how you could use the SSS Congruence Postulate to prove that $\triangle ADE \cong \triangle DBF$.

35. **A-FRAME HOUSE** In the A-frame house shown, the floor of the second level, labeled $\overline{PQ}$, is closer to the first floor, $\overline{RS}$, than midsegment $\overline{MN}$ is. If $\overline{RS}$ is 24 feet long, can $\overline{PQ}$ be 10 feet long? 12 feet long? 14 feet long? 24 feet long? Explain.

Test Preparation

36. **MULTI-STEP PROBLEM** The diagram below shows the points $D(2, 4)$, $E(3, 2)$, and $F(4, 5)$, which are midpoints of the sides of $\triangle ABC$. The directions below show how to use equations of lines to reconstruct the original $\triangle ABC$.



a. Plot D, E, and F in a coordinate plane.

b. Find the slope m_1 of one midsegment, say $\overline{DE}$.

c. The line containing side $\overline{CB}$ will have the same slope as $\overline{DE}$. Because $\overline{CB}$ contains $F(4, 5)$, an equation of $\overleftrightarrow{CB}$ in *point-slope form* is $y - 5 = m_1(x - 4)$. Write an equation of $\overleftrightarrow{CB}$.

d. Find the slopes m_2 and m_3 of the other two midsegments. Use these slopes to find equations of the lines containing the other two sides of $\triangle ABC$.

e. Rewrite your equations from parts (c) and (d) in *slope-intercept form*.

f. Use substitution to solve systems of equations to find the intersection of each pair of lines. Plot these points A, B, and C on your graph.

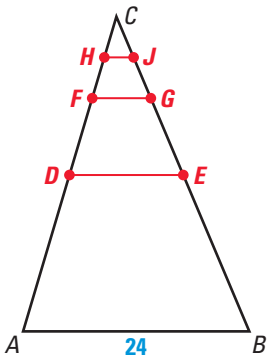
STUDENT HELP

Skills Review
For help with writing an equation of a line, see page 795.

★ Challenge

37. FINDING A PATTERN In $\triangle ABC$, the length of $\overline{AB}$ is 24. In the triangle, a succession of midsegments are formed.

- At Stage 1, draw the midsegment of $\triangle ABC$. Label it $\overline{DE}$.
- At Stage 2, draw the midsegment of $\triangle DEC$. Label it $\overline{FG}$.
- At Stage 3, draw the midsegment of $\triangle FGC$. Label it $\overline{HJ}$.



Copy and complete the table showing the length of the midsegment at each stage.

Stage n	0	1	2	3	4	5
Midsegment length	24	?	?	?	?	?

38. USING ALGEBRA In Exercise 37, let y represent the length of the midsegment at Stage n. Construct a scatter plot for the data given in the table. Then find a function that gives the length of the midsegment at Stage n.

EXTRA CHALLENGE
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MIXED REVIEW

SOLVING EQUATIONS Solve the equation and state a reason for each step. (Review 2.4)

39. $x - 3 = 11$

40. $3x + 13 = 46$

41. $8x - 1 = 2x + 17$

42. $5x + 12 = 9x - 4$

43. $2(4x - 1) = 14$

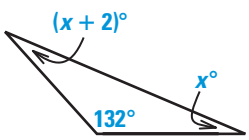
44. $9(3x + 10) = 27$

45. $-2(x + 1) + 3 = 23$

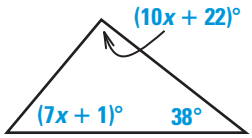
46. $3x + 2(x + 5) = 40$

USING ALGEBRA Find the value of x. (Review 4.1 for 5.5)

47.



48.



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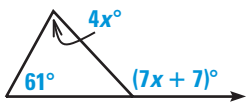


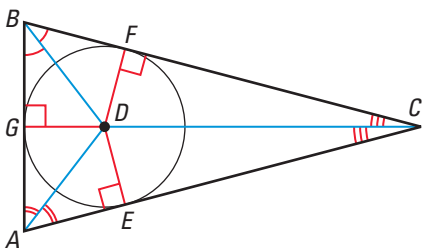
ANGLE BISECTORS $\overrightarrow{AD}$, $\overrightarrow{BD}$, and $\overrightarrow{CD}$ are angle bisectors of $\triangle ABC$. (Review 5.2)

50. Explain why $\angle CAD \cong \angle BAD$ and $\angle BCD \cong \angle ACD$.

51. Is point D the *circumcenter* or *incenter* of $\triangle ABC$?

52. Explain why $\overline{DE} \cong \overline{DG} \cong \overline{DF}$.

53. Suppose $CD = 10$ and $EC = 8$. Find DF.



5.5 Inequalities in One Triangle

What you should learn

GOAL 1 Use triangle measurements to decide which side is longest or which angle is largest, as applied in **Example 2**.

GOAL 2 Use the Triangle Inequality.

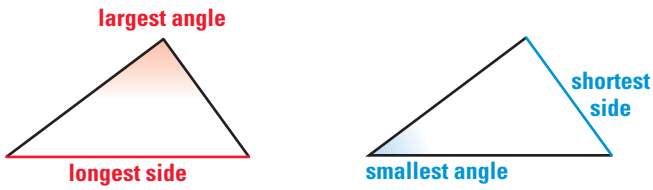
Why you should learn it

▼ To solve **real-life** problems, such as describing the motion of a crane as it clears the sediment from the mouth of a river in **Exs. 29–31**.

REAL LIFE

GOAL 1 COMPARING MEASUREMENTS OF A TRIANGLE

In Activity 5.5, you may have discovered a relationship between the positions of the longest and shortest sides of a triangle and the positions of its angles.

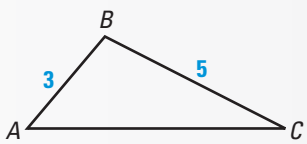


The diagrams illustrate the results stated in the theorems below.

THEOREMS

THEOREM 5.10

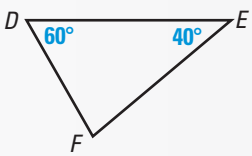
If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.



$m\angle A > m\angle C$

THEOREM 5.11

If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.



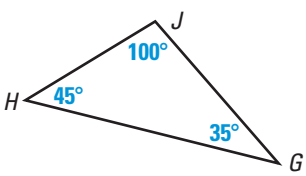
$EF > DF$

You can write the measurements of a triangle in order from least to greatest.

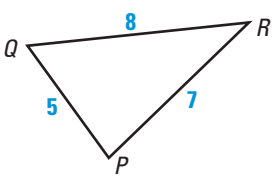
EXAMPLE 1 *Writing Measurements in Order from Least to Greatest*

Write the measurements of the triangles in order from least to greatest.

a.



b.



SOLUTION

a. $m\angle G < m\angle H < m\angle J$
$JH < JG < GH$

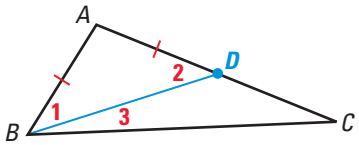
b. $QP < PR < QR$
$m\angle R < m\angle Q < m\angle P$

Theorem 5.11 will be proved in Lesson 5.6, using a technique called *indirect proof*. Theorem 5.10 can be proved using the diagram shown below.

Proof

GIVEN ▶ $AC > AB$

PROVE ▶ $m\angle ABC > m\angle C$



Paragraph Proof Use the Ruler Postulate to locate a point D on $\overline{AC}$ such that $DA = BA$. Then draw the segment $\overline{BD}$. In the isosceles triangle $\triangle ABD$, $\angle 1 \cong \angle 2$. Because $m\angle ABC = m\angle 1 + m\angle 3$, it follows that $m\angle ABC > m\angle 1$. Substituting $m\angle 2$ for $m\angle 1$ produces $m\angle ABC > m\angle 2$. Because $m\angle 2 = m\angle 3 + m\angle C$, $m\angle 2 > m\angle C$. Finally, because $m\angle ABC > m\angle 2$ and $m\angle 2 > m\angle C$, you can conclude that $m\angle ABC > m\angle C$.

. . . . . .

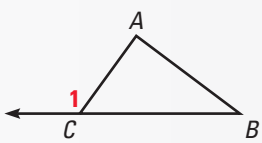
The proof of Theorem 5.10 above uses the fact that $\angle 2$ is an exterior angle for $\triangle BDC$, so its measure is the sum of the measures of the two nonadjacent interior angles. Then $m\angle 2$ must be greater than the measure of either nonadjacent interior angle. This result is stated below as Theorem 5.12.

THEOREM

THEOREM 5.12 ***Exterior Angle Inequality***

The measure of an exterior angle of a triangle is greater than the measure of either of the two nonadjacent interior angles.

$m\angle 1 > m\angle A$ and $m\angle 1 > m\angle B$

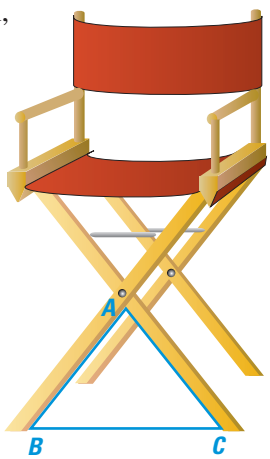


You can use Theorem 5.10 to determine possible angle measures in a chair or other real-life object.

EXAMPLE 2 *Using Theorem 5.10*

REAL LIFE

DIRECTOR'S CHAIR In the director's chair shown, $\overline{AB} \cong \overline{AC}$ and $BC > AB$. What can you conclude about the angles in $\triangle ABC$?



SOLUTION

Because $\overline{AB} \cong \overline{AC}$, $\triangle ABC$ is isosceles, so $\angle B \cong \angle C$. Therefore, $m\angle B = m\angle C$. Because $BC > AB$, $m\angle A > m\angle C$ by Theorem 5.10. By substitution, $m\angle A > m\angle B$. In addition, you can conclude that $m\angle A > 60°$, $m\angle B < 60°$, and $m\angle C < 60°$.

GOAL 2 USING THE TRIANGLE INEQUALITY

Not every group of three segments can be used to form a triangle. The lengths of the segments must fit a certain relationship.

EXAMPLE 3 *Constructing a Triangle*

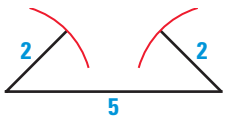
Construct a triangle with the given group of side lengths, if possible.

a. 2 cm, 2 cm, 5 cm **b.** 3 cm, 2 cm, 5 cm **c.** 4 cm, 2 cm, 5 cm

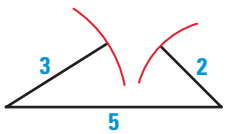
SOLUTION

Try drawing triangles with the given side lengths. Only group (c) is possible. The sum of the first and second lengths must be greater than the third length.

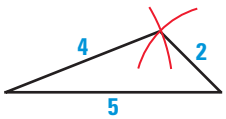
a.



b.



c.



The result of Example 3 is summarized as Theorem 5.13. Exercise 34 asks you to write a proof of this theorem.

THEOREM

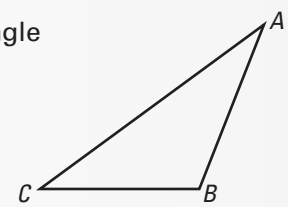
THEOREM 5.13 ***Triangle Inequality***

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

$AB + BC > AC$

$AC + BC > AB$

$AB + AC > BC$



EXAMPLE 4 *Finding Possible Side Lengths*

A triangle has one side of 10 centimeters and another of 14 centimeters. Describe the possible lengths of the third side.

SOLUTION

Let x represent the length of the third side. Using the Triangle Inequality, you can write and solve inequalities.

$x + 10 > 14$ $\qquad$ $10 + 14 > x$

$x > 4$ $\qquad$ $24 > x$

So, the length of the third side must be greater than 4 centimeters and less than 24 centimeters.

STUDENT HELP

Skills Review
For help with solving inequalities, see p. 791.

GUIDED PRACTICE

Vocabulary Check ✓

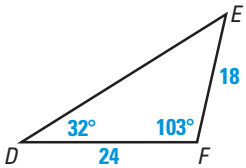
1. $\triangle ABC$ has side lengths of 1 inch, $1\frac{7}{8}$ inches, and $2\frac{1}{8}$ inches and angle measures of 90°, 28°, and 62°. Which side is *opposite* each angle?

Concept Check ✓

2. Is it possible to draw a triangle with side lengths of 5 inches, 2 inches, and 8 inches? Explain why or why not.

Skill Check ✓

In Exercises 3 and 4, use the figure shown at the right.



3. Name the smallest and largest angles of $\triangle DEF$.

4. Name the shortest and longest sides of $\triangle DEF$.

5. **GEOGRAPHY** Suppose you know the following information about distances between cities in the Philippine Islands:

   Cadiz to Masbate: 99 miles

   Cadiz to Guiuan: 165 miles

   Describe the range of possible distances from Guiuan to Masbate.

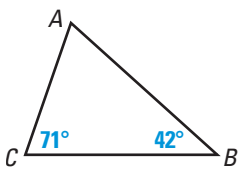
PRACTICE AND APPLICATIONS

STUDENT HELP

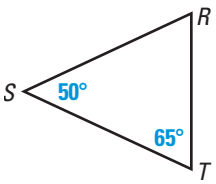
Extra Practice to help you master skills is on p. 812.

COMPARING SIDE LENGTHS **Name the shortest and longest sides of the triangle.**

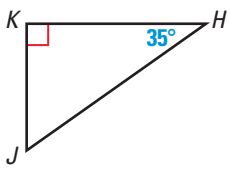
6.



7.

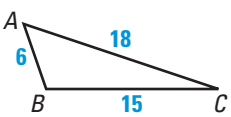


8.

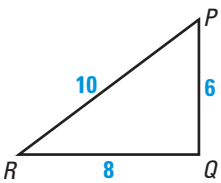


COMPARING ANGLE MEASURES **Name the smallest and largest angles of the triangle.**

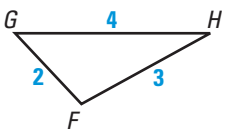
9.



10.



11.



STUDENT HELP

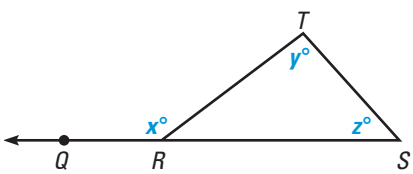
HOMEWORK HELP

Example 1: Exs. 6–19
Example 2: Exs. 6–19
Example 3: Exs. 20–23
Example 4: Exs. 24, 25

USING ALGEBRA **Use the diagram of $\triangle RST$ with exterior angle $\angle QRT$.**

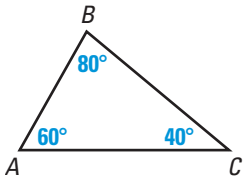
12. Write an equation about the angle measures labeled in the diagram.

13. Write two inequalities about the angle measures labeled in the diagram.

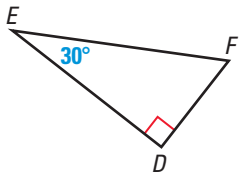


ORDERING SIDES **List the sides in order from shortest to longest.**

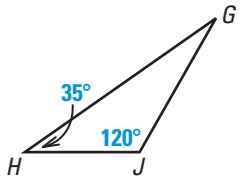
14.



15.

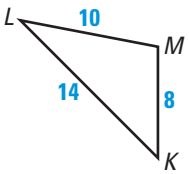


16.

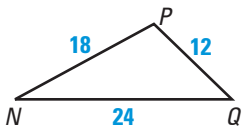


ORDERING ANGLES **List the angles in order from smallest to largest.**

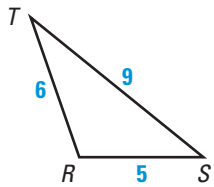
17.



18.



19.



FORMING TRIANGLES **In Exercises 20–23, you are given an 18 inch piece of wire. You want to bend the wire to form a triangle so that the length of each side is a whole number.**

20. Sketch four possible isosceles triangles and label each side length.

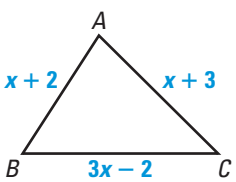
21. Sketch a possible acute scalene triangle.

22. Sketch a possible obtuse scalene triangle.

23. List three combinations of segment lengths that will not produce triangles.

USING ALGEBRA **In Exercises 24 and 25, solve the inequality** ***AB + AC > BC.***

24.



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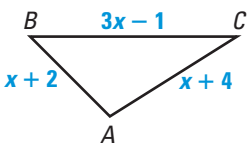
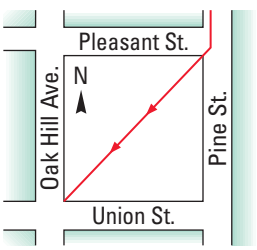


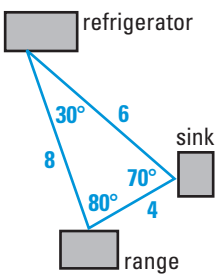
FOCUS ON APPLICATIONS

REAL LIFE WORK TRIANGLES
For ease of movement among appliances, the perimeter of an ideal kitchen work triangle should be less than 22 ft and more than 15 ft.

26. **TAKING A SHORTCUT** Look at the diagram shown. Suppose you are walking south on the sidewalk of Pine Street. When you reach Pleasant Street, you cut across the empty lot to go to the corner of Oak Hill Avenue and Union Street. Explain why this route is shorter than staying on the sidewalks.



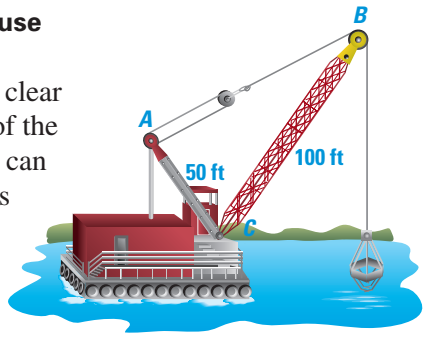
KITCHEN TRIANGLE **In Exercises 27 and 28, use the following information.**
The term "kitchen triangle" refers to the imaginary triangle formed by three kitchen appliances: the refrigerator, the sink, and the range. The distances shown are measured in feet.



27. What is wrong with the labels on the kitchen triangle?

28. Can a kitchen triangle have the following side lengths: 9 feet, 3 feet, and 5 feet? Explain why or why not.

CHANNEL DREDGING **In Exercises 29–31, use the figure shown and the given information.**
The crane is used in dredging mouths of rivers to clear out the collected debris. By adjusting the length of the boom lines from A to B, the operator of the crane can raise and lower the boom. Suppose the mast $\overline{AC}$ is 50 feet long and the boom $\overline{BC}$ is 100 feet long.



29. Is the boom *raised* or *lowered* when the boom lines are shortened?

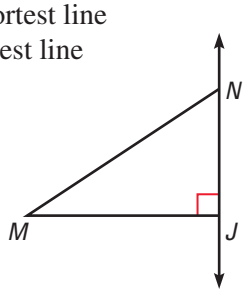
30. AB must be less than __?__ feet.

31. As the boom and shovel are raised or lowered, is $\angle ACB$ ever larger than $\angle BAC$? Explain.

32. **LOGICAL REASONING** In Example 4 on page 297, only two inequalities were needed to solve the problem. Write the third inequality. Why is that inequality not helpful in determining the range of values of x?

STUDENT HELP
HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with proof.

33. **PROOF** Prove that a perpendicular segment is the shortest line segment from a point to a line. Prove that $\overline{MJ}$ is the shortest line segment from M to $\overleftrightarrow{JN}$.

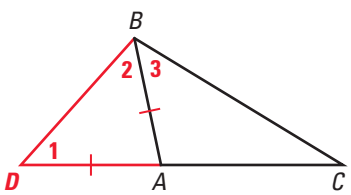


GIVEN ▶ $\overline{MJ} \perp \overleftrightarrow{JN}$

PROVE ▶ $MN > MJ$

Plan for Proof Show that $m\angle MJN > m\angle MNJ$, so $MN > MJ$.

34. **DEVELOPING PROOF** Complete the proof of Theorem 5.13, the Triangle Inequality.



GIVEN ▶ $\triangle ABC$

PROVE ▶ (1) $AB + BC > AC$
(2) $AC + BC > AB$
(3) $AB + AC > BC$

Plan for Proof One side, say $\overline{BC}$, is longer than or is at least as long as each of the other sides. Then (1) and (2) are true. The proof for (3) is as follows.

Statements	Reasons
1. $\triangle ABC$	**1.** Given
2. Extend $\overline{AC}$ to D such that $\overline{AB} \cong \overline{AD}$.	**2.** __?__
3. $AD + AC =$ __?__	**3.** Segment Addition Postulate
4. $\angle 1 \cong \angle 2$	**4.** __?__
5. $m\angle DBC >$ __?__	**5.** Protractor Postulate
6. $m\angle DBC > m\angle 1$	**6.** __?__
7. $DC > BC$	**7.** __?__
8. __?__ + __?__ $> BC$	**8.** Substitution property of equality
9. $AB + AC >$ __?__	**9.** Substitution property of equality

Test Preparation

QUANTITATIVE COMPARISON In Exercises 35–37, use the diagram to choose the statement that is true about the given quantities.

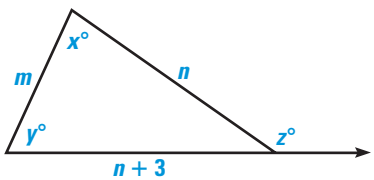
(A) The quantity in column A is greater.

(B) The quantity in column B is greater.

(C) The two quantities are equal.

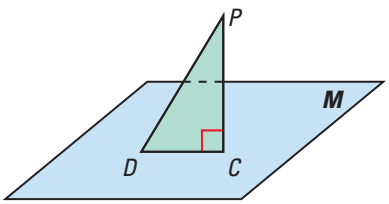
(D) The relationship cannot be determined from the given information.

	Column A	Column B
35.	x	y
36.	x	z
37.	m	n



★ Challenge

38. PROOF Use the diagram shown to prove that a perpendicular segment is the shortest segment from a point to a plane.



GIVEN ▶ $\overline{PC} \perp$ plane M

PROVE ▶ $PD > PC$

EXTRA CHALLENGE

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MIXED REVIEW

RECOGNIZING PROOFS In Exercises 39–41, look through your textbook to find an example of the type of proof. (Review Chapters 2–5 for 5.6)

39. two-column proof

40. paragraph proof

41. flow proof

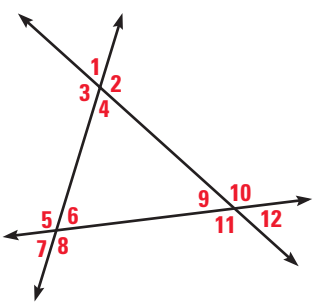
ANGLE RELATIONSHIPS Complete each statement. (Review 3.1)

42. $\angle 5$ and _?_ are corresponding angles. So are $\angle 5$ and _?_.

43. $\angle 12$ and _?_ are vertical angles.

44. $\angle 6$ and _?_ are alternate interior angles. So are $\angle 6$ and _?_.

45. $\angle 7$ and _?_ are alternate exterior angles. So are $\angle 7$ and _?_.



USING ALGEBRA In Exercises 46–49, you are given the coordinates of the midpoints of the sides of a triangle. Find the coordinates of the vertices of the triangle. (Review 5.4)

46. $L(-2, 1)$, $M(2, 3)$, $N(3, -1)$

47. $L(-3, 5)$, $M(-2, 2)$, $N(-6, 0)$

48. $L(3, 6)$, $M(9, 5)$, $N(8, 1)$

49. $L(3, -2)$, $M(0, -4)$, $N(3, -6)$

5.6 Indirect Proof and Inequalities in Two Triangles

What you should learn

GOAL 1 Read and write an indirect proof.

GOAL 2 Use the Hinge Theorem and its converse to compare side lengths and angle measures.

Why you should learn it

▼ To solve **real-life** problems, such as deciding which of two planes is farther from an airport in **Example 4** and **Exs. 28 and 29**.

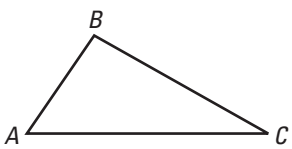
REAL LIFE

GOAL 1 USING INDIRECT PROOF

Up to now, all of the proofs in this textbook have used the Laws of Syllogism and Detachment to obtain conclusions directly. In this lesson, you will study *indirect proofs*. An **indirect proof** is a proof in which you prove that a statement is true by first assuming that its opposite is true. If this assumption leads to an impossibility, then you have proved that the original statement is true.

EXAMPLE 1 *Using Indirect Proof*

Use an indirect proof to prove that a triangle cannot have more than one obtuse angle.



SOLUTION

GIVEN ▶ $\triangle ABC$

PROVE ▶ $\triangle ABC$ does not have more than one obtuse angle.

Begin by assuming that $\triangle ABC$ *does* have more than one obtuse angle.

$m\angle A > 90°$ and $m\angle B > 90°$	**Assume $\triangle ABC$ has two obtuse angles.**
$m\angle A + m\angle B > 180°$	**Add the two given inequalities.**

You know, however, that the sum of the measures of all *three* angles is 180°.

$m\angle A + m\angle B + m\angle C = 180°$	**Triangle Sum Theorem**
$m\angle A + m\angle B = 180° - m\angle C$	**Subtraction property of equality**

So, you can substitute $180° - m\angle C$ for $m\angle A + m\angle B$ in $m\angle A + m\angle B > 180°$.

$180° - m\angle C > 180°$	**Substitution property of equality**
$0° > m\angle C$	**Simplify.**

The last statement is *not possible*; angle measures in triangles cannot be negative.

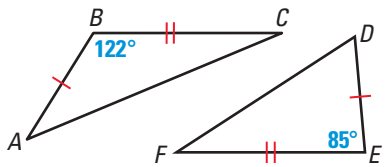
▶ So, you can conclude that the original assumption must be false. That is, $\triangle ABC$ cannot have more than one obtuse angle.

CONCEPT SUMMARY — GUIDELINES FOR WRITING AN INDIRECT PROOF

1. Identify the statement that you want to prove is true.
2. Begin by assuming the statement is false; assume its opposite is true.
3. Obtain statements that logically follow from your assumption.
4. If you obtain a contradiction, then the original statement must be true.

GOAL 2 USING THE HINGE THEOREM

In the two triangles shown, notice that $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, but $m\angle B$ is greater than $m\angle E$.



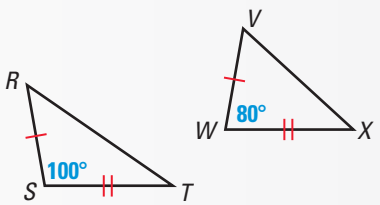
It appears that the side opposite the 122° angle is longer than the side opposite the 85° angle. This relationship is guaranteed by the Hinge Theorem below.

Exercise 31 asks you to write a proof of Theorem 5.14. Theorem 5.15 can be proved using Theorem 5.14 and indirect proof, as shown in Example 2.

THEOREMS

THEOREM 5.14 ***Hinge Theorem***

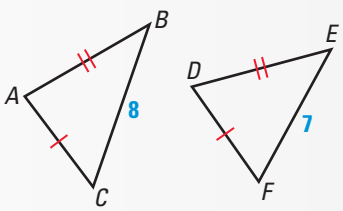
If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, then the third side of the first is longer than the third side of the second.



$RT > VX$

THEOREM 5.15 ***Converse of the Hinge Theorem***

If two sides of one triangle are congruent to two sides of another triangle, and the third side of the first is longer than the third side of the second, then the included angle of the first is larger than the included angle of the second.

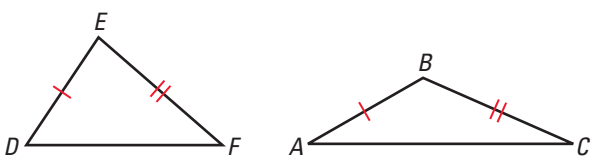


$m\angle A > m\angle D$

EXAMPLE 2 *Indirect Proof of Theorem 5.15*

GIVEN ▶ $\overline{AB} \cong \overline{DE}$
$\overline{BC} \cong \overline{EF}$
$AC > DF$

PROVE ▶ $m\angle B > m\angle E$



SOLUTION Begin by assuming that $m\angle B \ngtr m\angle E$. Then, it follows that either $m\angle B = m\angle E$ or $m\angle B < m\angle E$.

Case 1 If $m\angle B = m\angle E$, then $\angle B \cong \angle E$. So, $\triangle ABC \cong \triangle DEF$ by the SAS Congruence Postulate and $AC = DF$.

Case 2 If $m\angle B < m\angle E$, then $AC < DF$ by the Hinge Theorem.

Both conclusions contradict the given information that $AC > DF$. So the original assumption that $m\angle B \ngtr m\angle E$ cannot be correct. Therefore, $m\angle B > m\angle E$.

STUDENT HELP

Study Tip
The symbol $\ngtr$ is read as "is not greater than."

EXAMPLE 3 *Finding Possible Side Lengths and Angle Measures*

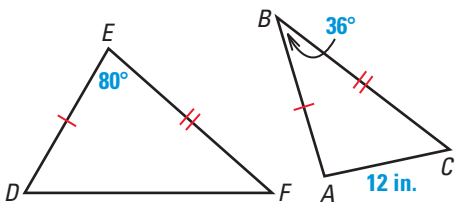
You can use the Hinge Theorem and its converse to choose possible side lengths or angle measures from a given list.

a. $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $AC = 12$ inches, $m\angle B = 36°$, and $m\angle E = 80°$. Which of the following is a possible length for $\overline{DF}$: 8 in., 10 in., 12 in., or 23 in.?

b. In a $\triangle RST$ and a $\triangle XYZ$, $\overline{RT} \cong \overline{XZ}$, $\overline{ST} \cong \overline{YZ}$, $RS = 3.7$ centimeters, $XY = 4.5$ centimeters, and $m\angle Z = 75°$. Which of the following is a possible measure for $\angle T$: 60°, 75°, 90°, or 105°?

SOLUTION

a. Because the included angle in $\triangle DEF$ is larger than the included angle in $\triangle ABC$, the third side $\overline{DF}$ must be longer than $\overline{AC}$. So, of the four choices, the only possible length for $\overline{DF}$ is 23 inches. A diagram of the triangles shows that this is plausible.



b. Because the third side in $\triangle RST$ is shorter than the third side in $\triangle XYZ$, the included angle $\angle T$ must be smaller than $\angle Z$. So, of the four choices, the only possible measure for $\angle T$ is 60°.

EXAMPLE 4 *Comparing Distances*

TRAVEL DISTANCES You and a friend are flying separate planes. You leave the airport and fly 120 miles due west. You then change direction and fly W 30° N for 70 miles. (W 30° N indicates a north-west direction that is 30° north of due west.) Your friend leaves the airport and flies 120 miles due east. She then changes direction and flies E 40° S for 70 miles. Each of you has flown 190 miles, but which plane is farther from the airport?

FOCUS ON CAREERS

AIR TRAFFIC CONTROLLERS help ensure the safety of airline passengers and crews by developing air traffic flight paths that keep planes a safe distance apart.

INTERNET CAREER LINK
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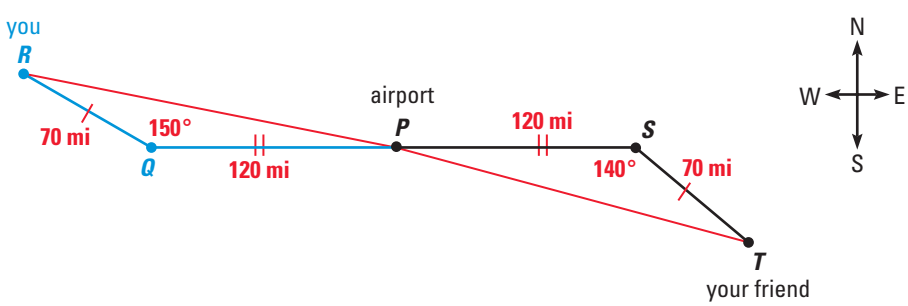
SOLUTION

Begin by drawing a diagram, as shown below. Your flight is represented by $\triangle PQR$ and your friend's flight is represented by $\triangle PST$.



Because these two triangles have two sides that are congruent, you can apply the Hinge Theorem to conclude that $\overline{RP}$ is longer than $\overline{TP}$.

▶ So, your plane is farther from the airport than your friend's plane.

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain why an indirect proof might also be called a *proof by contradiction*.

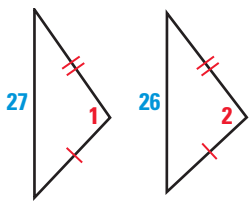
Concept Check ✓

2. To use an indirect proof to show that two lines m and n are parallel, you would first make the assumption that __?__.

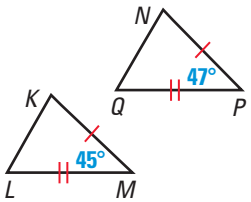
Skill Check ✓

In Exercises 3–5, complete with <, >, or =.

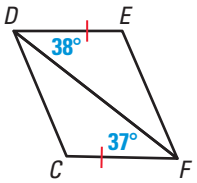
3. $m\angle 1$ __?__ $m\angle 2$



4. KL __?__ NQ



5. DC __?__ FE



6. Suppose that in a $\triangle ABC$, you want to prove that $BC > AC$. What are the two cases you would use in an indirect proof?

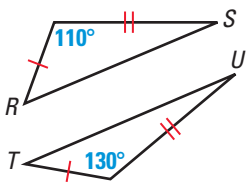
PRACTICE AND APPLICATIONS

STUDENT HELP

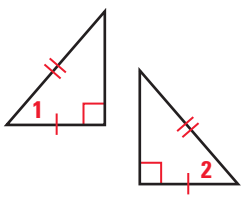
Extra Practice to help you master skills is on p. 812.

USING THE HINGE THEOREM AND ITS CONVERSE **Complete with <, >, or =.**

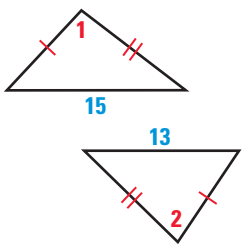
7. RS __?__ TU



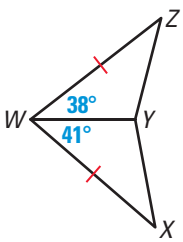
8. $m\angle 1$ __?__ $m\angle 2$



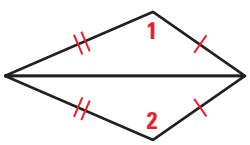
9. $m\angle 1$ __?__ $m\angle 2$



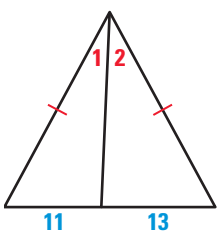
10. XY __?__ ZY



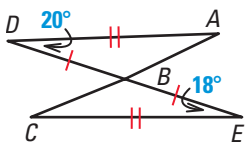
11. $m\angle 1$ __?__ $m\angle 2$



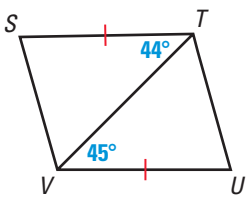
12. $m\angle 1$ __?__ $m\angle 2$



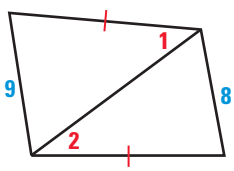
13. AB __?__ CB



14. UT __?__ SV



15. $m\angle 1$ __?__ $m\angle 2$



STUDENT HELP

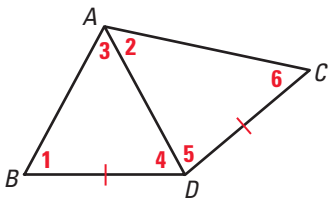
HOMEWORK HELP

Example 1: Exs. 21–24
Example 2: Exs. 25–27
Example 3: Exs. 7–17
Example 4: Exs. 28, 29

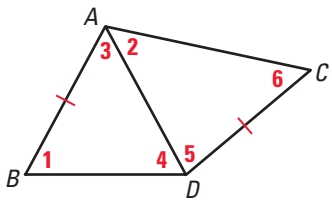
LOGICAL REASONING **In Exercises 16 and 17, match the given information with conclusion A, B, or C. Explain your reasoning.**

A. $AD > CD$ **B.** $AC > BD$ **C.** $m\angle 4 < m\angle 5$

16. $AC > AB$, $BD = CD$

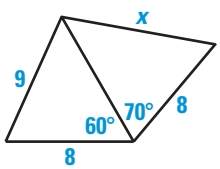


17. $AB = DC$, $m\angle 3 < m\angle 5$



USING ALGEBRA **Use an inequality to describe a restriction on the value of *x* as determined by the Hinge Theorem or its converse.**

18.



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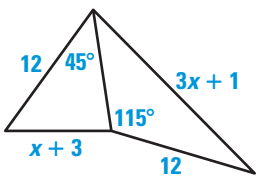

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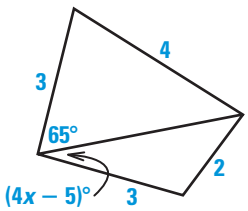


STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with negations in Exs. 21–23.

ASSUMING THE NEGATION OF THE CONCLUSION **In Exercises 21–23, write the first statement for an indirect proof of the situation.**

21. If $RS + ST \neq 12$ in. and $ST = 5$ in., then $RS \neq 7$ in.

22. In $\triangle MNP$, if Q is the midpoint of $\overline{NP}$, then $\overline{MQ}$ is a median.

23. In $\triangle ABC$, if $m\angle A + m\angle B = 90°$, then $m\angle C = 90°$.

24. **DEVELOPING PROOF** Arrange statements A–D in correct order to write an indirect proof of Postulate 7 from page 73: *If two lines intersect, then their intersection is exactly one point.*

GIVEN ▶ line m, line n

PROVE ▶ Lines m and n intersect in exactly one point.

A. But this contradicts Postulate 5, which states that there is exactly one line through any two points.

B. Then there are two lines (m and n) through points P and Q.

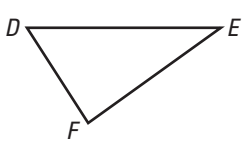
C. Assume that there are two points, P and Q, where m and n intersect.

D. It is false that m and n can intersect in two points, so they must intersect in exactly one point.

25. **PROOF** Write an indirect proof of Theorem 5.11 on page 295.

GIVEN ▶ $m\angle D > m\angle E$

PROVE ▶ $EF > DF$

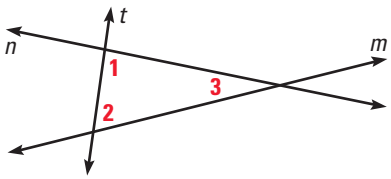


Plan for Proof In Case 1, assume that $EF < DF$. In Case 2, assume that $EF = DF$. Show that neither case can be true, so $EF > DF$.

PROOF **Write an indirect proof in paragraph form. The diagrams, which illustrate negations of the conclusions, may help you.**

26. GIVEN ▶ $\angle 1$ and $\angle 2$ are supplementary.

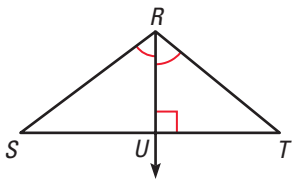
PROVE ▶ $m \parallel n$



Begin by assuming that $m \nparallel n$.

27. GIVEN ▶ $\overline{RU}$ is an altitude, $\overrightarrow{RU}$ bisects $\angle SRT$.

PROVE ▶ $\triangle RST$ is isosceles.



Begin by assuming that $RS > RT$.

COMPARING DISTANCES **In Exercises 28 and 29, consider the flight paths described. Explain how to use the Hinge Theorem to determine who is farther from the airport.**

28. *Your flight:* 100 miles due west, then 50 miles N 20° W
Friend's flight: 100 miles due north, then 50 miles N 30° E

29. *Your flight:* 210 miles due south, then 80 miles S 70° W
Friend's flight: 80 miles due north, then 210 miles N 50° E

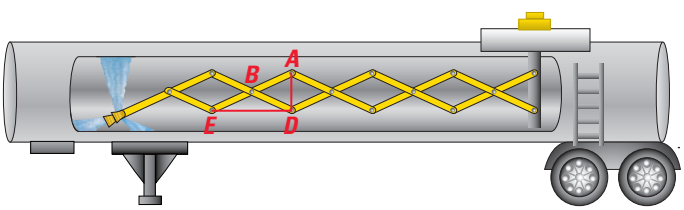
Test Preparation

30. MULTI-STEP PROBLEM Use the diagram of the tank cleaning system's expandable arm shown below.

a. As the cleaning system arm expands, $\overline{ED}$ gets longer. As ED increases, what happens to $m\angle EBD$? What happens to $m\angle DBA$?

b. Name a distance that decreases as $\overline{ED}$ gets longer.

c. *Writing* Explain how the cleaning arm illustrates the Hinge Theorem.

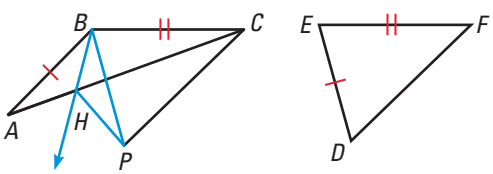


★ Challenge

31. PROOF Prove Theorem 5.14, the Hinge Theorem.

GIVEN ▶ $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $m\angle ABC > m\angle DEF$

PROVE ▶ $AC > DF$



Plan for Proof

1. Locate a point P outside $\triangle ABC$ so you can construct $\triangle PBC \cong \triangle DEF$.

2. Show that $\triangle PBC \cong \triangle DEF$ by the SAS Congruence Postulate.

3. Because $m\angle ABC > m\angle DEF$, locate a point H on $\overline{AC}$ so that $\overrightarrow{BH}$ bisects $\angle PBA$.

4. Give reasons for each equality or inequality below to show that $AC > DF$.

$$AC = AH + HC = PH + HC > PC = DF$$

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

CLASSIFYING TRIANGLES **State whether the triangle described is *isosceles, equiangular, equilateral,* or *scalene*.** (Review 4.1 for 6.1)

32. Side lengths: 3 cm, 5 cm, 3 cm

33. Side lengths: 5 cm, 5 cm, 5 cm

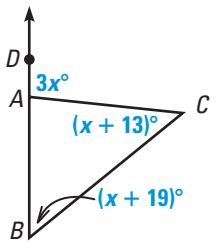
34. Side lengths: 5 cm, 6 cm, 8 cm

35. Angle measures: 30°, 30°, 120°

36. Angle measures: 60°, 60°, 60°

37. Angle measures: 65°, 50°, 65°

USING ALGEBRA **In Exercises 38–41, use the diagram shown at the right.** (Review 4.1 for 6.1)



38. Find the value of x.

39. Find $m\angle B$.

40. Find $m\angle C$.

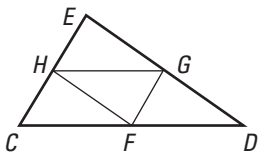
41. Find $m\angle BAC$.

42. **DESCRIBING A SEGMENT** Draw any equilateral triangle $\triangle RST$. Draw a line segment from vertex R to the midpoint of side $\overline{ST}$. State everything that you know about the line segment you have drawn. (Review 5.3)

QUIZ 2

Self-Test for Lessons 5.4–5.6

In Exercises 1–3, use the triangle shown at the right. The midpoints of the sides of $\triangle CDE$ are *F, G,* and *H*. (Lesson 5.4)



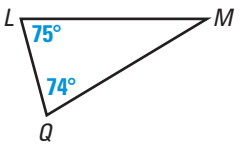
1. $\overline{FG} \parallel$ __?__

2. If $FG = 8$, then $CE =$ __?__.

3. If the perimeter of $\triangle CDE = 42$, then the perimeter of $\triangle GHF =$ __?__.

In Exercises 4–6, list the sides in order from shortest to longest. (Lesson 5.5)

4.



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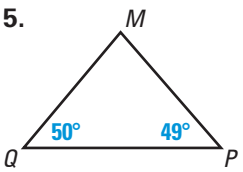

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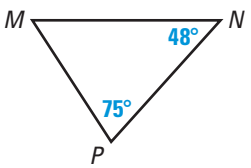
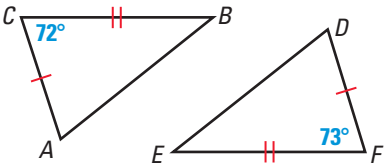
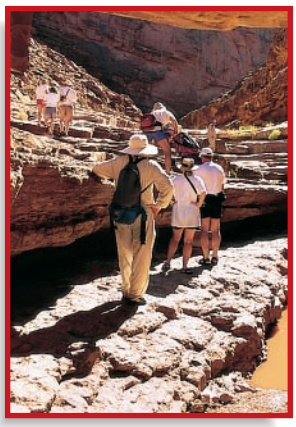


7. In $\triangle ABC$ and $\triangle DEF$ shown at the right, which is longer, $\overline{AB}$ or $\overline{DE}$? (Lesson 5.6)



8. **HIKING** Two groups of hikers leave from the same base camp and head in opposite directions. The first group walks 4.5 miles due east, then changes direction and walks E 45° N for 3 miles. The second group walks 4.5 miles due west, then changes direction and walks W 20° S for 3 miles. Each group has walked 7.5 miles, but which is farther from the base camp? (Lesson 5.6)



Hikers in the Grand Canyon

CHAPTER 5

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Use properties of perpendicular bisectors and angle bisectors. **(5.1)**	Decide where a hockey goalie should be positioned to defend the goal. **(p. 270)**
Use properties of perpendicular bisectors and angle bisectors of a triangle. **(5.2)**	Find the center of a mushroom ring. **(p. 277)**
Use properties of medians and altitudes of a triangle. **(5.3)**	Find points in a triangle used to measure a person's heart fitness. **(p. 283)**
Use properties of midsegments of a triangle. **(5.4)**	Determine the length of the crossbar of a swing set. **(p. 292)**
Compare the lengths of the sides or the measures of the angles of a triangle. **(5.5)**	Determine how the lengths of the boom lines of a crane affect the position of the boom. **(p. 300)**
Understand and write indirect proofs. **(5.6)**	Prove theorems that cannot be easily proved directly.
Use the Hinge Theorem and its converse to compare side lengths and angle measures of triangles. **(5.6)**	Decide which of two airplanes is farther from an airport. **(p. 304)**

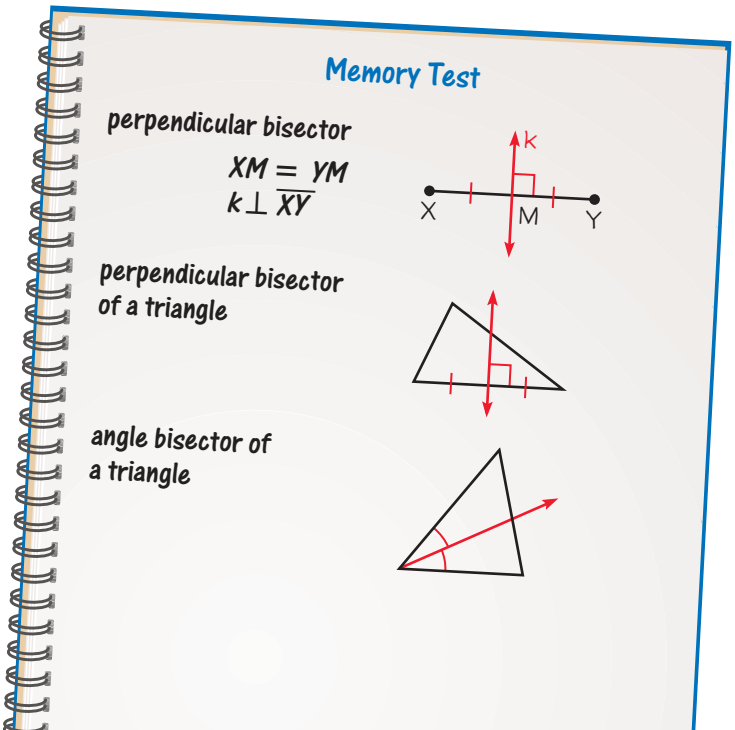
How does Chapter 5 fit into the BIGGER PICTURE of geometry?

In this chapter, you studied properties of special segments of triangles, which are an important building block for more complex figures that you will explore in later chapters. The special segments of a triangle have applications in many areas such as demographics (p. 280), medicine (p. 283), and room design (p. 299).

STUDY STRATEGY

Did you test your memory?

The list of important vocabulary terms and skills you made, following the **Study Strategy** on page 262, may resemble this one.



CHAPTER 5

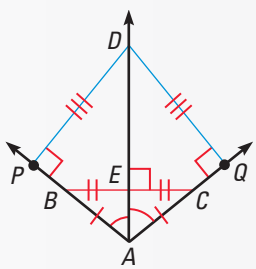
Chapter Review

- perpendicular bisector, p. 264
- equidistant from two points, p. 264
- distance from a point to a line, p. 266
- equidistant from two lines, p. 266
- perpendicular bisector of a triangle, p. 272
- concurrent lines, p. 272
- point of concurrency, p. 272
- circumcenter of a triangle, p. 273
- angle bisector of a triangle, p. 274
- incenter of a triangle, p. 274
- median of a triangle, p. 279
- centroid of a triangle, p. 279
- altitude of a triangle, p. 281
- orthocenter of a triangle, p. 281
- midsegment of a triangle, p. 287
- indirect proof, p. 302

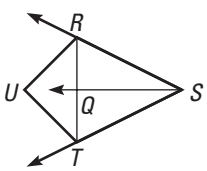
5.1 Perpendiculars and Bisectors

Examples on pp. 264–267

EXAMPLES In the figure, $\overrightarrow{AD}$ is the angle bisector of $\angle BAC$ and the perpendicular bisector of $\overline{BC}$. You know that $BE = CE$ by the definition of perpendicular bisector and that $AB = AC$ by the Perpendicular Bisector Theorem. Because $\overline{DP} \perp \overrightarrow{AP}$ and $\overline{DQ} \perp \overrightarrow{AQ}$, then DP and DQ are the distances from D to the sides of $\angle PAQ$ and you know that $DP = DQ$ by the Angle Bisector Theorem.



In Exercises 1–3, use the diagram.



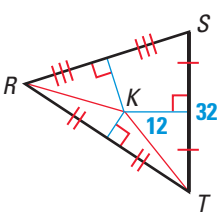
1. If $\overrightarrow{SQ}$ is the perpendicular bisector of $\overline{RT}$, explain how you know that $\overline{RQ} \cong \overline{TQ}$ and $\overline{RS} \cong \overline{TS}$.
2. If $\overline{UR} \cong \overline{UT}$, what can you conclude about U?
3. If Q is equidistant from $\overrightarrow{SR}$ and $\overrightarrow{ST}$, what can you conclude about Q?

5.2 Bisectors of a Triangle

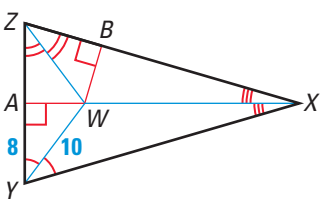
Examples on pp. 272–274

EXAMPLES The perpendicular bisectors of a triangle intersect at the *circumcenter*, which is equidistant from the vertices of the triangle. The angle bisectors of a triangle intersect at the *incenter*, which is equidistant from the sides of the triangle.

4. The perpendicular bisectors of $\triangle RST$ intersect at K. Find KR.



5. The angle bisectors of $\triangle XYZ$ intersect at W. Find WB.

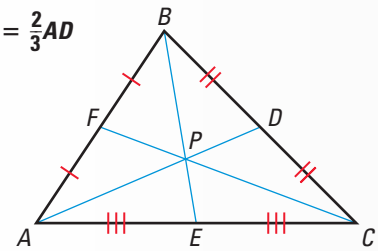


5.3 MEDIANS AND ALTITUDES OF A TRIANGLE

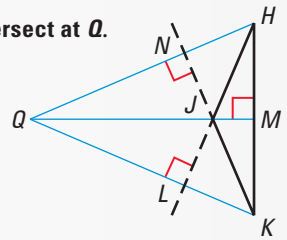
Examples on pp. 279–281

EXAMPLES The medians of a triangle intersect at the centroid. The lines containing the altitudes of a triangle intersect at the orthocenter.

$AP = \frac{2}{3}AD$

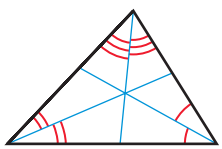


$\overleftrightarrow{HN}$, $\overleftrightarrow{JM}$, and $\overleftrightarrow{KL}$ intersect at Q.



Name the special segments and point of concurrency of the triangle.

6.



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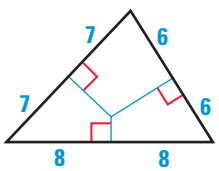

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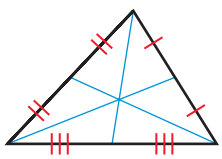

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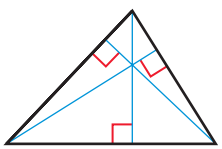


$\triangle XYZ$ has vertices $X(0, 0)$, $Y(-4, 0)$, and $Z(0, 6)$. Find the coordinates of the indicated point.

10. the centroid of $\triangle XYZ$

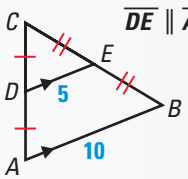
11. the orthocenter of $\triangle XYZ$

5.4 MIDSEGMENT THEOREM

Examples on pp. 287–289

EXAMPLES A midsegment of a triangle connects the midpoints of two sides of the triangle. By the Midsegment Theorem, a midsegment of a triangle is parallel to the third side and its length is half the length of the third side.

$\overline{DE} \parallel \overline{AB}$, $DE = \frac{1}{2}AB$

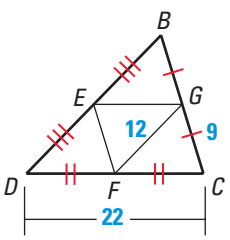


In Exercises 12 and 13, the midpoints of the sides of $\triangle HJK$ are $L(4, 3)$, $M(8, 3)$, and $N(6, 1)$.

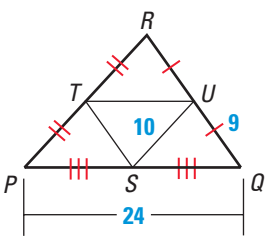
12. Find the coordinates of the vertices of the triangle.

13. Show that each midsegment is parallel to a side of the triangle.

14. Find the perimeter of $\triangle BCD$.



15. Find the perimeter of $\triangle STU$.



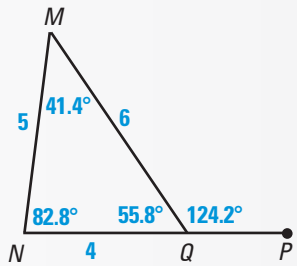
5.5 INEQUALITIES IN ONE TRIANGLE

Examples on pp. 295–297

EXAMPLES In a triangle, the side and the angle of greatest measurement are always opposite each other. In the diagram, the largest angle, $\angle MNQ$, is opposite the longest side, $\overline{MQ}$.

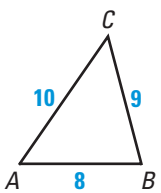
By the Exterior Angle Inequality, $m\angle MQP > m\angle N$ and $m\angle MQP > m\angle M$.

By the Triangle Inequality, $MN + NQ > MQ$, $NQ + MQ > MN$, and $MN + MQ > NQ$.

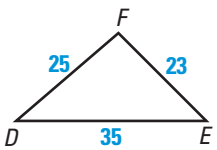


In Exercises 16–19, write the angle and side measurements in order from least to greatest.

16.



17.



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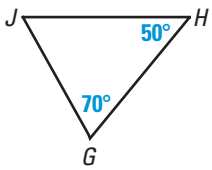

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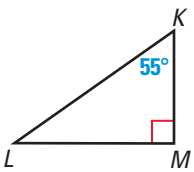


20. **FENCING A GARDEN** You are enclosing a triangular garden region with a fence. You have measured two sides of the garden to be 100 feet and 200 feet. What is the maximum length of fencing you need? Explain.

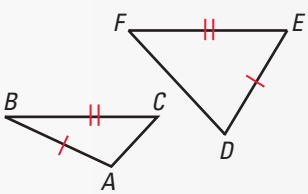
5.6 INDIRECT PROOF AND INEQUALITIES IN TWO TRIANGLES

Examples on pp. 302–304

EXAMPLES $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$

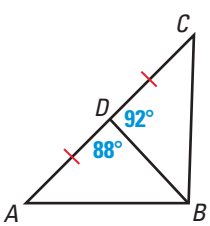
Hinge Theorem: If $m\angle E > m\angle B$, then $DF > AC$.

Converse of the Hinge Theorem: If $DF > AC$, then $m\angle E > m\angle B$.

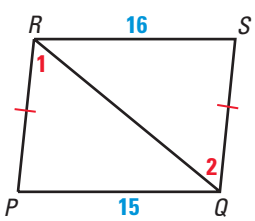


In Exercises 21–23, complete with <, >, or =.

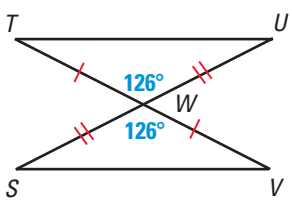
21. AB ? CB



22. $m\angle 1$? $m\angle 2$



23. TU ? VS



24. Write the first statement for an indirect proof of this situation: In a $\triangle MPQ$, if $\angle M \cong \angle Q$, then $\triangle MPQ$ is isosceles.

25. Write an indirect proof to show that no triangle has two right angles.

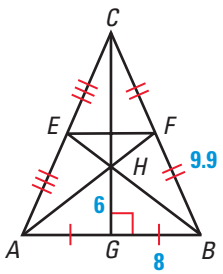
CHAPTER 5

Chapter Test

In Exercises 1–5, complete the statement with the word *always, sometimes,* or *never*.

1. If P is the circumcenter of $\triangle RST$, then PR, PS, and PT are __?__ equal.
2. If $\overrightarrow{BD}$ bisects $\angle ABC$, then $\overline{AD}$ and $\overline{CD}$ are __?__ congruent.
3. The incenter of a triangle __?__ lies outside the triangle.
4. The length of a median of a triangle is __?__ equal to the length of a midsegment.
5. If $\overline{AM}$ is the altitude to side $\overline{BC}$ of $\triangle ABC$, then $\overline{AM}$ is __?__ shorter than $\overline{AB}$.

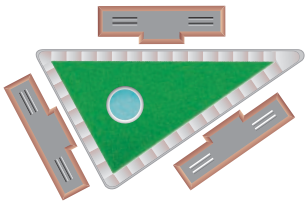
In Exercises 6–10, use the diagram.



6. Find each length.

   a. HC **b.** HB **c.** HE **d.** BC

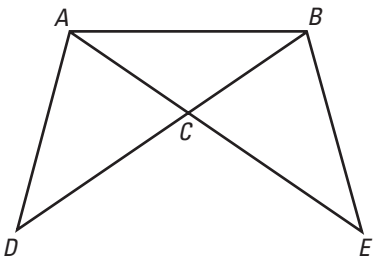
7. Point H is the __?__ of the triangle.
8. $\overline{CG}$ is a(n) __?__, __?__, __?__, and __?__ of $\triangle ABC$.
9. $EF =$ __?__ and $\overline{EF} \parallel$ __?__ by the __?__ Theorem.
10. Compare the measures of $\angle ACB$ and $\angle BAC$. Justify your answer.
11. **LANDSCAPE DESIGN** You are designing a circular swimming pool for a triangular lawn surrounded by apartment buildings. You want the center of the pool to be equidistant from the three sidewalks. Explain how you can locate the center of the pool.



In Exercises 12–14, use the photo of the three-legged tripod.

12. As the legs of a tripod are spread apart, which theorem guarantees that the angles between each pair of legs get larger?
13. Each leg of a tripod can extend to a length of 5 feet. What is the maximum possible distance between the ends of two legs?
14. Let $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ represent the legs of a tripod. Draw and label a sketch. Suppose the legs are congruent and $m\angle AOC > m\angle BOC$. Compare the lengths of $\overline{AC}$ and $\overline{BC}$.

In Exercises 15 and 16, use the diagram at the right.



15. Write a two-column proof.

    GIVEN ▶ $AC = BC$

    PROVE ▶ $BE < AE$

16. Write an indirect proof.

    GIVEN ▶ $AD \neq AB$

    PROVE ▶ $m\angle D \neq m\angle ABC$

6.1 Polygons

What you should learn

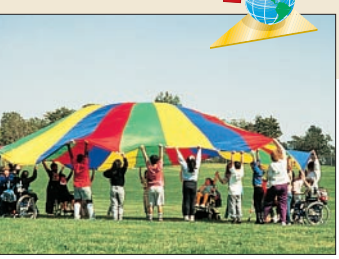
GOAL 1 Identify, name, and describe polygons such as the building shapes in **Example 2**.

GOAL 2 Use the sum of the measures of the interior angles of a quadrilateral.

Why you should learn it

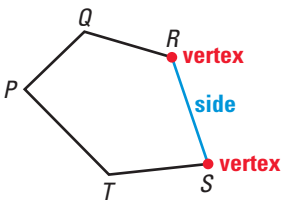
▼ To describe **real-life** objects, such as the parachute in **Exs. 21–23**.

REAL LIFE



GOAL 1 DESCRIBING POLYGONS

A **polygon** is a plane figure that meets the following conditions.

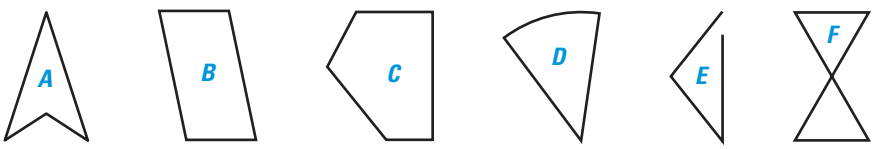


1. It is formed by three or more segments called **sides**, such that no two sides with a common endpoint are collinear.
2. Each side intersects exactly two other sides, one at each endpoint.

Each endpoint of a side is a **vertex** of the polygon. The plural of *vertex* is *vertices*. You can name a polygon by listing its vertices *consecutively*. For instance, *PQRST* and *QPTSR* are two correct names for the polygon above.

EXAMPLE 1 *Identifying Polygons*

State whether the figure is a polygon. If it is not, explain why.



SOLUTION

Figures *A*, *B*, and *C* *are* polygons.

- Figure *D* *is not* a polygon because it has a side that is not a segment.
- Figure *E* *is not* a polygon because two of the sides intersect only one other side.
- Figure *F* *is not* a polygon because some of its sides intersect more than two other sides.

Polygons are named by the number of sides they have.

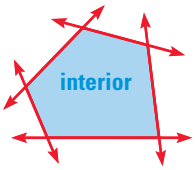
Number of sides	Type of polygon
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon

Number of sides	Type of polygon
8	Octagon
9	Nonagon
10	Decagon
12	Dodecagon
n	n-gon

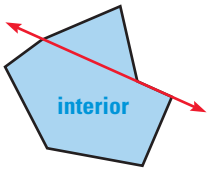
STUDENT HELP

Study Tip
To name a polygon not listed in the table, use the number of sides. For example, a polygon with 14 sides is a 14-gon.

A polygon is **convex** if no line that contains a side of the polygon contains a point in the interior of the polygon. A polygon that is not convex is called **nonconvex** or **concave**.



convex polygon

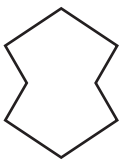


concave polygon

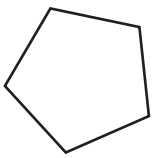
EXAMPLE 2 *Identifying Convex and Concave Polygons*

Identify the polygon and state whether it is convex or concave.

a.



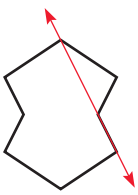
b.



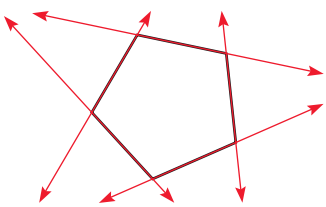
This tile pattern in Iran contains both convex and concave polygons.

SOLUTION

a. The polygon has 8 sides, so it is an octagon. When extended, some of the sides intersect the interior, so the polygon is concave.



b. The polygon has 5 sides, so it is a pentagon. When extended, none of the sides intersect the interior, so the polygon is convex.

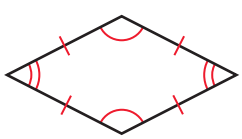


A polygon is **equilateral** if all of its sides are congruent. A polygon is **equiangular** if all of its interior angles are congruent. A polygon is **regular** if it is equilateral and equiangular.

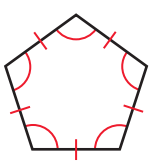
EXAMPLE 3 *Identifying Regular Polygons*

Decide whether the polygon is regular.

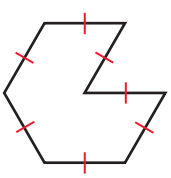
a.



b.



c.



SOLUTION

a. The polygon is an equilateral quadrilateral, but not equiangular. So, it is not a regular polygon.

b. This pentagon is equilateral and equiangular. So, it is a regular polygon.

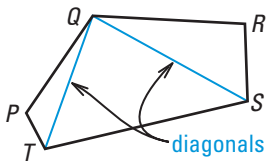
c. This heptagon is equilateral, but not equiangular. So, it is not regular.

GOAL 2 INTERIOR ANGLES OF QUADRILATERALS

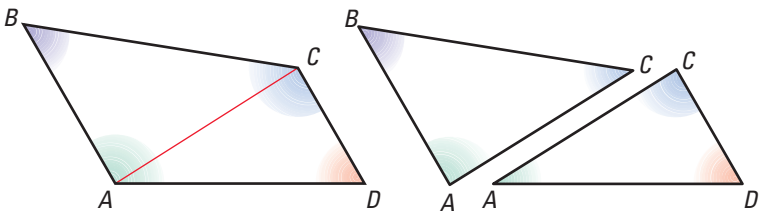
STUDENT HELP

Study Tip
Two vertices that are endpoints of the same side are called *consecutive vertices*. For example, *P* and *Q* are consecutive vertices.

A **diagonal** of a polygon is a segment that joins two *nonconsecutive* vertices. Polygon *PQRST* has 2 diagonals from point *Q*, $\overline{QT}$ and $\overline{QS}$.



Like triangles, quadrilaterals have both *interior* and *exterior* angles. If you draw a diagonal in a quadrilateral, you divide it into two triangles, each of which has interior angles with measures that add up to 180°. So you can conclude that the sum of the measures of the interior angles of a quadrilateral is 2(180°), or 360°.

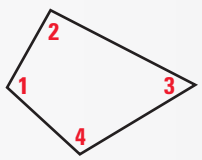


THEOREM

THEOREM 6.1 ***Interior Angles of a Quadrilateral***

The sum of the measures of the interior angles of a quadrilateral is 360°.

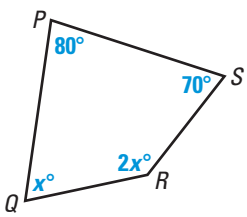
$$m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 = 360°$$



EXAMPLE 4 *Interior Angles of a Quadrilateral*

Using Algebra

Find $m\angle Q$ and $m\angle R$.



SOLUTION

Find the value of x. Use the sum of the measures of the interior angles to write an equation involving x. Then, solve the equation.

$x° + 2x° + 70° + 80° = 360°$	**Sum of measures of int. ∠s of a quad. is 360°.**
$3x + 150 = 360$	**Combine like terms.**
$3x = 210$	**Subtract 150 from each side.**
$x = 70$	**Divide each side by 3.**

Find $m\angle Q$ and $m\angle R$.

$m\angle Q = x° = 70°$

$m\angle R = 2x° = 140°$

▶ So, $m\angle Q = 70°$ and $m\angle R = 140°$.

GUIDED PRACTICE

Vocabulary Check ✓

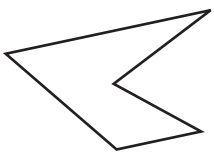
1. What is the plural of *vertex*?
2. What do you call a polygon with 8 sides? a polygon with 15 sides?

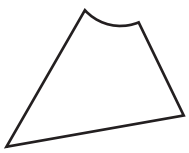
Concept Check ✓

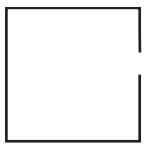
3. Suppose you could tie a string tightly around a convex polygon. Would the length of the string be equal to the perimeter of the polygon? What if the polygon were concave? Explain.

Skill Check ✓

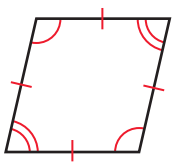
Decide whether the figure is a polygon. If it is not, explain why.

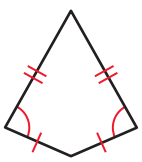
4. 

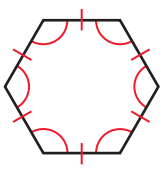
5. 

6. 

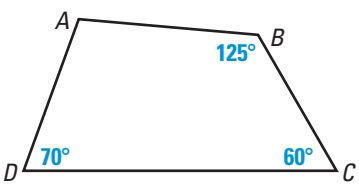
Tell whether the polygon is best described as *equiangular, equilateral, regular,* or *none of these*.

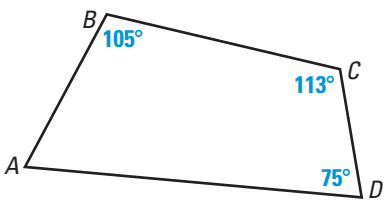
7. 

8. 

9. 

Use the information in the diagram to find $m\angle A$.

10. 

11. 

PRACTICE AND APPLICATIONS

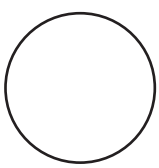
STUDENT HELP

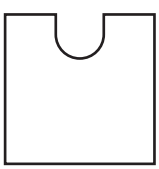
Extra Practice to help you master skills is on p. 813.

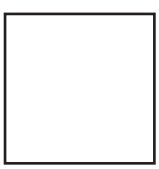
RECOGNIZING POLYGONS **Decide whether the figure is a polygon.**

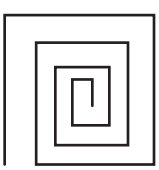
12. 

13. 

14. 

15. 

16. 

17. 

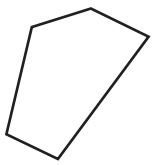
STUDENT HELP

HOMEWORK HELP

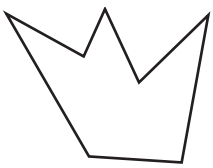
Example 1: Exs. 12–17, 48–51
Example 2: Exs. 18–20, 48–51
Example 3: Exs. 24–30, 48–51
Example 4: Exs. 36–46

CONVEX OR CONCAVE **Use the number of sides to tell what kind of polygon the shape is. Then state whether the polygon is *convex* or *concave.***

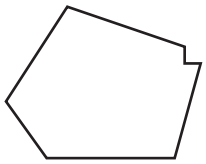
18.



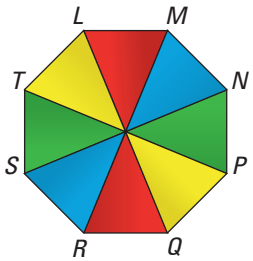
19.



20.



PARACHUTES **Some gym classes use parachutes that look like the polygon at the right.**



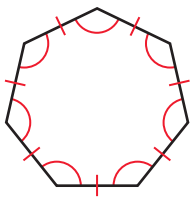
21. Is the polygon a *heptagon*, *octagon*, or *nonagon*?

22. Polygon *LMNPQRST* is one name for the polygon. State two other names.

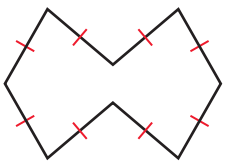
23. Name all of the diagonals that have vertex *M* as an endpoint. Not all of the diagonals are shown.

RECOGNIZING PROPERTIES **State whether the polygon is best described as *equilateral, equiangular, regular,* or *none of these*.**

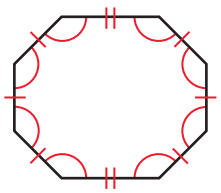
24.



25.



26.



FOCUS ON APPLICATIONS

REAL LIFE ROAD SIGNS The shape of a sign tells what it is for. For example, triangular signs like the one above are used internationally as warning signs.

TRAFFIC SIGNS **Use the number of sides of the traffic sign to tell what kind of polygon it is. Is it *equilateral, equiangular, regular,* or *none of these*?**

27.

28.

29.

30.

DRAWING **Draw a figure that fits the description.**

31. A convex heptagon

32. A concave nonagon

33. An equilateral hexagon that is not equiangular

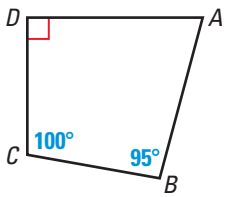
34. An equiangular polygon that is not equilateral

35. **LOGICAL REASONING** Is every triangle convex? Explain your reasoning.

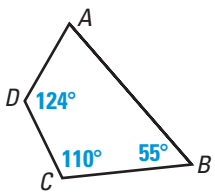
36. **LOGICAL REASONING** Quadrilateral *ABCD* is regular. What is the measure of $\angle ABC$? How do you know?

ANGLE MEASURES **Use the information in the diagram to find $m\angle A$.**

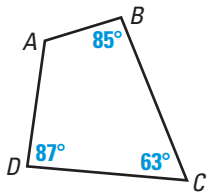
37.



38.



39.



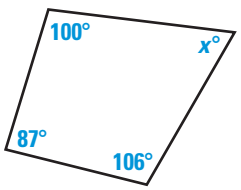
STUDENT HELP

SOFTWARE HELP Visit our Web site www.mcdougallittell.com to see instructions for several software applications.

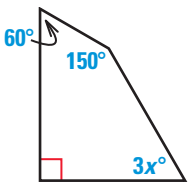
40. **TECHNOLOGY** Use geometry software to draw a quadrilateral. Measure each interior angle and calculate the sum. What happens to the sum as you drag the vertices of the quadrilateral?

USING ALGEBRA **Use the information in the diagram to solve for x.**

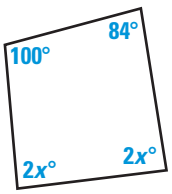
41.



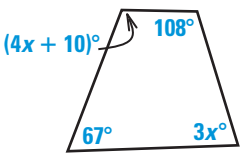
42.



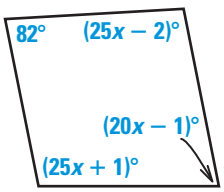
43.



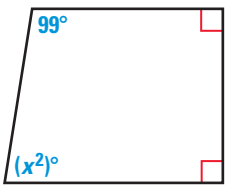
44.



45.



46.



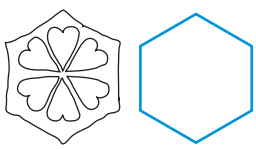
47. **LANGUAGE CONNECTION** A ***deca**gon* has ten sides and a ***deca**de* has ten years. The prefix *deca-* comes from Greek. It means *ten*. What does the prefix *tri-* mean? List four words that use *tri-* and explain what they mean.

PLANT SHAPES **In Exercises 48–51, use the following information.** Cross sections of seeds and fruits often resemble polygons. Next to each cross section is the polygon it resembles. Describe each polygon. Tell what kind of polygon it is, whether it is *concave* or *convex*, and whether it appears to be *equilateral*, *equiangular*, *regular*, or *none of these*. ▶ Source: The History and Folklore of N. American Wildflowers

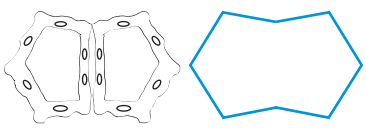
FOCUS ON APPLICATIONS

REAL LIFE **CARAMBOLA**, or star fruit, has a cross section shaped like a 5 pointed star. The fruit comes from an evergreen tree whose leaflets may fold at night or when the tree is shaken.

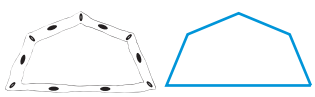
48. Virginia Snakeroot



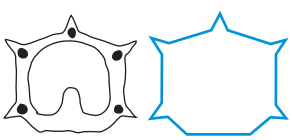
49. Caraway



50. Fennel

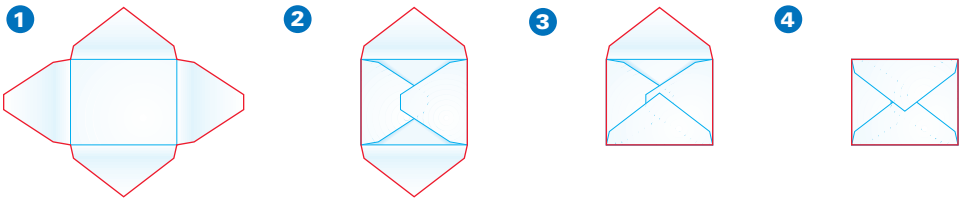


51. Poison Hemlock



Test Preparation

52. MULTI-STEP PROBLEM Envelope manufacturers fold a specially-shaped piece of paper to make an envelope, as shown below.



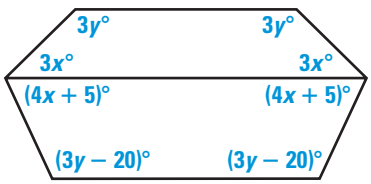
a. What type of polygon is formed by the outside edges of the paper before it is folded? Is the polygon convex?

b. Tell what type of polygon is formed at each step. Which of the polygons are convex?

c. *Writing* Explain the reason for the V-shaped notches that are at the ends of the folds.

★ Challenge

53. FINDING VARIABLES Find the values of x and y in the diagram at the right. Check your answer. Then copy the shape and write the measure of each angle on your diagram.



EXTRA CHALLENGE
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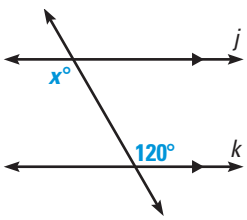
MIXED REVIEW

PARALLEL LINES **In the diagram, $j \parallel k$. Find the value of x.** (Review 3.3 for 6.2)

54.



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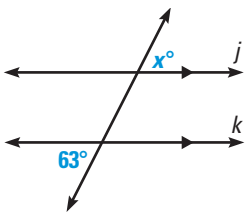

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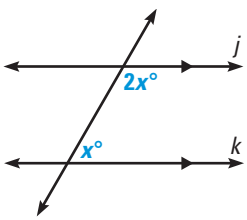

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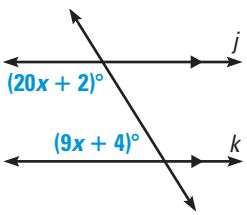

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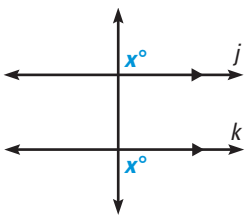

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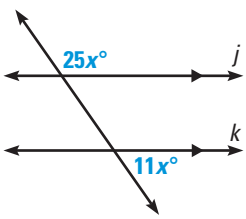


COORDINATE GEOMETRY **You are given the midpoints of the sides of a triangle. Find the coordinates of the vertices of the triangle.** (Review 5.4)

60. $L(-3, 7)$, $M(-5, 1)$, $N(-8, 8)$

61. $L(-4, -1)$, $M(3, 6)$, $N(-2, -8)$

62. $L(2, 4)$, $M(-1, 2)$, $N(0, 7)$

63. $L(-1, 3)$, $M(6, 7)$, $N(3, -5)$

64. **USING ALGEBRA** Use the Distance Formula to find the lengths of the diagonals of a polygon with vertices $A(0, 3)$, $B(3, 3)$, $C(4, 1)$, $D(0, -1)$, and $E(-2, 1)$. (Review 1.3)

6.2 Properties of Parallelograms

What you should learn

GOAL 1 Use some properties of parallelograms.

GOAL 2 Use properties of parallelograms in **real-life** situations, such as the drafting table shown in **Example 6**.

Why you should learn it

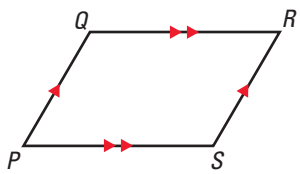
▼ You can use properties of parallelograms to understand how a scissors lift works in **Exs. 51–54**.

REAL LIFE

GOAL 1 PROPERTIES OF PARALLELOGRAMS

In this lesson and in the rest of the chapter you will study special quadrilaterals. A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.

When you mark diagrams of quadrilaterals, use matching arrowheads to indicate which sides are parallel. For example, in the diagram at the right, $\overline{PQ} \parallel \overline{RS}$ and $\overline{QR} \parallel \overline{SP}$. The symbol ▱$PQRS$ is read "parallelogram $PQRS$."

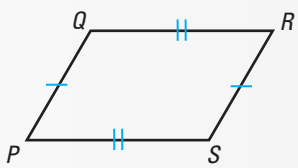


THEOREMS ABOUT PARALLELOGRAMS

THEOREM 6.2

If a quadrilateral is a parallelogram, then its **opposite sides** are congruent.

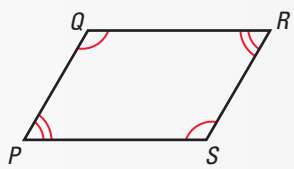
$\overline{PQ} \cong \overline{RS}$ and $\overline{SP} \cong \overline{QR}$



THEOREM 6.3

If a quadrilateral is a parallelogram, then its **opposite angles** are congruent.

$\angle P \cong \angle R$ and $\angle Q \cong \angle S$

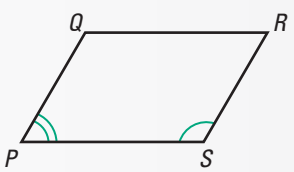


THEOREM 6.4

If a quadrilateral is a parallelogram, then its **consecutive angles** are supplementary.

$m\angle P + m\angle Q = 180°$, $m\angle Q + m\angle R = 180°$,

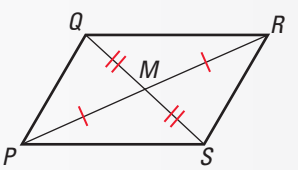
$m\angle R + m\angle S = 180°$, $m\angle S + m\angle P = 180°$



THEOREM 6.5

If a quadrilateral is a parallelogram, then its diagonals bisect each other.

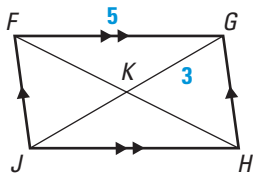
$\overline{QM} \cong \overline{SM}$ and $\overline{PM} \cong \overline{RM}$



Theorem 6.2 is proved in Example 5. You are asked to prove Theorem 6.3, Theorem 6.4, and Theorem 6.5 in Exercises 38–44.

EXAMPLE 1 *Using Properties of Parallelograms*

FGHJ is a parallelogram. Find the unknown length. Explain your reasoning.



a. JH

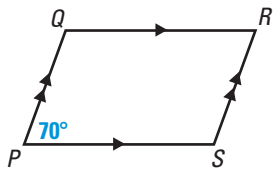
b. JK

SOLUTION

a. $JH = FG$	**Opposite sides of a ▱ are ≅.**
$JH = 5$	**Substitute 5 for *FG*.**
b. $JK = GK$	**Diagonals of a ▱ bisect each other.**
$JK = 3$	**Substitute 3 for *GK*.**

EXAMPLE 2 *Using Properties of Parallelograms*

PQRS is a parallelogram. Find the angle measure.



a. $m\angle R$

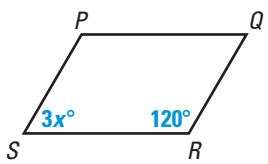
b. $m\angle Q$

SOLUTION

a. $m\angle R = m\angle P$	**Opposite angles of a ▱ are ≅.**
$m\angle R = 70°$	**Substitute 70° for *m*∠*P*.**
b. $m\angle Q + m\angle P = 180°$	**Consecutive ∠s of a ▱ are supplementary.**
$m\angle Q + 70° = 180°$	**Substitute 70° for *m*∠*P*.**
$m\angle Q = 110°$	**Subtract 70° from each side.**

EXAMPLE 3 *Using Algebra with Parallelograms*

PQRS is a parallelogram. Find the value of x.



SOLUTION

$m\angle S + m\angle R = 180°$	**Consecutive angles of a ▱ are supplementary.**
$3x + 120 = 180$	**Substitute 3*x* for *m*∠*S* and 120 for *m*∠*R*.**
$3x = 60$	**Subtract 120 from each side.**
$x = 20$	**Divide each side by 3.**

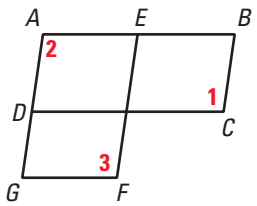
GOAL 2 REASONING ABOUT PARALLELOGRAMS

EXAMPLE 4 *Proving Facts about Parallelograms*

GIVEN ▶ $ABCD$ and $AEFG$ are parallelograms.

PROVE ▶ $\angle 1 \cong \angle 3$

Plan Show that both angles are congruent to $\angle 2$. Then use the Transitive Property of Congruence.



SOLUTION

Method 1 Write a two-column proof.

Statements	Reasons
1. $ABCD$ is a ▱. $AEFG$ is a ▱.	**1.** Given
2. $\angle 1 \cong \angle 2$, $\angle 2 \cong \angle 3$	**2.** Opposite angles of a ▱ are $\cong$.
3. $\angle 1 \cong \angle 3$	**3.** Transitive Property of Congruence

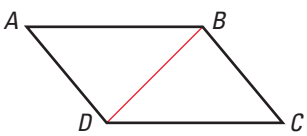
Method 2 Write a paragraph proof.

$ABCD$ is a parallelogram, so $\angle 1 \cong \angle 2$ because opposite angles of a parallelogram are congruent. $AEFG$ is a parallelogram, so $\angle 2 \cong \angle 3$. By the Transitive Property of Congruence, $\angle 1 \cong \angle 3$.

EXAMPLE 5 *Proving Theorem 6.2*

GIVEN ▶ $ABCD$ is a parallelogram.

PROVE ▶ $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$



SOLUTION

Statements	Reasons
1. $ABCD$ is a ▱.	**1.** Given
2. Draw $\overline{BD}$.	**2.** Through any two points there exists exactly one line.
3. $\overline{AB} \parallel \overline{CD}$, $\overline{AD} \parallel \overline{CB}$	**3.** Definition of parallelogram
4. $\angle ABD \cong \angle CDB$, $\angle ADB \cong \angle CBD$	**4.** Alternate Interior Angles Theorem
5. $\overline{DB} \cong \overline{DB}$	**5.** Reflexive Property of Congruence
6. $\triangle ADB \cong \triangle CBD$	**6.** ASA Congruence Postulate
7. $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$	**7.** Corresponding parts of $\cong$ $\triangle$s are $\cong$.

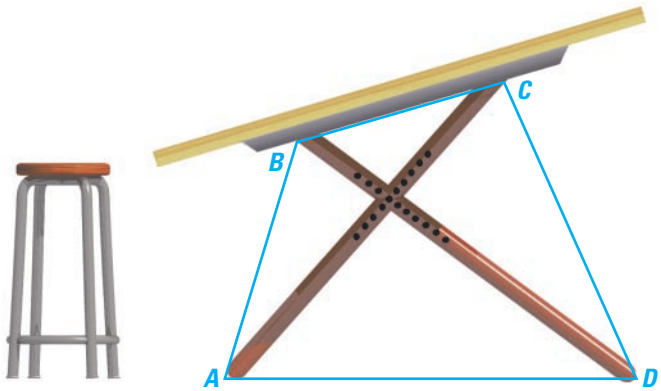
FOCUS ON CAREERS

FURNITURE DESIGN
Furniture designers use geometry, trigonometry, and other skills to create designs for furniture.

CAREER LINK
www.mcdougallittell.com

EXAMPLE 6 *Using Parallelograms in Real Life*

FURNITURE DESIGN A drafting table is made so that the legs can be joined in different ways to change the slope of the drawing surface. In the arrangement below, the legs $\overline{AC}$ and $\overline{BD}$ do *not* bisect each other. Is *ABCD* a parallelogram?



SOLUTION

No. If *ABCD* were a parallelogram, then by Theorem 6.5 $\overline{AC}$ would bisect $\overline{BD}$ and $\overline{BD}$ would bisect $\overline{AC}$.

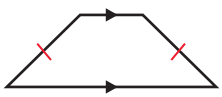
GUIDED PRACTICE

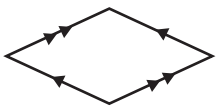
Vocabulary Check ✓

1. Write a definition of *parallelogram*.

Concept Check ✓

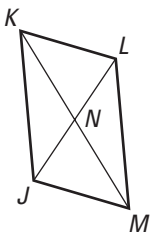
Decide whether the figure is a parallelogram. If it is not, explain why not.

2. 

3. 

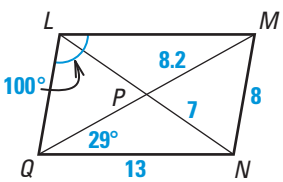
Skill Check ✓

IDENTIFYING CONGRUENT PARTS **Use the diagram of parallelogram *JKLM* at the right. Complete the statement, and give a reason for your answer.**



4. $\overline{JK} \cong$ __?__
5. $\overline{MN} \cong$ __?__
6. $\angle MLK \cong$ __?__
7. $\angle JKL \cong$ __?__
8. $\overline{JN} \cong$ __?__
9. $\overline{KL} \cong$ __?__
10. $\angle MNL \cong$ __?__
11. $\angle MKL \cong$ __?__

Find the measure in parallelogram *LMNQ*. Explain your reasoning.



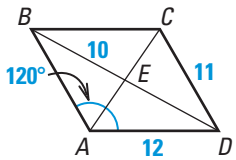
12. LM
13. LP
14. LQ
15. QP
16. $m\angle LMN$
17. $m\angle NQL$
18. $m\angle MNQ$
19. $m\angle LMQ$

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 813.

FINDING MEASURES **Find the measure in parallelogram *ABCD*. Explain your reasoning.**



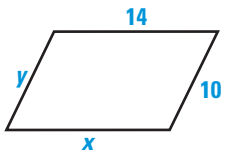
20. DE **21.** BA

22. BC **23.** $m\angle CDA$

24. $m\angle ABC$ **25.** $m\angle BCD$

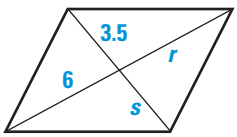
USING ALGEBRA **Find the value of each variable in the parallelogram.**

26.



27.

28.



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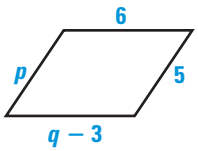

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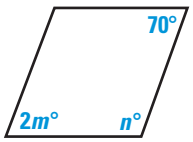

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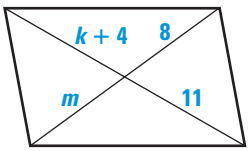
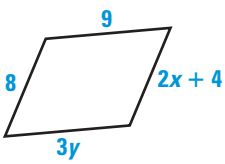


USING ALGEBRA **Find the value of each variable in the parallelogram.**

32.



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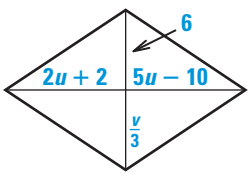

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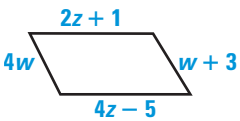

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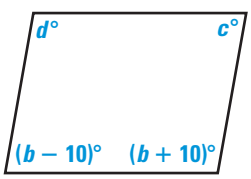

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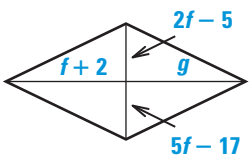

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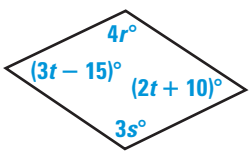
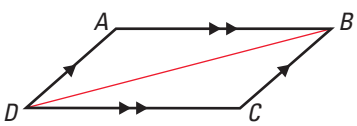


38. **PROVING THEOREM 6.3** Copy and complete the proof of Theorem 6.3: If a quadrilateral is a parallelogram, then its opposite angles are congruent.

GIVEN ▶ $ABCD$ is a ▱.

PROVE ▶ $\angle A \cong \angle C$, $\angle B \cong \angle D$



Paragraph Proof Opposite sides of a parallelogram are congruent, so ___a.___ and ___b.___. By the Reflexive Property of Congruence, ___c.___. $\triangle ABD \cong \triangle CDB$ because of the ___d.___ Congruence Postulate. Because ___e.___ parts of congruent triangles are congruent, $\angle A \cong \angle C$.

To prove that $\angle B \cong \angle D$, draw ___f.___ and use the same reasoning.

STUDENT HELP

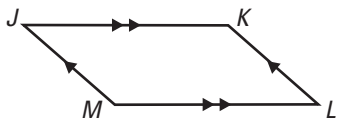
HOMEWORK HELP

Example 1: Exs. 20–22
Example 2: Exs. 23–25
Example 3: Exs. 26–37
Example 4: Exs. 55–58
Example 5: Exs. 38–44
Example 6: Exs. 45–54

39. **PROVING THEOREM 6.4** Copy and complete the two-column proof of Theorem 6.4: If a quadrilateral is a parallelogram, then its consecutive angles are supplementary.

GIVEN ▶ $JKLM$ is a ▱.

PROVE ▶ $\angle J$ and $\angle K$ are supplementary.



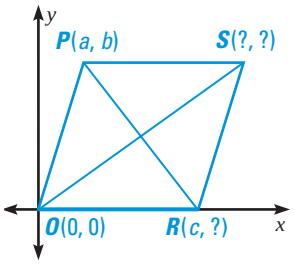
Statements	Reasons
1. ___?___	**1.** Given
2. $m\angle J = m\angle L$, $m\angle K = m\angle M$	**2.** ___?___
3. $m\angle J + m\angle L + m\angle K + m\angle M =$ ___?___	**3.** Sum of measures of int. ∠s of a quad. is 360°.
4. $m\angle J + m\angle J + m\angle K + m\angle K = 360°$	**4.** ___?___
5. $2($___?___ $+$ ___?___$) = 360°$	**5.** Distributive property
6. $m\angle J + m\angle K = 180°$	**6.** ___?___ prop. of equality
7. $\angle J$ and $\angle K$ are supplementary.	**7.** ___?___

You can use the same reasoning to prove any other pair of consecutive angles in ▱$JKLM$ are supplementary.

DEVELOPING COORDINATE PROOF **Copy and complete the coordinate proof of Theorem 6.5.**

GIVEN ▶ $PORS$ is a ▱.

PROVE ▶ $\overline{PR}$ and $\overline{OS}$ bisect each other.



Plan for Proof Find the coordinates of the midpoints of the diagonals of ▱$PORS$ and show that they are the same.

40. Point R is on the x-axis, and the length of $\overline{OR}$ is c units. What are the coordinates of point R?

41. The length of $\overline{PS}$ is also c units, and $\overline{PS}$ is horizontal. What are the coordinates of point S?

42. What are the coordinates of the midpoint of $\overline{PR}$?

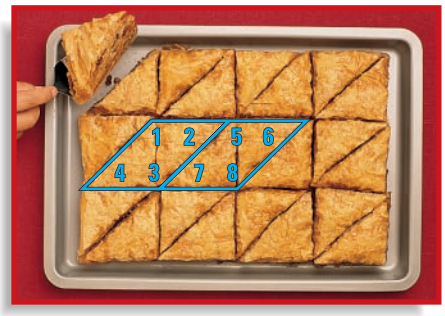
43. What are the coordinates of the midpoint of $\overline{OS}$?

44. *Writing* How do you know that $\overline{PR}$ and $\overline{OS}$ bisect each other?

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with the coordinate proof in Exs. 40–44.

BAKING **In Exercises 45 and 46, use the following information.**
In a recipe for baklava, the pastry should be cut into triangles that form congruent parallelograms, as shown. Write a paragraph proof to prove the statement.

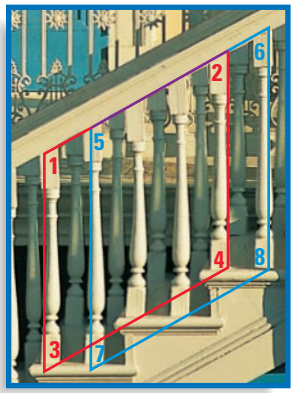


45. $\angle 3$ is supplementary to $\angle 6$.

46. $\angle 4$ is supplementary to $\angle 5$.

STAIR BALUSTERS **In Exercises 47–50, use the following information.**

In the diagram at the right, the slope of the handrail is equal to the slope of the stairs. The balusters (vertical posts) support the handrail.



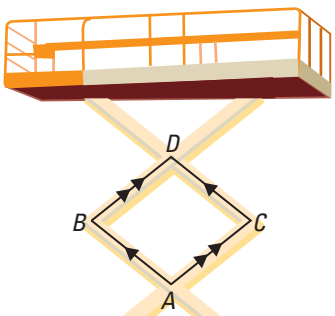
47. Which angle in the red parallelogram is congruent to $\angle 1$?

48. Which angles in the blue parallelogram are supplementary to $\angle 6$?

49. Which postulate can be used to prove that $\angle 1 \cong \angle 5$?

50. *Writing* Is the red parallelogram congruent to the blue parallelogram? Explain your reasoning.

SCISSORS LIFT **Photographers can use scissors lifts for overhead shots, as shown at the left. The crossing beams of the lift form parallelograms that move together to raise and lower the platform. In Exercises 51–54, use the diagram of parallelogram *ABDC* at the right.**



51. What is $m\angle B$ when $m\angle A = 120°$?

52. Suppose you decrease $m\angle A$. What happens to $m\angle B$?

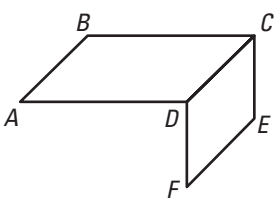
53. Suppose you decrease $m\angle A$. What happens to AD?

54. Suppose you decrease $m\angle A$. What happens to the overall height of the scissors lift?

TWO-COLUMN PROOF **Write a two-column proof.**

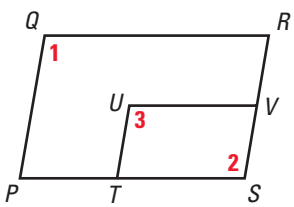
55. **GIVEN** ▶ $ABCD$ and $CEFD$ are ▱s.

PROVE ▶ $\overline{AB} \cong \overline{FE}$



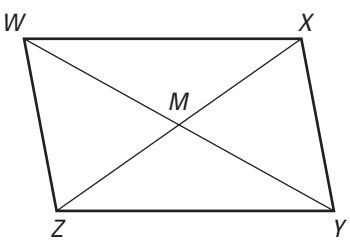
56. **GIVEN** ▶ $PQRS$ and $TUVS$ are ▱s.

PROVE ▶ $\angle 1 \cong \angle 3$



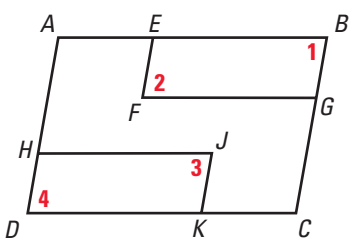
57. **GIVEN** ▶ $WXYZ$ is a ▱.

PROVE ▶ $\triangle WMZ \cong \triangle YMX$

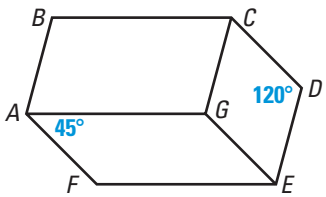


58. **GIVEN** ▶ $ABCD$, $EBGF$, $HJKD$ are ▱s.

PROVE ▶ $\angle 2 \cong \angle 3$

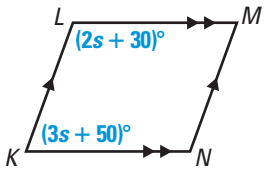


59. *Writing* In the diagram, *ABCG, CDEG,* and *AGEF* are parallelograms. Copy the diagram and add as many other angle measures as you can. Then describe how you know the angle measures you added are correct.



Test Preparation

60. MULTIPLE CHOICE In ▱*KLMN*, what is the value of *s*?



Ⓐ 5 Ⓑ 20 Ⓒ 40

Ⓓ 52 Ⓔ 70

61. MULTIPLE CHOICE In ▱*ABCD*, point *E* is the intersection of the diagonals. Which of the following is *not* necessarily true?

Ⓐ $AB = CD$ Ⓑ $AC = BD$ Ⓒ $AE = CE$ Ⓓ $AD = BC$ Ⓔ $DE = BE$

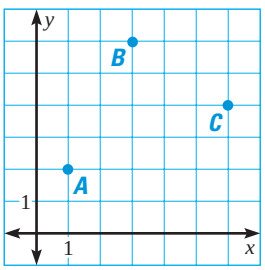
★ Challenge

USING ALGEBRA Suppose points *A*(1, 2), *B*(3, 6), and *C*(6, 4) are three vertices of a parallelogram.

62. Give the coordinates of a point that could be the fourth vertex. Sketch the parallelogram in a coordinate plane.

63. Explain how to check to make sure the figure you drew in Exercise 62 is a parallelogram.

64. How many different parallelograms can be formed using *A*, *B*, and *C* as vertices? Sketch each parallelogram and label the coordinates of the fourth vertex.



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

USING ALGEBRA Use the Distance Formula to find *AB*. (Review 1.3 for 6.3)

65. $A(2, 1)$, $B(6, 9)$ **66.** $A(-4, 2)$, $B(2, -1)$ **67.** $A(-8, -4)$, $B(-1, -3)$

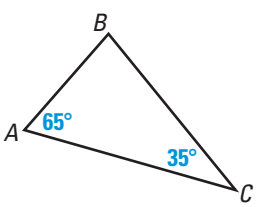
USING ALGEBRA Find the slope of $\overline{AB}$. (Review 3.6 for 6.3)

68. $A(2, 1)$, $B(6, 9)$ **69.** $A(-4, 2)$, $B(2, -1)$ **70.** $A(-8, -4)$, $B(-1, -3)$

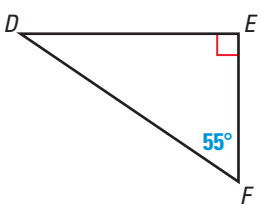
71. PARKING CARS In a parking lot, two guidelines are painted so that they are both perpendicular to the line along the curb. Are the guidelines parallel? Explain why or why not. (Review 3.5)

Name the shortest and longest sides of the triangle. Explain. (Review 5.5)

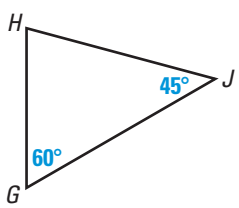
72.



73.



74.



6.3 Proving Quadrilaterals are Parallelograms

What you should learn

GOAL 1 Prove that a quadrilateral is a parallelogram.

GOAL 2 Use coordinate geometry with parallelograms.

Why you should learn it

▼ To understand how **real-life** tools work, such as the bicycle derailleur in **Ex. 27**, which lets you change gears when you are biking uphill.

REAL LIFE

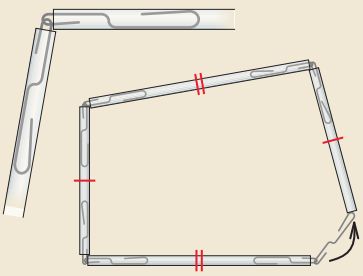
GOAL 1 PROVING QUADRILATERALS ARE PARALLELOGRAMS

The activity illustrates one way to prove that a quadrilateral is a parallelogram.

ACTIVITY Developing Concepts

Investigating Properties of Parallelograms

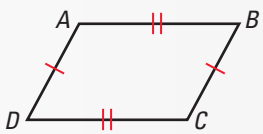
1. Cut four straws to form two congruent pairs.
2. Partly unbend two paperclips, link their smaller ends, and insert the larger ends into two cut straws, as shown. Join the rest of the straws to form a quadrilateral with opposite sides congruent, as shown.
3. Change the angles of your quadrilateral. Is your quadrilateral always a parallelogram?



THEOREMS

THEOREM 6.6

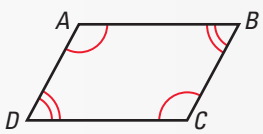
If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.



***ABCD* is a parallelogram.**

THEOREM 6.7

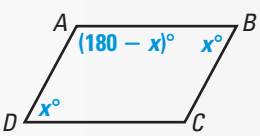
If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.



***ABCD* is a parallelogram.**

THEOREM 6.8

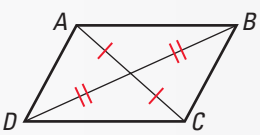
If an angle of a quadrilateral is supplementary to both of its consecutive angles, then the quadrilateral is a parallelogram.



***ABCD* is a parallelogram.**

THEOREM 6.9

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



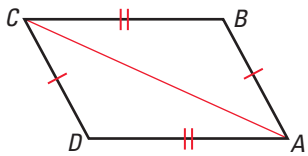
***ABCD* is a parallelogram.**

The proof of Theorem 6.6 is given in Example 1. You will be asked to prove Theorem 6.7, Theorem 6.8, and Theorem 6.9 in Exercises 32–36.

Proof

EXAMPLE 1 *Proof of Theorem 6.6*

Prove Theorem 6.6.



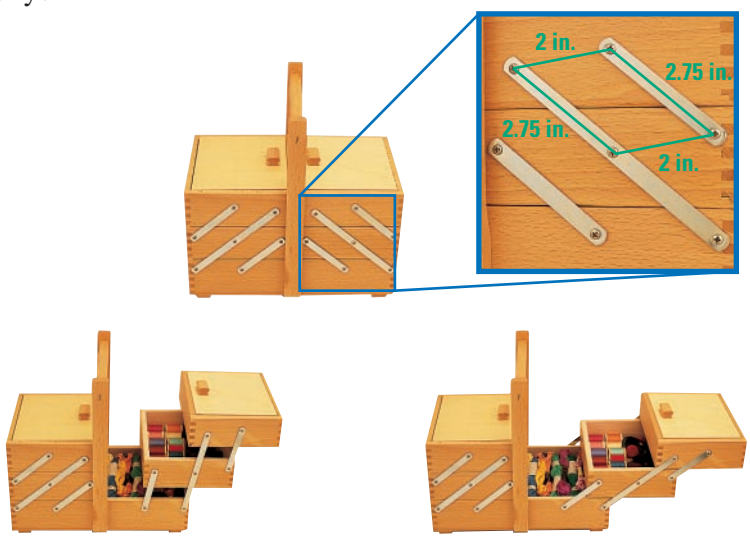
GIVEN ▶ $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$

PROVE ▶ $ABCD$ is a parallelogram.

Statements	Reasons
1. $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$	**1.** Given
2. $\overline{AC} \cong \overline{AC}$	**2.** Reflexive Property of Congruence
3. $\triangle ABC \cong \triangle CDA$	**3.** SSS Congruence Postulate
4. $\angle BAC \cong \angle DCA$, $\angle DAC \cong \angle BCA$	**4.** Corresponding parts of $\cong$ $\triangle$s are $\cong$.
5. $\overline{AB} \parallel \overline{CD}$, $\overline{AD} \parallel \overline{CB}$	**5.** Alternate Interior Angles Converse
6. $ABCD$ is a $\square$.	**6.** Definition of parallelogram

EXAMPLE 2 *Proving Quadrilaterals are Parallelograms*

As the sewing box below is opened, the trays are always parallel to each other. Why?



FOCUS ON APPLICATIONS

CONTAINERS Many containers, such as tackle boxes, jewelry boxes, and tool boxes, use parallelograms in their design to ensure that the trays stay level.

SOLUTION

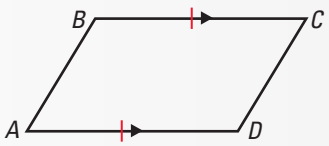
Each pair of hinges are opposite sides of a quadrilateral. The 2.75 inch sides of the quadrilateral are opposite and congruent. The 2 inch sides are also opposite and congruent. Because opposite sides of the quadrilateral are congruent, it is a parallelogram. By the definition of a parallelogram, opposite sides are parallel, so the trays of the sewing box are always parallel.

Theorem 6.10 gives another way to prove a quadrilateral is a parallelogram.

THEOREM

THEOREM 6.10

If one pair of opposite sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.

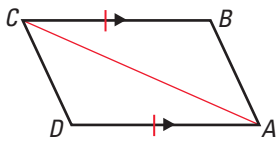


***ABCD* is a parallelogram.**

EXAMPLE 3 *Proof of Theorem 6.10*

Proof

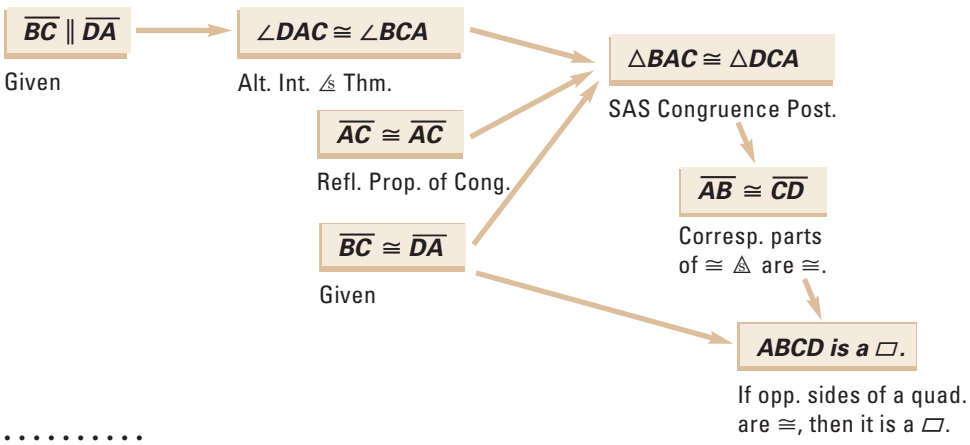
Prove Theorem 6.10.



GIVEN ▶ $\overline{BC} \parallel \overline{DA}$, $\overline{BC} \cong \overline{DA}$

PROVE ▶ $ABCD$ is a parallelogram.

Plan for Proof Show that $\triangle BAC \cong \triangle DCA$, so $\overline{AB} \cong \overline{CD}$. Use Theorem 6.6.



You have studied several ways to prove that a quadrilateral is a parallelogram. In the box below, the first way is also the definition of a parallelogram.

CONCEPT SUMMARY **PROVING QUADRILATERALS ARE PARALLELOGRAMS**

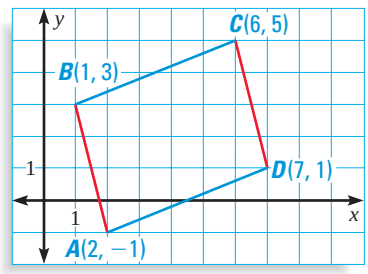
- Show that both pairs of opposite sides are parallel.
- Show that both pairs of opposite sides are congruent.
- Show that both pairs of opposite angles are congruent.
- Show that one angle is supplementary to both consecutive angles.
- Show that the diagonals bisect each other.
- Show that one pair of opposite sides are congruent and parallel.

GOAL 2 USING COORDINATE GEOMETRY

When a figure is in the coordinate plane, you can use the Distance Formula to prove that sides are congruent and you can use the slope formula to prove that sides are parallel.

EXAMPLE 4 *Using Properties of Parallelograms*

Show that $A(2, -1)$, $B(1, 3)$, $C(6, 5)$, and $D(7, 1)$ are the vertices of a parallelogram.



SOLUTION

There are many ways to solve this problem.

Method 1 Show that opposite sides have the same slope, so they are parallel.

$$\text{Slope of } \overline{AB} = \frac{3-(-1)}{1-2} = -4$$

$$\text{Slope of } \overline{CD} = \frac{1-5}{7-6} = -4$$

$$\text{Slope of } \overline{BC} = \frac{5-3}{6-1} = \frac{2}{5}$$

$$\text{Slope of } \overline{DA} = \frac{-1-1}{2-7} = \frac{2}{5}$$

$\overline{AB}$ and $\overline{CD}$ have the same slope so they are parallel. Similarly, $\overline{BC} \parallel \overline{DA}$.

▶ Because opposite sides are parallel, $ABCD$ is a parallelogram.

Method 2 Show that opposite sides have the same length.

$$AB = \sqrt{(1-2)^2 + [3-(-1)]^2} = \sqrt{17}$$

$$CD = \sqrt{(7-6)^2 + (1-5)^2} = \sqrt{17}$$

$$BC = \sqrt{(6-1)^2 + (5-3)^2} = \sqrt{29}$$

$$DA = \sqrt{(2-7)^2 + (-1-1)^2} = \sqrt{29}$$

▶ $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$. Because both pairs of opposite sides are congruent, $ABCD$ is a parallelogram.

Method 3 Show that one pair of opposite sides is congruent and parallel.

Find the slopes and lengths of $\overline{AB}$ and $\overline{CD}$ as shown in Methods 1 and 2.

$$\text{Slope of } \overline{AB} = \text{Slope of } \overline{CD} = -4$$

$$AB = CD = \sqrt{17}$$

▶ $\overline{AB}$ and $\overline{CD}$ are congruent and parallel, so $ABCD$ is a parallelogram.

STUDENT HELP

Study Tip
Because you don't know the measures of the angles of $ABCD$, you can *not* use Theorems 6.7 or 6.8 in Example 4.

STUDENT HELP

INTERNET **HOMEWORK HELP**
Visit our Web site www.mcdougallittell.com for extra examples.

GUIDED PRACTICE

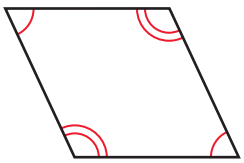
Concept Check ✓

1. Is a hexagon with opposite sides parallel called a parallelogram? Explain.

Skill Check ✓

Decide whether you are given enough information to determine that the quadrilateral is a parallelogram. Explain your reasoning.

2.



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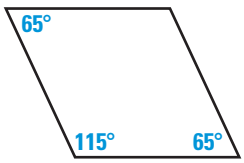

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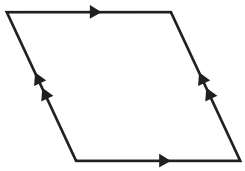
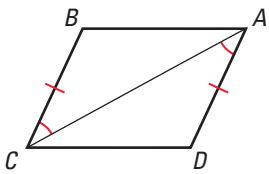


Describe how you would prove that *ABCD* is a parallelogram.

5.



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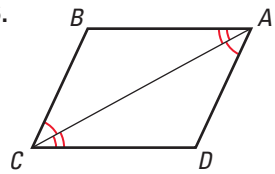
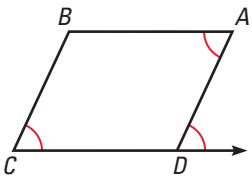


7.



8. Describe at least three ways to show that $A(0, 0)$, $B(2, 6)$, $C(5, 7)$, and $D(3, 1)$ are the vertices of a parallelogram.

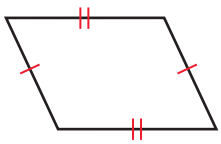
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 813.

LOGICAL REASONING **Are you given enough information to determine whether the quadrilateral is a parallelogram? Explain.**

9.



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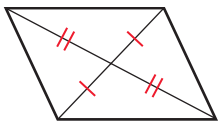

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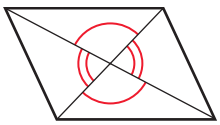
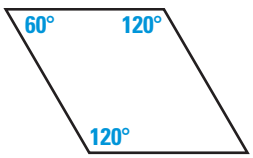


12.



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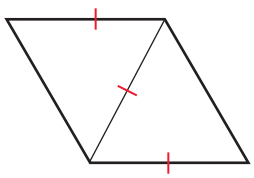

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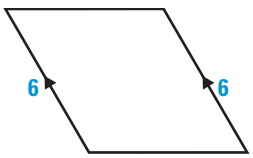


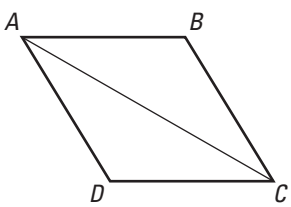
STUDENT HELP

HOMEWORK HELP

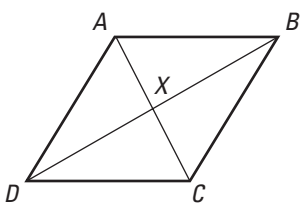
Example 1: Exs. 15, 16, 32, 33
Example 2: Exs. 21, 28, 31
Example 3: Exs. 32, 33
Example 4: Exs. 21–26, 34–36

LOGICAL REASONING **Describe how to prove that *ABCD* is a parallelogram. Use the given information.**

15. $\triangle ABC \cong \triangle CDA$

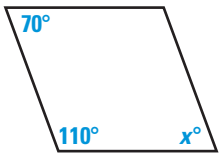


16. $\triangle AXB \cong \triangle CXD$

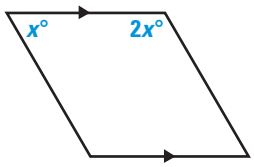


USING ALGEBRA **What value of x will make the polygon a parallelogram?**

17.



18.



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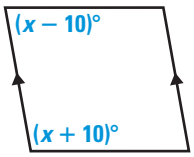


20. VISUAL THINKING Draw a quadrilateral that has one pair of congruent sides and one pair of parallel sides but that is not a parallelogram.

COORDINATE GEOMETRY **Use the given definition or theorem to prove that *ABCD* is a parallelogram. Use $A(-1, 6)$, $B(3, 5)$, $C(5, -3)$, and $D(1, -2)$.**

21. Theorem 6.6

22. Theorem 6.9

23. definition of a parallelogram

24. Theorem 6.10

USING COORDINATE GEOMETRY **Prove that the points represent the vertices of a parallelogram. Use a different method for each exercise.**

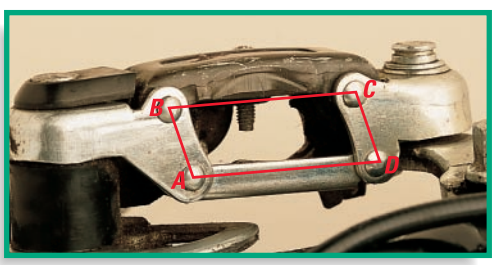
25. $J(-6, 2)$, $K(-1, 3)$, $L(2, -3)$, $M(-3, -4)$

26. $P(2, 5)$, $Q(8, 4)$, $R(9, -4)$, $S(3, -3)$

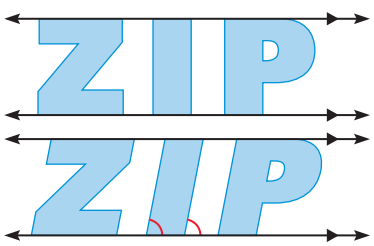
FOCUS ON APPLICATIONS

REAL LIFE **DERAILLEURS** (named from the French word meaning 'derail') move the chain among two to six sprockets of different diameters to change gears.

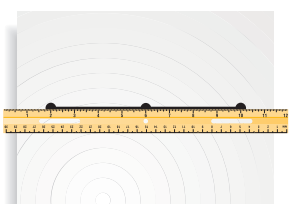
27. CHANGING GEARS When you change gears on a bicycle, the *derailleur* moves the chain to the new gear. For the derailleur at the right, $AB = 1.8$ cm, $BC = 3.6$ cm, $CD = 1.8$ cm, and $DA = 3.6$ cm. Explain why $\overline{AB}$ and $\overline{CD}$ are always parallel when the derailleur moves.



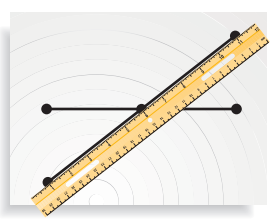
28. COMPUTERS Many word processors have a feature that allows a regular letter to be changed to an oblique (slanted) letter. The diagram at the right shows some regular letters and their oblique versions. Explain how you can prove that the oblique I is a parallelogram.



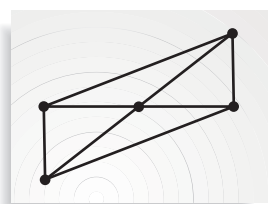
29. VISUAL REASONING Explain why the following method of drawing a parallelogram works. State a theorem to support your answer.



1 Use a ruler to draw a segment and its midpoint.



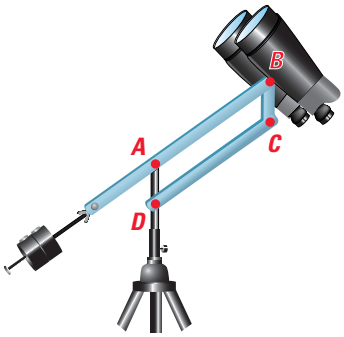
2 Draw another segment so the midpoints coincide.



3 Connect the endpoints of the segments.

30. **CONSTRUCTION** There are many ways to use a compass and straightedge to construct a parallelogram. Describe a method that uses Theorem 6.6, Theorem 6.8, or Theorem 6.10. Then use your method to construct a parallelogram.

31. **BIRD WATCHING** You are designing a binocular mount that will keep the binoculars pointed in the same direction while they are raised and lowered for different viewers. If $\overline{BC}$ is always vertical, the binoculars will always point in the same direction. How can you design the mount so $\overline{BC}$ is always vertical? Justify your answer.



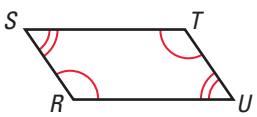
PROVING THEOREMS 6.7 AND 6.8 **Write a proof of the theorem.**

32. Prove Theorem 6.7.

GIVEN ▶ $\angle R \cong \angle T$, $\angle S \cong \angle U$

PROVE ▶ $RSTU$ is a parallelogram.

Plan for Proof Show that the sum $2(m\angle S) + 2(m\angle T) = 360°$, so $\angle S$ and $\angle T$ are supplementary and $\overline{SR} \parallel \overline{UT}$.

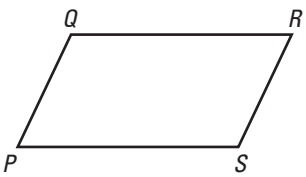


33. Prove Theorem 6.8.

GIVEN ▶ $\angle P$ is supplementary to $\angle Q$ and $\angle S$.

PROVE ▶ $PQRS$ is a parallelogram.

Plan for Proof Show that opposite sides of $PQRS$ are parallel.



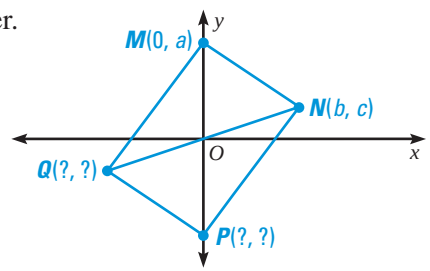
PROVING THEOREM 6.9 **In Exercises 34–36, complete the coordinate proof of Theorem 6.9.**

GIVEN ▶ Diagonals $\overline{MP}$ and $\overline{NQ}$ bisect each other.

PROVE ▶ $MNPQ$ is a parallelogram.

Plan for Proof Show that opposite sides of $MNPQ$ have the same slope.

Place $MNPQ$ in the coordinate plane so the diagonals intersect at the origin and $\overline{MP}$ lies on the y-axis. Let the coordinates of M be $(0, a)$ and the coordinates of N be (b, c). Copy the graph at the right.



34. What are the coordinates of P? Explain your reasoning and label the coordinates on your graph.

35. What are the coordinates of Q? Explain your reasoning and label the coordinates on your graph.

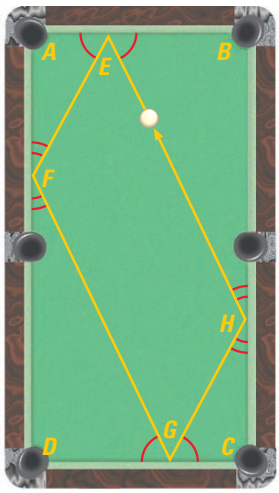
36. Find the slope of each side of $MNPQ$ and show that the slopes of opposite sides are equal.

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with the coordinate proof in Exs. 34–36.

Test Preparation

37. MULTI-STEP PROBLEM You shoot a pool ball as shown at the right and it rolls back to where it started. The ball bounces off each wall at the same angle at which it hit the wall. Copy the diagram and add each angle measure as you know it.



a. The ball hits the first wall at an angle of about 63°. So $m\angle AEF = m\angle BEH \approx 63°$. Explain why $m\angle AFE \approx 27°$.

b. Explain why $m\angle FGD \approx 63°$.

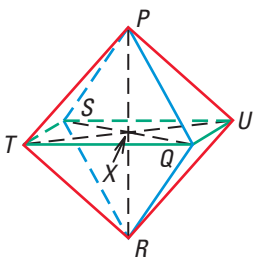
c. What is $m\angle GHC$? $m\angle EHB$?

d. Find the measure of each interior angle of *EFGH*. What kind of shape is *EFGH*? How do you know?

★ Challenge

EXTRA CHALLENGE
www.mcdougallittell.com

38. VISUAL THINKING *PQRS* is a parallelogram and *QTSU* is a parallelogram. Use the diagonals of the parallelograms to explain why *PTRU* is a parallelogram.



MIXED REVIEW

USING ALGEBRA Rewrite the biconditional statement as a conditional statement and its converse. (Review 2.2 for 6.4)

39. $x^2 + 2 = 2$ if and only if $x = 0$.

40. $4x + 7 = x + 37$ if and only if $x = 10$.

41. A quadrilateral is a parallelogram if and only if each pair of opposite sides are parallel.

WRITING BICONDITIONAL STATEMENTS Write the pair of theorems from Lesson 5.1 as a single biconditional statement. (Review 2.2, 5.1 for 6.4)

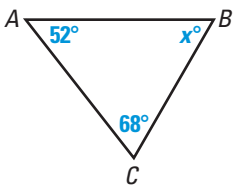
42. Theorems 5.1 and 5.2

43. Theorems 5.3 and 5.4

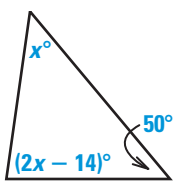
44. Write an equation of the line that is perpendicular to $y = -4x + 2$ and passes through the point $(1, -2)$. (Review 3.7)

ANGLE MEASURES Find the value of *x*. (Review 4.1)

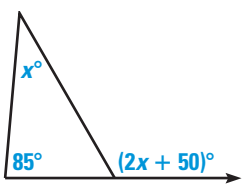
45.



46.



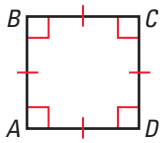
47.



QUIZ 1

Self-Test for Lessons 6.1–6.3

1. Choose the words that describe the quadrilateral at the right: *concave*, *convex*, *equilateral*, *equiangular*, and *regular*. **(Lesson 6.1)**

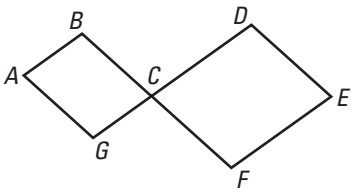


2. Find the value of x. Explain your reasoning. **(Lesson 6.1)**

3. Write a proof. **(Lesson 6.2)**

GIVEN ▶ *ABCG* and *CDEF* are parallelograms.

PROVE ▶ $\angle A \cong \angle E$



4. Describe two ways to show that $A(-4, 1)$, $B(3, 0)$, $C(5, -7)$, and $D(-2, -6)$ are the vertices of a parallelogram. **(Lesson 6.3)**

MATH & History — History of Finding Area

APPLICATION LINK
www.mcdougallittell.com

THEN

THOUSANDS OF YEARS AGO, the Egyptians needed to find the area of the land they were farming. The mathematical methods they used are described in a papyrus dating from about 1650 B.C.

NOW

TODAY, satellites and aerial photographs can be used to measure the areas of large or inaccessible regions.

1. Find the area of the trapezoid outlined on the aerial photograph. The formula for the area of a trapezoid appears on page 374.

c.1650 B.C.

This Egyptian papyrus includes methods for finding area.

Methods for finding area are recorded in this Chinese manuscript.

c. 300 B.C.–A.D. 200

Surveyors use signals from satellites to measure large areas.

1990s

6.4 Rhombuses, Rectangles, and Squares

What you should learn

GOAL 1 Use properties of sides and angles of rhombuses, rectangles, and squares.

GOAL 2 Use properties of diagonals of rhombuses, rectangles, and squares.

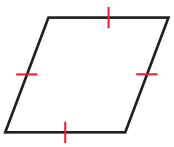
Why you should learn it

▼ To simplify **real-life** tasks, such as checking whether a theater flat is rectangular in **Example 6**.

REAL LIFE

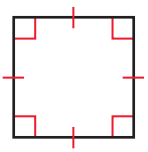
GOAL 1 PROPERTIES OF SPECIAL PARALLELOGRAMS

In this lesson you will study three special types of parallelograms: *rhombuses*, *rectangles*, and *squares*.



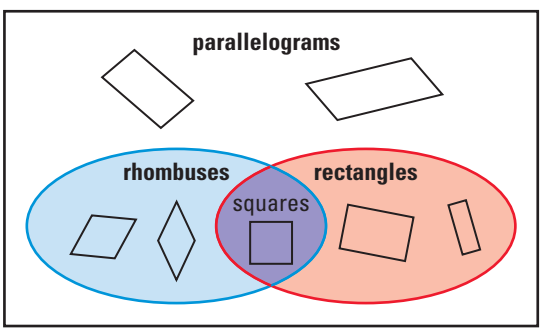
A **rhombus** is a parallelogram with four congruent sides.

A **rectangle** is a parallelogram with four right angles.



A **square** is a parallelogram with four congruent sides and four right angles.

The *Venn diagram* at the right shows the relationships among parallelograms, rhombuses, rectangles, and squares. Each shape has the properties of every group that it belongs to. For instance, a square is a rectangle, a rhombus, and a parallelogram, so it has all of the properties of each of those shapes.

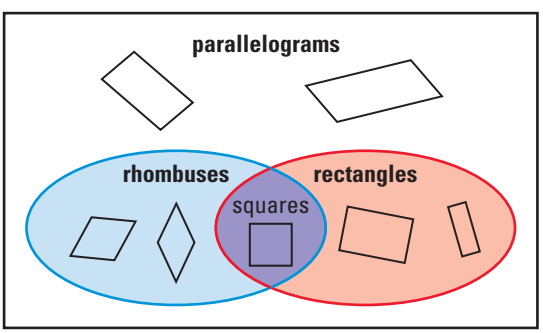


EXAMPLE 1 *Describing a Special Parallelogram*

Decide whether the statement is *always*, *sometimes*, or *never* true.

a. A rhombus is a rectangle.

b. A parallelogram is a rectangle.



SOLUTION

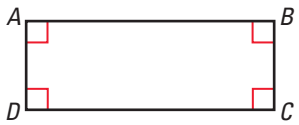
a. The statement is *sometimes* true. In the Venn diagram, the regions for rhombuses and rectangles overlap. If the rhombus is a square, it is a rectangle.

b. The statement is *sometimes* true. Some parallelograms are rectangles. In the Venn diagram, you can see that some of the shapes in the parallelogram box are in the region for rectangles, but many aren't.

Logical Reasoning

EXAMPLE 2 *Using Properties of Special Parallelograms*

ABCD is a rectangle. What else do you know about *ABCD*?



SOLUTION

Because *ABCD* is a rectangle, it has four right angles by the definition. The definition also states that rectangles are parallelograms, so *ABCD* has all the properties of a parallelogram:

- Opposite sides are parallel and congruent.
- Opposite angles are congruent and consecutive angles are supplementary.
- Diagonals bisect each other.

. . . . . .

A rectangle is defined as a *parallelogram* with four right angles. But *any quadrilateral* with four right angles is a rectangle because any quadrilateral with four right angles is a parallelogram. In Exercises 48–50 you will justify the following corollaries to the definitions of rhombus, rectangle, and square.

STUDENT HELP

Look Back
For help with biconditional statements, see p. 80.

COROLLARIES ABOUT SPECIAL QUADRILATERALS

RHOMBUS COROLLARY

A quadrilateral is a rhombus if and only if it has four congruent sides.

RECTANGLE COROLLARY

A quadrilateral is a rectangle if and only if it has four right angles.

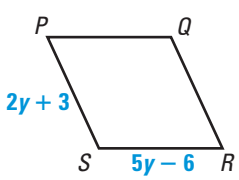
SQUARE COROLLARY

A quadrilateral is a square if and only if it is a rhombus and a rectangle.

You can use these corollaries to prove that a quadrilateral is a rhombus, rectangle, or square without proving first that the quadrilateral is a parallelogram.

EXAMPLE 3 *Using Properties of a Rhombus*

In the diagram at the right, *PQRS* is a rhombus. What is the value of *y*?



SOLUTION

All four sides of a rhombus are congruent, so $RS = PS$.

$5y - 6 = 2y + 3$	**Equate lengths of congruent sides.**
$5y = 2y + 9$	**Add 6 to each side.**
$3y = 9$	**Subtract 2*y* from each side.**
$y = 3$	**Divide each side by 3.**

GOAL 2 USING DIAGONALS OF SPECIAL PARALLELOGRAMS

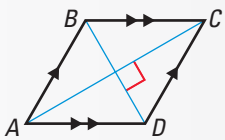
The following theorems are about diagonals of rhombuses and rectangles. You are asked to prove Theorems 6.12 and 6.13 in Exercises 51, 52, 59, and 60.

THEOREMS

THEOREM 6.11

A parallelogram is a rhombus if and only if its diagonals are perpendicular.

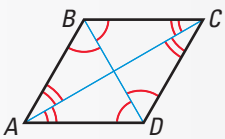
ABCD is a rhombus if and only if $\overline{AC} \perp \overline{BD}$.



THEOREM 6.12

A parallelogram is a rhombus if and only if each diagonal bisects a pair of opposite angles.

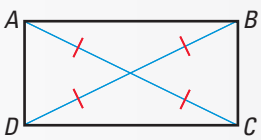
ABCD is a rhombus if and only if $\overline{AC}$ bisects $\angle DAB$ and $\angle BCD$ and $\overline{BD}$ bisects $\angle ADC$ and $\angle CBA$.



THEOREM 6.13

A parallelogram is a rectangle if and only if its diagonals are congruent.

ABCD is a rectangle if and only if $\overline{AC} \cong \overline{BD}$.



You can rewrite Theorem 6.11 as a conditional statement and its converse.

Conditional statement: If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

Converse: If a parallelogram is a rhombus, then its diagonals are perpendicular.

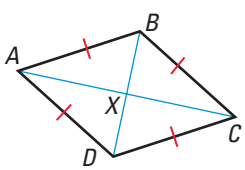
To prove the theorem, you must prove both statements.

EXAMPLE 4 *Proving Theorem 6.11*

Write a paragraph proof of the converse above.

GIVEN ▶ *ABCD* is a rhombus.

PROVE ▶ $\overline{AC} \perp \overline{BD}$



SOLUTION

Paragraph Proof *ABCD* is a rhombus, so $\overline{AB} \cong \overline{CB}$. Because *ABCD* is a parallelogram, its diagonals bisect each other so $\overline{AX} \cong \overline{CX}$ and $\overline{BX} \cong \overline{BX}$. Use the SSS Congruence Postulate to prove $\triangle AXB \cong \triangle CXB$, so $\angle AXB \cong \angle CXB$. Then, because $\overline{AC}$ and $\overline{BD}$ intersect to form congruent adjacent angles, $\overline{AC} \perp \overline{BD}$.

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

EXAMPLE 5 *Coordinate Proof of Theorem 6.11*

Proof

In Example 4, a paragraph proof was given for part of Theorem 6.11. Write a coordinate proof of the original conditional statement.

GIVEN ▶ $ABCD$ is a parallelogram, $\overline{AC} \perp \overline{BD}$.

PROVE ▶ $ABCD$ is a rhombus.

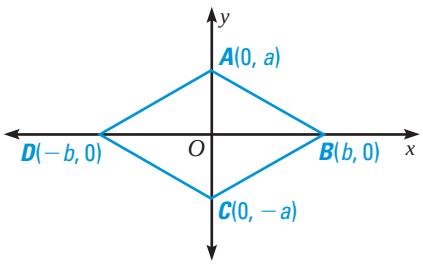
SOLUTION

Assign coordinates Because $\overline{AC} \perp \overline{BD}$, place $ABCD$ in the coordinate plane so $\overline{AC}$ and $\overline{BD}$ lie on the axes and their intersection is at the origin.

Let $(0, a)$ be the coordinates of A, and let $(b, 0)$ be the coordinates of B.

Because $ABCD$ is a parallelogram, the diagonals bisect each other and $OA = OC$. So, the coordinates of C are $(0, -a)$.

Similarly, the coordinates of D are $(-b, 0)$.



Find the lengths of the sides of $ABCD$. Use the Distance Formula.

$$AB = \sqrt{(b-0)^2 + (0-a)^2} = \sqrt{b^2 + a^2}$$

$$BC = \sqrt{(0-b)^2 + (-a-0)^2} = \sqrt{b^2 + a^2}$$

$$CD = \sqrt{(-b-0)^2 + [0-(-a)]^2} = \sqrt{b^2 + a^2}$$

$$DA = \sqrt{[0-(-b)]^2 + (a-0)^2} = \sqrt{b^2 + a^2}$$

▶ All of the side lengths are equal, so $ABCD$ is a rhombus.

FOCUS ON APPLICATIONS

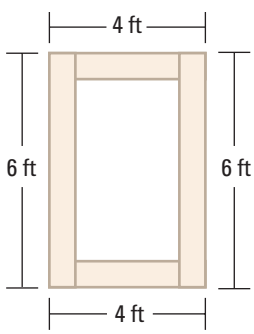
CARPENTRY If a screen door is not rectangular, you can use a piece of hardware called a *turnbuckle* to shorten the longer diagonal until the door is rectangular.

EXAMPLE 6 *Checking a Rectangle*

CARPENTRY You are building a rectangular frame for a theater set.

a. First, you nail four pieces of wood together, as shown at the right. What is the shape of the frame?

b. To make sure the frame is a rectangle, you measure the diagonals. One is 7 feet 4 inches and the other is 7 feet 2 inches. Is the frame a rectangle? Explain.



SOLUTION

a. Opposite sides are congruent, so the frame is a parallelogram.

b. The parallelogram is not a rectangle. If it were a rectangle, the diagonals would be congruent.

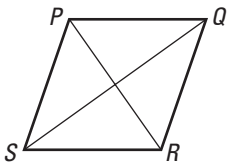
GUIDED PRACTICE

Vocabulary Check ✓

1. What is another name for an *equilateral quadrilateral*?

Concept Check ✓

2. Theorem 6.12 is a biconditional statement. Rewrite the theorem as a conditional statement and its converse, and tell what each statement means for parallelogram *PQRS*.



Skill Check ✓

Decide whether the statement is *sometimes, always,* or *never* true.

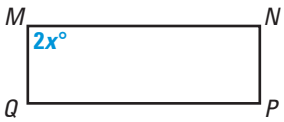
3. A rectangle is a parallelogram.
4. A parallelogram is a rhombus.
5. A rectangle is a rhombus.
6. A square is a rectangle.

Which of the following quadrilaterals have the given property?

7. All sides are congruent.
8. All angles are congruent.
9. The diagonals are congruent.
10. Opposite angles are congruent.

A. Parallelogram
B. Rectangle
C. Rhombus
D. Square

11. *MNPQ* is a rectangle. What is the value of x?



PRACTICE AND APPLICATIONS

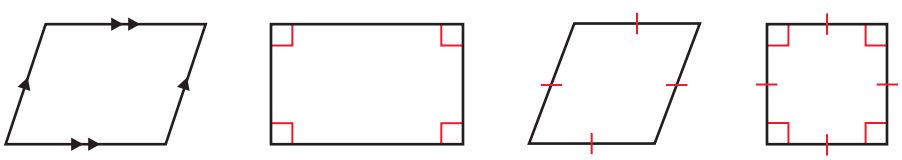
STUDENT HELP

Extra Practice to help you master skills is on p. 814.

RECTANGLE **For any rectangle *ABCD*, decide whether the statement is *always, sometimes,* or *never* true. Draw a sketch and explain your answer.**

12. $\angle A \cong \angle B$
13. $\overline{AB} \cong \overline{BC}$
14. $\overline{AC} \cong \overline{BD}$
15. $\overline{AC} \perp \overline{BD}$

PROPERTIES **List each quadrilateral for which the statement is true.**



parallelogram rectangle rhombus square

16. It is equiangular.
17. It is equiangular and equilateral.
18. The diagonals are perpendicular.
19. Opposite sides are congruent.
20. The diagonals bisect each other.
21. The diagonals bisect opposite angles.

PROPERTIES **Sketch the quadrilateral and list everything you know about it.**

22. parallelogram *FGHI*
23. rhombus *PQRS*
24. square *ABCD*

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 12–15, 27–32
Example 2: Exs. 27–32, 51
Example 3: Exs. 33–43
Example 4: Exs. 44–52
Example 5: Exs. 55–60
Example 6: Exs. 61, 62

LOGICAL REASONING **Give another name for the quadrilateral.**

25. equiangular quadrilateral

26. regular quadrilateral

RHOMBUS **For any rhombus *ABCD*, decide whether the statement is *always*, *sometimes*, or *never* true. Draw a sketch and explain your answer.**

27. $\angle A \cong \angle C$

28. $\angle A \cong \angle B$

29. $\angle ABD \cong \angle CBD$

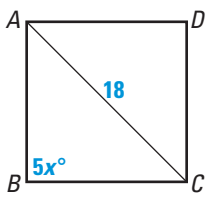
30. $\overline{AB} \cong \overline{BC}$

31. $\overline{AC} \cong \overline{BD}$

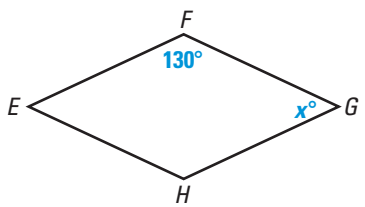
32. $\overline{AD} \cong \overline{CD}$

USING ALGEBRA **Find the value of *x*.**

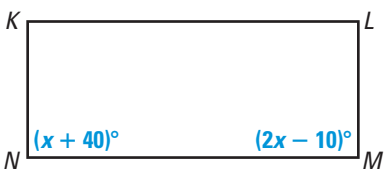
33. *ABCD* is a square.



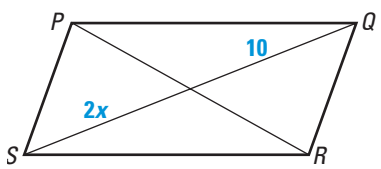
34. *EFGH* is a rhombus.



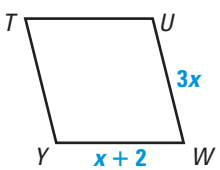
35. *KLMN* is a rectangle.



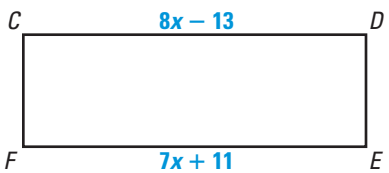
36. *PQRS* is a parallelogram.



37. *TUWY* is a rhombus.



38. *CDEF* is a rectangle.



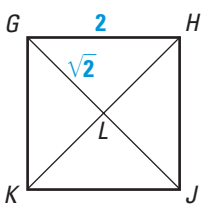
COMPLETING STATEMENTS ***GHJK* is a square with diagonals intersecting at *L*. Given that *GH* = 2 and *GL* = $\sqrt{2}$, complete the statement.**

39. $HK =$ __?__

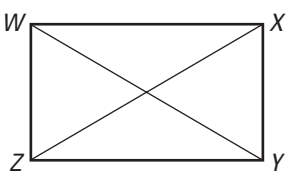
40. $m\angle KLJ =$ __?__

41. $m\angle HJG =$ __?__

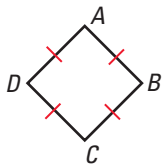
42. Perimeter of $\triangle HJK =$ __?__



43. **USING ALGEBRA** *WXYZ* is a rectangle. The perimeter of $\triangle XYZ$ is 24. $XY + YZ = 5x - 1$ and $XZ = 13 - x$. Find *WY*.



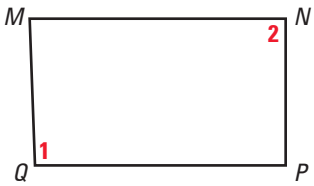
44. **LOGICAL REASONING** What additional information do you need to prove that *ABCD* is a square?



PROOF In Exercises 45 and 46, write any kind of proof.

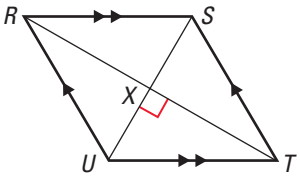
45. GIVEN ▶ $\overline{MN} \parallel \overline{PQ}$, $\angle 1 \ncong \angle 2$

PROVE ▶ $\overline{MQ}$ is not parallel to $\overline{PN}$.

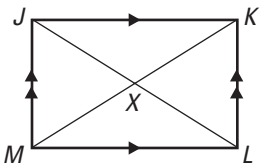


46. GIVEN ▶ *RSTU* is a ▱, $\overline{SU} \perp \overline{RT}$

PROVE ▶ $\angle STR \cong \angle UTR$



47. **BICONDITIONAL STATEMENTS** Rewrite Theorem 6.13 as a conditional statement and its converse. Tell what each statement means for parallelogram *JKLM*.



LOGICAL REASONING Write the corollary as a conditional statement and its converse. Then explain why each statement is true.

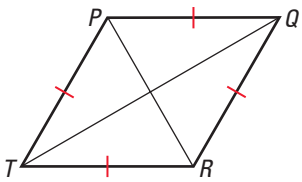
48. Rhombus corollary **49.** Rectangle corollary **50.** Square corollary

PROVING THEOREM 6.12 Prove both conditional statements of Theorem 6.12.

51. GIVEN ▶ *PQRT* is a rhombus.

PROVE ▶ $\overline{PR}$ bisects $\angle TPQ$ and $\angle QRT$. $\overline{TQ}$ bisects $\angle PTR$ and $\angle RQP$.

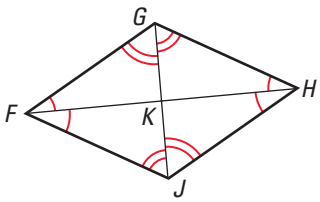
Plan for Proof To prove that $\overline{PR}$ bisects $\angle TPQ$ and $\angle QRT$, first prove that $\triangle PRQ \cong \triangle PRT$.



52. GIVEN ▶ *FGHJ* is a parallelogram. $\overline{FH}$ bisects $\angle JFG$ and $\angle GHJ$. $\overline{JG}$ bisects $\angle FJH$ and $\angle HGF$.

PROVE ▶ *FGHJ* is a rhombus.

Plan for Proof Prove $\triangle FHJ \cong \triangle FHG$ so $\overline{JH} \cong \overline{GH}$. Then use the fact that $\overline{JH} \cong \overline{FG}$ and $\overline{GH} \cong \overline{FJ}$.



CONSTRUCTION Explain how to construct the figure using a straightedge and a compass. Use a definition or theorem from this lesson to explain why your method works.

53. a rhombus that is not a square **54.** a rectangle that is not a square

COORDINATE GEOMETRY **It is given that *PQRS* is a parallelogram. Graph ▱*PQRS*. Decide whether it is a *rectangle*, a *rhombus*, a *square*, or *none of the above*. Justify your answer using theorems about quadrilaterals.**

55. $P(3, 1)$
$Q(3, -3)$
$R(-2, -3)$
$S(-2, 1)$

56. $P(5, 2)$
$Q(1, 9)$
$R(-3, 2)$
$S(1, -5)$

57. $P(-1, 4)$
$Q(-3, 2)$
$R(2, -3)$
$S(4, -1)$

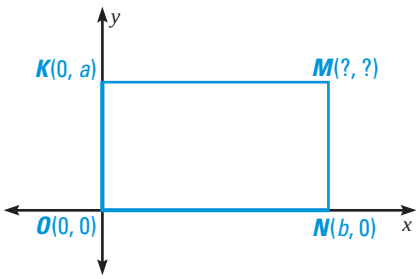
58. $P(5, 2)$
$Q(2, 5)$
$R(-1, 2)$
$S(2, -1)$

COORDINATE PROOF OF THEOREM 6.13 **In Exercises 59 and 60, you will complete a coordinate proof of one conditional statement of Theorem 6.13.**

GIVEN ▶ $KMNO$ is a rectangle.

PROVE ▶ $\overline{OM} \cong \overline{KN}$

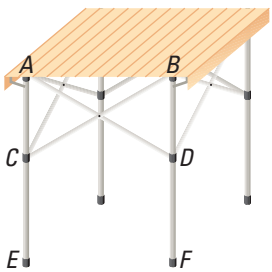
Because $\angle O$ is a right angle, place $KMNO$ in the coordinate plane so O is at the origin, $\overline{ON}$ lies on the x-axis and $\overline{OK}$ lies on the y-axis. Let the coordinates of K be $(0, a)$ and let the coordinates of N be $(b, 0)$.



59. What are the coordinates of M? Explain your reasoning.

60. Use the Distance Formula to prove that $\overline{OM} \cong \overline{KN}$.

PORTABLE TABLE **The legs of the table shown at the right are all the same length. The cross braces are all the same length and bisect each other.**



61. Show that the edge of the tabletop $\overline{AB}$ is perpendicular to legs $\overline{AE}$ and $\overline{BF}$.

62. Show that $\overline{AB}$ is parallel to $\overline{EF}$.

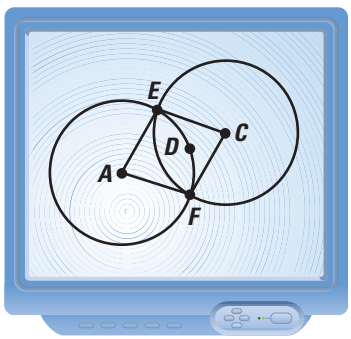
STUDENT HELP

INTERNET SOFTWARE HELP Visit our Web site www.mcdougallittell.com to see instructions for several software applications.

TECHNOLOGY **In Exercises 63–65, use geometry software.**

Draw a segment $\overline{AB}$ and a point C on the segment. Construct the midpoint D of $\overline{AB}$. Then hide $\overline{AB}$ and point B so only points A, D, and C are visible.

Construct two circles with centers A and C using the length $\overline{AD}$ as the radius of each circle. Label the points of intersection E and F. Draw $\overline{AE}$, $\overline{CE}$, $\overline{CF}$, and $\overline{AF}$.



63. What kind of shape is $AECF$? How do you know? What happens to the shape as you drag A? drag C?

64. Hide the circles and point D, and draw diagonals $\overline{EF}$ and $\overline{AC}$. Measure $\angle EAC$, $\angle FAC$, $\angle AEF$, and $\angle CEF$. What happens to the measures as you drag A? drag C?

65. Which theorem does this construction illustrate?

Test Preparation

66. MULTIPLE CHOICE In rectangle $ABCD$, if $AB = 7x - 3$ and $CD = 4x + 9$, then $x =$?.

Ⓐ 1 Ⓑ 2 Ⓒ 3 Ⓓ 4 Ⓔ 5

67. MULTIPLE CHOICE In parallelogram $KLMN$, $KM = LN$, $m\angle KLM = 2xy$, and $m\angle LMN = 9x + 9$. Find the value of y.

Ⓐ 9 Ⓑ 5 Ⓒ 18

Ⓓ 10 Ⓔ Cannot be determined.

68. *Writing* Explain why a parallelogram with one right angle is a rectangle.

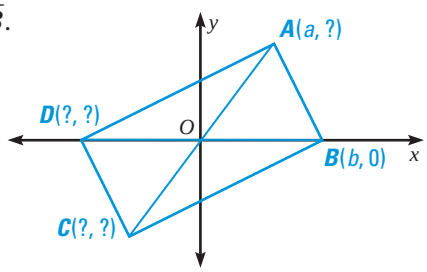
★ Challenge

COORDINATE PROOF OF THEOREM 6.13 Complete the coordinate proof of one conditional statement of Theorem 6.13.

GIVEN ▶ $ABCD$ is a parallelogram, $\overline{AC} \cong \overline{DB}$.

PROVE ▶ $ABCD$ is a rectangle.

Place $ABCD$ in the coordinate plane so $\overline{DB}$ lies on the x-axis and the diagonals intersect at the origin. Let the coordinates of B be $(b, 0)$ and let the x-coordinate of A be a as shown.



69. Explain why $OA = OB = OC = OD$.

70. Write the y-coordinate of A in terms of a and b. Explain your reasoning.

71. Write the coordinates of C and D in terms of a and b. Explain your reasoning.

72. Find and compare the slopes of the sides to prove that $ABCD$ is a rectangle.

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

USING THE SAS CONGRUENCE POSTULATE Decide whether enough information is given to determine that $\triangle ABC \cong \triangle DEF$. (Review 4.3)

73. $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$

74. $\overline{AB} \cong \overline{BC}$, $\overline{BC} \cong \overline{CA}$, $\angle A \cong \angle D$

75. $\angle B \cong \angle E$, $\overline{AC} \cong \overline{DF}$, $\overline{AB} \cong \overline{DE}$

76. $\overline{EF} \cong \overline{BC}$, $\overline{DF} \cong \overline{AB}$, $\angle A \cong \angle E$

77. $\angle C \cong \angle F$, $\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$

78. $\angle B \cong \angle E$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$

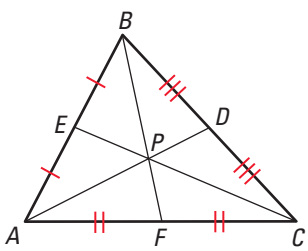
CONCURRENCY PROPERTY FOR MEDIANS Use the information given in the diagram to fill in the blanks. (Review 5.3)

79. $AP = 1$, $PD =$?

80. $PC = 6.6$, $PE =$?

81. $PB = 6$, $FB =$?

82. $AD = 39$, $PD =$?



83. INDIRECT PROOF Write an indirect proof to show that there is no quadrilateral with four acute angles. (Review 6.1 for 6.5)

6.5 Trapezoids and Kites

What you should learn

GOAL 1 Use properties of trapezoids.

GOAL 2 Use properties of kites.

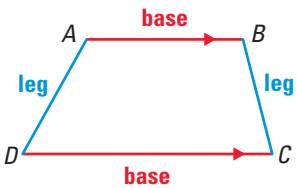
Why you should learn it

▼ To solve **real-life** problems, such as planning the layers of a layer cake in **Example 3**.

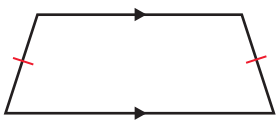
REAL LIFE

GOAL 1 USING PROPERTIES OF TRAPEZOIDS

A **trapezoid** is a quadrilateral with exactly one pair of parallel sides. The parallel sides are the **bases**. A trapezoid has two pairs of **base angles**. For instance, in trapezoid $ABCD$, $\angle D$ and $\angle C$ are one pair of base angles. The other pair is $\angle A$ and $\angle B$. The nonparallel sides are the **legs** of the trapezoid.



If the legs of a trapezoid are congruent, then the trapezoid is an **isosceles trapezoid**.



isosceles trapezoid

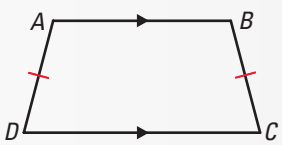
You are asked to prove the following theorems in the exercises.

THEOREMS

THEOREM 6.14

If a trapezoid is isosceles, then each pair of base angles is congruent.

$\angle A \cong \angle B$, $\angle C \cong \angle D$



THEOREM 6.15

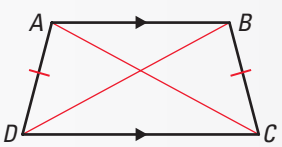
If a trapezoid has a pair of congruent base angles, then it is an isosceles trapezoid.

$ABCD$ is an isosceles trapezoid.

THEOREM 6.16

A trapezoid is isosceles if and only if its diagonals are congruent.

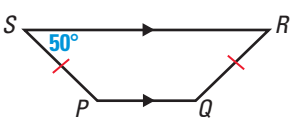
$ABCD$ is isosceles if and only if $\overline{AC} \cong \overline{BD}$.



$AD \cong BC$

EXAMPLE 1 *Using Properties of Isosceles Trapezoids*

$PQRS$ is an isosceles trapezoid. Find $m\angle P$, $m\angle Q$, and $m\angle R$.



SOLUTION $PQRS$ is an isosceles trapezoid, so $m\angle R = m\angle S = 50°$. Because $\angle S$ and $\angle P$ are consecutive interior angles formed by parallel lines, they are supplementary. So, $m\angle P = 180° - 50° = 130°$, and $m\angle Q = m\angle P = 130°$.

STUDENT HELP

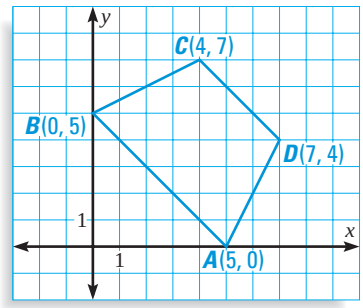
INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

EXAMPLE 2 *Using Properties of Trapezoids*

Show that $ABCD$ is a trapezoid.

SOLUTION

Compare the slopes of opposite sides.



$$\text{The slope of } \overline{AB} = \frac{5-0}{0-5} = \frac{5}{-5} = -1.$$

$$\text{The slope of } \overline{CD} = \frac{4-7}{7-4} = \frac{-3}{3} = -1.$$

The slopes of $\overline{AB}$ and $\overline{CD}$ are equal, so $\overline{AB} \parallel \overline{CD}$.

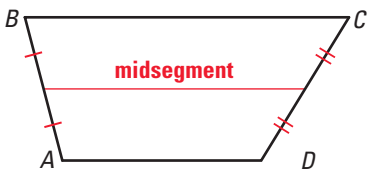
$$\text{The slope of } \overline{BC} = \frac{7-5}{4-0} = \frac{2}{4} = \frac{1}{2}.$$

$$\text{The slope of } \overline{AD} = \frac{4-0}{7-5} = \frac{4}{2} = 2.$$

The slopes of $\overline{BC}$ and $\overline{AD}$ are not equal, so $\overline{BC}$ is not parallel to $\overline{AD}$.

▶ So, because $\overline{AB} \parallel \overline{CD}$ and $\overline{BC}$ is not parallel to $\overline{AD}$, $ABCD$ is a trapezoid.

The **midsegment** of a trapezoid is the segment that connects the midpoints of its legs. Theorem 6.17 is similar to the Midsegment Theorem for triangles. You will justify part of this theorem in Exercise 42. A proof appears on page 839.

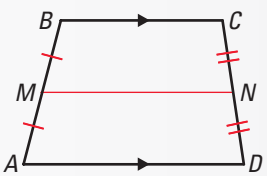


THEOREM

THEOREM 6.17 ***Midsegment Theorem for Trapezoids***

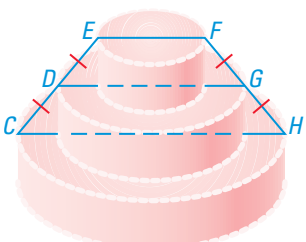
The midsegment of a trapezoid is parallel to each base and its length is one half the sum of the lengths of the bases.

$$\overline{MN} \parallel \overline{AD},\ \overline{MN} \parallel \overline{BC},\ MN = \frac{1}{2}(AD + BC)$$



EXAMPLE 3 *Finding Midsegment Lengths of Trapezoids*

REAL LIFE **LAYER CAKE** A baker is making a cake like the one at the right. The top layer has a diameter of 8 inches and the bottom layer has a diameter of 20 inches. How big should the middle layer be?



SOLUTION

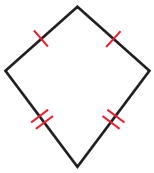
Use the Midsegment Theorem for Trapezoids.

$$DG = \frac{1}{2}(EF + CH) = \frac{1}{2}(8 + 20) = 14 \text{ inches}$$

The simplest of flying kites often use the geometric kite shape.

GOAL 2 USING PROPERTIES OF KITES

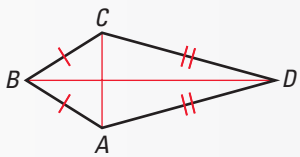
A **kite** is a quadrilateral that has two pairs of consecutive congruent sides, but opposite sides are not congruent. You are asked to prove Theorem 6.18 and Theorem 6.19 in Exercises 46 and 47.



THEOREMS ABOUT KITES

THEOREM 6.18

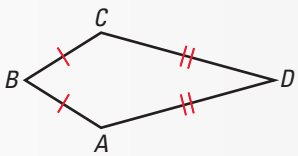
If a quadrilateral is a kite, then its diagonals are perpendicular.



$\overline{AC} \perp \overline{BD}$

THEOREM 6.19

If a quadrilateral is a kite, then exactly one pair of opposite angles are congruent.

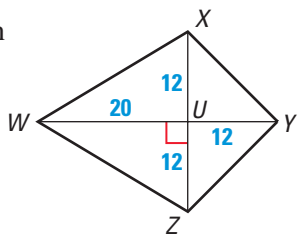


$\angle A \cong \angle C, \angle B \ncong \angle D$

EXAMPLE 4 *Using the Diagonals of a Kite*

Using Algebra

$WXYZ$ is a kite so the diagonals are perpendicular. You can use the Pythagorean Theorem to find the side lengths.



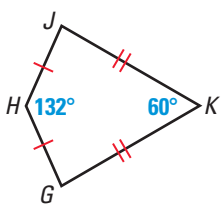
$$WX = \sqrt{20^2 + 12^2} \approx 23.32$$

$$XY = \sqrt{12^2 + 12^2} \approx 16.97$$

Because $WXYZ$ is a kite, $WZ = WX \approx 23.32$ and $ZY = XY \approx 16.97$.

EXAMPLE 5 *Angles of a Kite*

Find $m\angle G$ and $m\angle J$ in the diagram at the right.



SOLUTION

$GHJK$ is a kite, so $\angle G \cong \angle J$ and $m\angle G = m\angle J$.

$2(m\angle G) + 132° + 60° = 360°$	**Sum of measures of int. ∠s of a quad. is 360°.**
$2(m\angle G) = 168°$	**Simplify.**
$m\angle G = 84°$	**Divide each side by 2.**

▶ So, $m\angle J = m\angle G = 84°$.

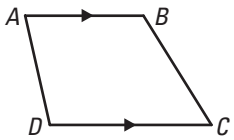
GUIDED PRACTICE

Vocabulary Check ✓

1. Name the bases of trapezoid *ABCD*.

Concept Check ✓

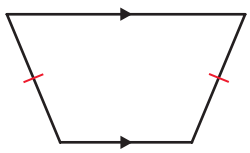
2. Explain why a rhombus is not a kite. Use the definition of a kite.



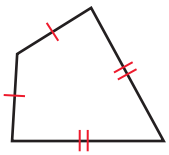
Skill Check ✓

Decide whether the quadrilateral is a *trapezoid*, an *isosceles trapezoid*, a *kite*, or *none of these*.

3.



4.



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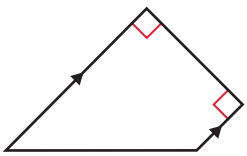
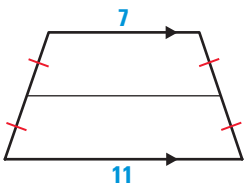


6. How can you prove that trapezoid *ABCD* in Example 2 is isosceles?

Find the length of the midsegment.

7.



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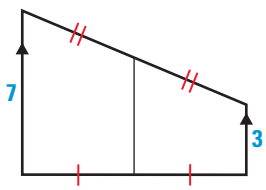

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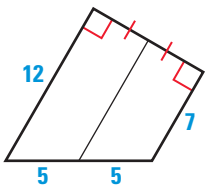


PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 814.

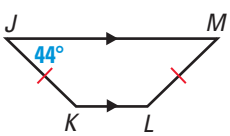
STUDYING A TRAPEZOID **Draw a trapezoid *PQRS* with $\overline{QR} \parallel \overline{PS}$. Identify the segments or angles of *PQRS* as *bases, consecutive sides, legs, diagonals, base angles,* or *opposite angles*.**

10. $\overline{QR}$ and $\overline{PS}$ **11.** $\overline{PQ}$ and $\overline{RS}$ **12.** $\overline{PQ}$ and $\overline{QR}$

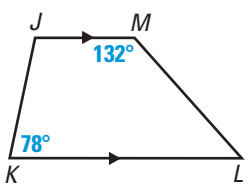
13. $\overline{QS}$ and $\overline{PR}$ **14.** $\angle Q$ and $\angle S$ **15.** $\angle S$ and $\angle P$

FINDING ANGLE MEASURES **Find the angle measures of *JKLM*.**

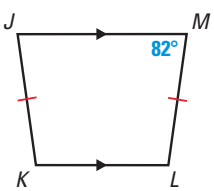
16.



17.



18.



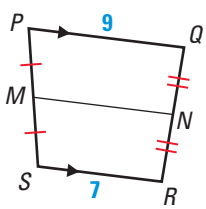
STUDENT HELP

HOMEWORK HELP

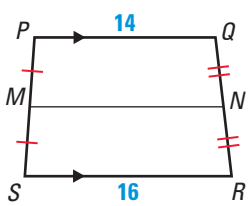
Example 1: Exs. 16–18
Example 2: Exs. 34, 37, 38, 48–50
Example 3: Exs. 19–24, 35, 39
Example 4: Exs. 28–30
Example 5: Exs. 31–33

FINDING MIDSEGMENTS **Find the length of the midsegment $\overline{MN}$.**

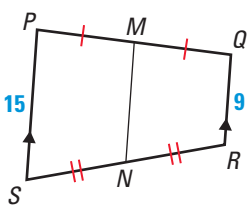
19.



20.

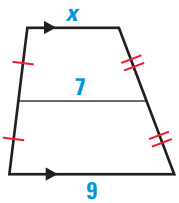


21.

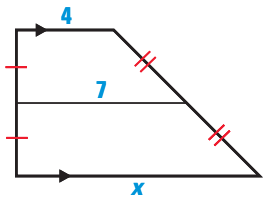


USING ALGEBRA **Find the value of *x*.**

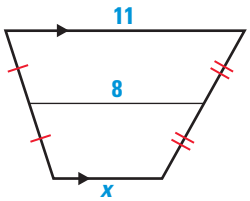
22.



23.



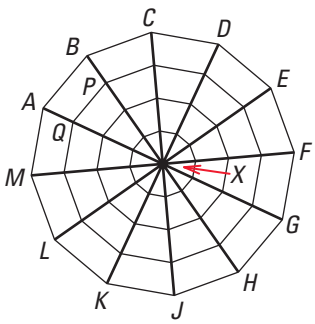
24.



FOCUS ON APPLICATIONS

WEBS The spider web above is called an orb web. Although it looks like concentric polygons, the spider actually followed a spiral path to spin the web.

CONCENTRIC POLYGONS **In the diagram, *ABCDEFGHJKLM* is a regular dodecagon, $\overline{AB} \parallel \overline{PQ}$, and *X* is equidistant from the vertices of the dodecagon.**



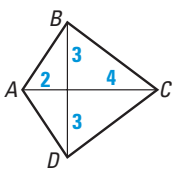
25. Are you given enough information to prove that *ABPQ* is isosceles? Explain your reasoning.

26. What is the measure of $\angle AXB$?

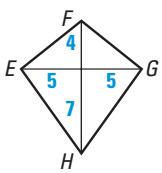
27. What is the measure of each interior angle of *ABPQ*?

USING ALGEBRA **What are the lengths of the sides of the kite? Give your answer to the nearest hundredth.**

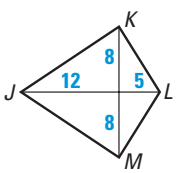
28.



29.

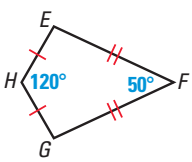


30.

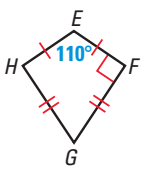


ANGLES OF KITES ***EFGH* is a kite. What is $m\angle G$?**

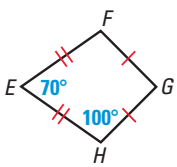
31.



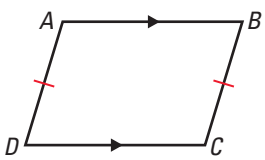
32.



33.



34. **ERROR ANALYSIS** A student says that parallelogram *ABCD* is an isosceles trapezoid because $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$. Explain what is wrong with this reasoning.



35. **CRITICAL THINKING** The midsegment of a trapezoid is 5 inches long. What are possible lengths of the bases?

36. **COORDINATE GEOMETRY** Determine whether the points $A(4, 5)$, $B(-3, 3)$, $C(-6, -13)$, and $D(6, -2)$ are the vertices of a kite. Explain your answer.

TRAPEZOIDS **Determine whether the given points represent the vertices of a trapezoid. If so, is the trapezoid isosceles? Explain your reasoning.**

37. $A(-2, 0)$, $B(0, 4)$, $C(5, 4)$, $D(8, 0)$

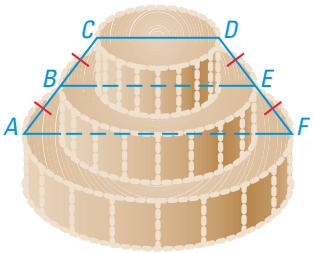
38. $E(1, 9)$, $F(4, 2)$, $G(5, 2)$, $H(8, 9)$

FOCUS ON CAREERS

CAKE DESIGNERS design cakes for many occasions, including weddings, birthdays, anniversaries, and graduations.

CAREER LINK
www.mcdougallittell.com

39. LAYER CAKE The top layer of the cake has a diameter of 10 inches. The bottom layer has a diameter of 22 inches. What is the diameter of the middle layer?

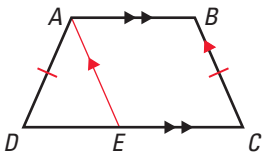


40. PROVING THEOREM 6.14 Write a proof of Theorem 6.14.

GIVEN ▶ $ABCD$ is an isosceles trapezoid.

$\overline{AB} \parallel \overline{DC}$, $\overline{AD} \cong \overline{BC}$

PROVE ▶ $\angle D \cong \angle C$, $\angle DAB \cong \angle B$



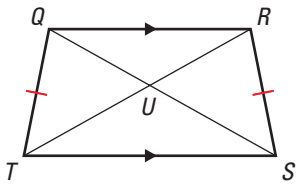
Plan for Proof To show $\angle D \cong \angle C$, first draw $\overline{AE}$ so $ABCE$ is a parallelogram. Then show $\overline{BC} \cong \overline{AE}$, so $\overline{AE} \cong \overline{AD}$ and $\angle D \cong \angle AED$. Finally, show $\angle D \cong \angle C$. To show $\angle DAB \cong \angle B$, use the consecutive interior angles theorem and substitution.

41. PROVING THEOREM 6.16 Write a proof of one conditional statement of Theorem 6.16.

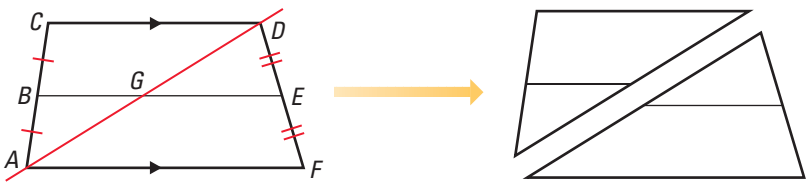
GIVEN ▶ $TQRS$ is an isosceles trapezoid.

$\overline{QR} \parallel \overline{TS}$ and $\overline{QT} \cong \overline{RS}$

PROVE ▶ $\overline{TR} \cong \overline{SQ}$



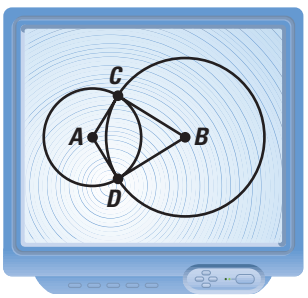
42. JUSTIFYING THEOREM 6.17 In the diagram below, $\overline{BG}$ is the midsegment of $\triangle ACD$ and $\overline{GE}$ is the midsegment of $\triangle ADF$. Explain why the midsegment of trapezoid $ACDF$ is parallel to each base and why its length is one half the sum of the lengths of the bases.



STUDENT HELP

SOFTWARE HELP Visit our Web site www.mcdougallittell.com to see instructions for several software applications.

USING TECHNOLOGY **In Exercises 43–45, use geometry software.** Draw points A, B, C and segments $\overline{AC}$ and $\overline{BC}$. Construct a circle with center A and radius AC. Construct a circle with center B and radius BC. Label the other intersection of the circles D. Draw $\overline{BD}$ and $\overline{AD}$.

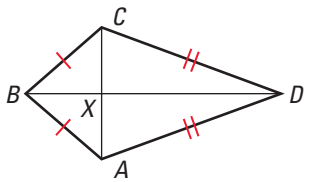


43. What kind of shape is $ACBD$? How do you know? What happens to the shape as you drag A? drag B? drag C?

44. Measure $\angle ACB$ and $\angle ADB$. What happens to the angle measures as you drag A, B, or C?

45. Which theorem does this construction illustrate?

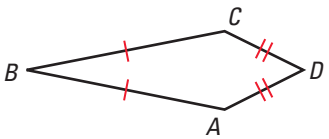
46. **PROVING THEOREM 6.18** Write a two-column proof of Theorem 6.18.



GIVEN ▶ $\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{CD}$

PROVE ▶ $\overline{AC} \perp \overline{BD}$

47. **PROVING THEOREM 6.19** Write a paragraph proof of Theorem 6.19.

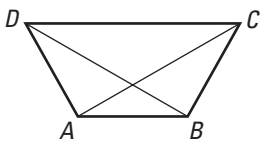


GIVEN ▶ $ABCD$ is a kite with $\overline{AB} \cong \overline{CB}$ and $\overline{AD} \cong \overline{CD}$.

PROVE ▶ $\angle A \cong \angle C$, $\angle B \ncong \angle D$

Plan for Proof First show that $\angle A \cong \angle C$. Then use an indirect argument to show $\angle B \ncong \angle D$: If $\angle B \cong \angle D$, then $ABCD$ is a parallelogram. But opposite sides of a parallelogram are congruent. This contradicts the definition of a kite.

TRAPEZOIDS **Decide whether you are given enough information to conclude that *ABCD* is an isosceles trapezoid. Explain your reasoning.**



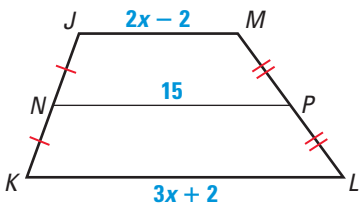
48. $\overline{AB} \parallel \overline{DC}$
$\overline{AD} \cong \overline{BC}$
$\overline{AD} \cong \overline{AB}$

49. $\overline{AB} \parallel \overline{DC}$
$\overline{AC} \cong \overline{BD}$
$\angle A \ncong \angle C$

50. $\angle A \cong \angle B$
$\angle D \cong \angle C$
$\angle A \ncong \angle C$

Test Preparation

51. **MULTIPLE CHOICE** In the trapezoid at the right, $NP = 15$. What is the value of x?



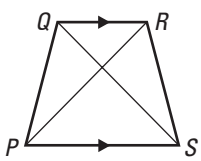
Ⓐ 2 Ⓑ 3 Ⓒ 4
Ⓓ 5 Ⓔ 6

52. **MULTIPLE CHOICE** Which one of the following can a trapezoid have?

Ⓐ congruent bases
Ⓑ diagonals that bisect each other
Ⓒ exactly two congruent sides
Ⓓ a pair of congruent opposite angles
Ⓔ exactly three congruent angles

★ Challenge

53. **PROOF** Prove one direction of Theorem 6.16: If the diagonals of a trapezoid are congruent, then the trapezoid is isosceles.



GIVEN ▶ $PQRS$ is a trapezoid.
$\overline{QR} \parallel \overline{PS}$, $\overline{PR} \cong \overline{SQ}$

PROVE ▶ $\overline{QP} \cong \overline{RS}$

Plan for Proof Draw a perpendicular segment from Q to $\overline{PS}$ and label the intersection M. Draw a perpendicular segment from R to $\overline{PS}$ and label the intersection N. Prove that $\triangle QMS \cong \triangle RNP$. Then prove that $\triangle QPS \cong \triangle RSP$.

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

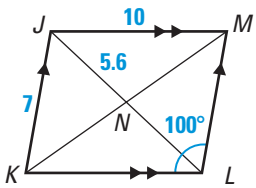
CONDITIONAL STATEMENTS **Rewrite the statement in if-then form.** (Review 2.1)

54. A scalene triangle has no congruent sides.

55. A kite has perpendicular diagonals.

56. A polygon is a pentagon if it has five sides.

FINDING MEASUREMENTS **Use the diagram to find the side length or angle measure.** (Review 6.2 for 6.6)



57. LN **58.** KL **59.** ML

60. JL **61.** $m\angle JML$ **62.** $m\angle MJK$

PARALLELOGRAMS **Determine whether the given points represent the vertices of a parallelogram. Explain your answer.** (Review 6.3 for 6.6)

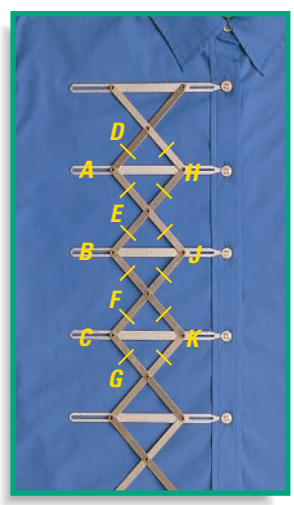
63. $A(-2, 8)$, $B(5, 8)$, $C(2, 0)$, $D(-5, 0)$

64. $P(4, -3)$, $Q(9, -1)$, $R(8, -6)$, $S(3, -8)$

QUIZ 2

Self-Test for Lessons 6.4 and 6.5

1. **POSITIONING BUTTONS** The tool at the right is used to decide where to put buttons on a shirt. The tool is stretched to fit the length of the shirt, and the pointers show where to put the buttons. Why are the pointers always evenly spaced? (*Hint:* You can prove that $\overline{HJ} \cong \overline{JK}$ if you know that $\triangle JFK \cong \triangle HEJ$.) (Lesson 6.4)



Determine whether the given points represent the vertices of a *rectangle*, a *rhombus*, a *square*, a *trapezoid*, or a *kite*. (Lessons 6.4, 6.5)

2. $P(2, 5)$, $Q(-4, 5)$, $R(2, -7)$, $S(-4, -7)$

3. $A(-3, 6)$, $B(0, 9)$, $C(3, 6)$, $D(0, -10)$

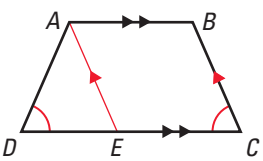
4. $J(-5, 6)$, $K(-4, -2)$, $L(4, -1)$, $M(3, 7)$

5. $P(-5, -3)$, $Q(1, -2)$, $R(6, 3)$, $S(7, 9)$

6. **PROVING THEOREM 6.15** Write a proof of Theorem 6.15.

GIVEN ▶ $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$.
$\angle D \cong \angle C$

PROVE ▶ $\overline{AD} \cong \overline{BC}$



Plan for Proof Draw $\overline{AE}$ so $ABCE$ is a parallelogram. Use the Transitive Property of Congruence to show $\angle AED \cong \angle D$. Then $\overline{AD} \cong \overline{AE}$, so $\overline{AD} \cong \overline{BC}$. (Lesson 6.5)

6.6 Special Quadrilaterals

What you should learn

GOAL 1 Identify special quadrilaterals based on limited information.

GOAL 2 Prove that a quadrilateral is a special type of quadrilateral, such as a rhombus or a trapezoid.

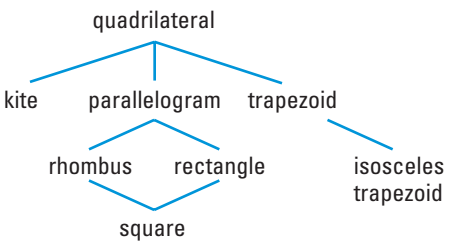
Why you should learn it

▼ To understand and describe **real-world** shapes such as gem facets in **Exs. 42 and 43**.

REAL LIFE

GOAL 1 SUMMARIZING PROPERTIES OF QUADRILATERALS

In this chapter, you have studied the seven special types of quadrilaterals at the right. Notice that each shape has all the properties of the shapes linked above it. For instance, squares have the properties of rhombuses, rectangles, parallelograms, and quadrilaterals.

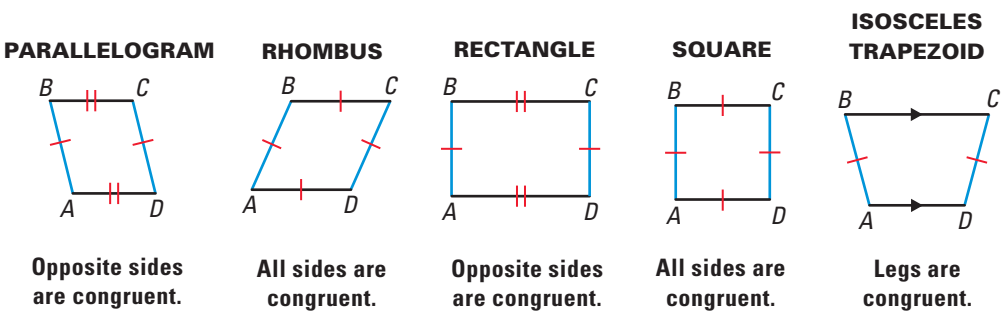


EXAMPLE 1 *Identifying Quadrilaterals*

Quadrilateral *ABCD* has at least one pair of opposite sides congruent. What kinds of quadrilaterals meet this condition?

SOLUTION

There are many possibilities.

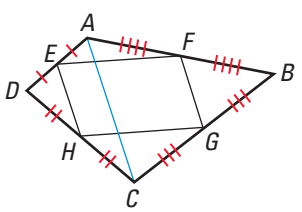


EXAMPLE 2 *Connecting Midpoints of Sides*

When you join the midpoints of the sides of any quadrilateral, what special quadrilateral is formed? Why?

SOLUTION

Let *E*, *F*, *G*, and *H* be the midpoints of the sides of any quadrilateral, *ABCD*, as shown.



If you draw $\overline{AC}$, the Midsegment Theorem for triangles says $\overline{FG} \parallel \overline{AC}$ and $\overline{EH} \parallel \overline{AC}$, so $\overline{FG} \parallel \overline{EH}$. Similar reasoning shows that $\overline{EF} \parallel \overline{HG}$.

▶ So, by definition, *EFGH* is a parallelogram.

GOAL 2 PROOF WITH SPECIAL QUADRILATERALS

When you want to prove that a quadrilateral has a specific shape, you can use either the definition of the shape as in Example 2, *or* you can use a theorem.

CONCEPT SUMMARY **PROVING QUADRILATERALS ARE RHOMBUSES**

You have learned three ways to prove that a quadrilateral is a rhombus.

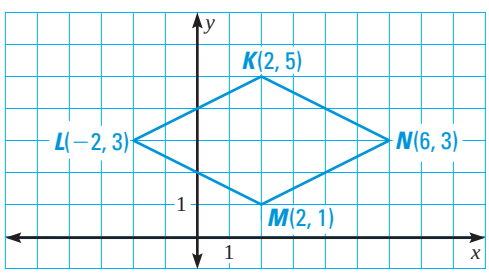
1. You can use the definition and show that the quadrilateral is a *parallelogram* that has four congruent sides. It is easier, however, to use the Rhombus Corollary and simply show that all four sides of the quadrilateral are congruent.
2. Show that the quadrilateral is a parallelogram *and* that the diagonals are perpendicular. (*Theorem 6.11*)
3. Show that the quadrilateral is a parallelogram *and* that each diagonal bisects a pair of opposite angles. (*Theorem 6.12*)

STUDENT HELP

Look Back
For help with proving a quadrilateral is a parallelogram, see pp. 338–341.

EXAMPLE 3 *Proving a Quadrilateral is a Rhombus*

Show that *KLMN* is a rhombus.



SOLUTION You can use any of the three ways described in the concept summary above. For instance, you could show that opposite sides have the same slope and that the diagonals are perpendicular. Another way, shown below, is to prove that all four sides have the same length.

$$LM = \sqrt{[2-(-2)]^2 + (1-3)^2} = \sqrt{4^2 + (-2)^2} = \sqrt{20}$$

$$NK = \sqrt{(2-6)^2 + (5-3)^2} = \sqrt{(-4)^2 + 2^2} = \sqrt{20}$$

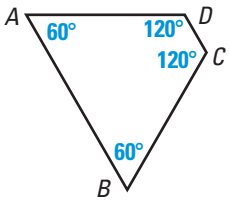
$$MN = \sqrt{(6-2)^2 + (3-1)^2} = \sqrt{4^2 + 2^2} = \sqrt{20}$$

$$KL = \sqrt{(-2-2)^2 + (3-5)^2} = \sqrt{(-4)^2 + (-2)^2} = \sqrt{20}$$

▶ So, because $LM = NK = MN = KL$, *KLMN* is a rhombus.

EXAMPLE 4 *Identifying a Quadrilateral*

What type of quadrilateral is *ABCD*? Explain your reasoning.



SOLUTION

$\angle A$ and $\angle D$ are supplementary, but $\angle A$ and $\angle B$ are not. So, $\overline{AB} \parallel \overline{DC}$ but $\overline{AD}$ is not parallel to $\overline{BC}$. By definition, *ABCD* is a trapezoid. Because base angles are congruent, *ABCD* is an isosceles trapezoid.

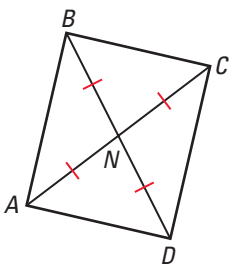
EXAMPLE 5 *Identifying a Quadrilateral*

The diagonals of quadrilateral *ABCD* intersect at point *N* to produce four congruent segments: $\overline{AN} \cong \overline{BN} \cong \overline{CN} \cong \overline{DN}$. What type of quadrilateral is *ABCD*? Prove that your answer is correct.

SOLUTION

Draw a diagram:

Draw the diagonals as described. Then connect the endpoints to draw quadrilateral *ABCD*.



Make a conjecture:

Quadrilateral *ABCD* looks like a rectangle.

Proof

Prove your conjecture:

GIVEN ▶ $\overline{AN} \cong \overline{BN} \cong \overline{CN} \cong \overline{DN}$

PROVE ▶ *ABCD* is a rectangle.

Paragraph Proof Because you are given information about the diagonals, show that *ABCD* is a parallelogram with congruent diagonals.

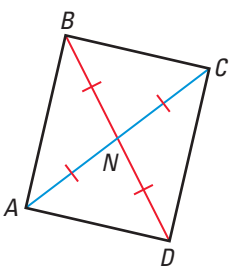
First prove that *ABCD* is a parallelogram.

Because $\overline{BN} \cong \overline{DN}$ and $\overline{AN} \cong \overline{CN}$, $\overline{BD}$ and $\overline{AC}$ bisect each other. Because the diagonals of *ABCD* bisect each other, *ABCD* is a parallelogram.

Then prove that the diagonals of *ABCD* are congruent.

From the given you can write $BN = AN$ and $DN = CN$ so, by the Addition Property of Equality, $BN + DN = AN + CN$. By the Segment Addition Postulate, $BD = BN + DN$ and $AC = AN + CN$ so, by substitution, $BD = AC$.

So, $\overline{BD} \cong \overline{AC}$.



▶ *ABCD* is a parallelogram with congruent diagonals, so *ABCD* is a rectangle.

GUIDED PRACTICE

Concept Check ✓

1. In Example 2, explain how to prove that $\overline{EF} \parallel \overline{HG}$.

Skill Check ✓

Copy the chart. Put an X in the box if the shape *always* has the given property.

	Property	▱	Rectangle	Rhombus	Square	Kite	Trapezoid
2.	Both pairs of opp. sides are ∥.	?	?	?	?	?	?
3.	Exactly 1 pair of opp. sides are ∥.	?	?	?	?	?	?
4.	Diagonals are ⊥.	?	?	?	?	?	?
5.	Diagonals are ≅.	?	?	?	?	?	?
6.	Diagonals bisect each other.	?	?	?	?	?	?

7. Which quadrilaterals can you form with four sticks of the same length? You must attach the sticks at their ends and cannot bend or break any of them.

PRACTICE AND APPLICATIONS

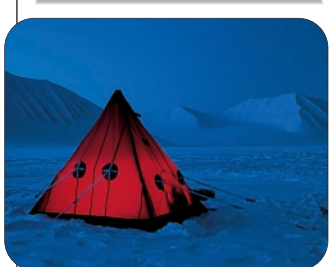
STUDENT HELP

Extra Practice to help you master skills is on p. 814.

PROPERTIES OF QUADRILATERALS **Copy the chart. Put an X in the box if the shape *always* has the given property.**

	Property	▱	Rectangle	Rhombus	Square	Kite	Trapezoid
8.	Both pairs of opp. sides are ≅.	?	?	?	?	?	?
9.	Exactly 1 pair of opp. sides are ≅.	?	?	?	?	?	?
10.	All sides are ≅.	?	?	?	?	?	?
11.	Both pairs of opp. ∠s are ≅.	?	?	?	?	?	?
12.	Exactly 1 pair of opp. ∠s are ≅.	?	?	?	?	?	?
13.	All ∠s are ≅.	?	?	?	?	?	?

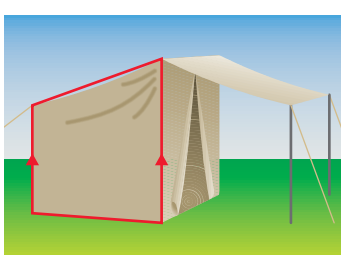
FOCUS ON APPLICATIONS



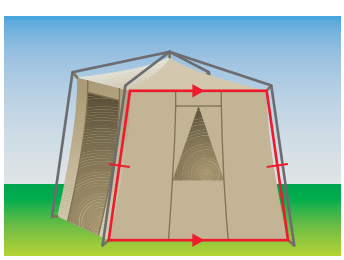
REAL LIFE Tents are designed differently for different climates. For example, winter tents are designed to shed snow. Desert tents can have flat roofs because they don't need to shed rain.

TENT SHAPES **What kind of special quadrilateral is the red shape?**

14.



15.

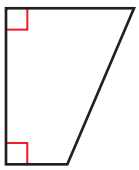


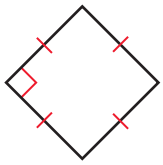
STUDENT HELP

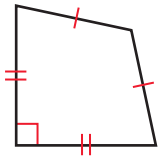
HOMEWORK HELP

Example 1: Exs. 8–24, 30–35
Example 2: Exs. 14–18, 42–44
Example 3: Exs. 25–29, 36–41
Example 4: Exs. 14–18, 42, 43
Example 5: Exs. 45–47

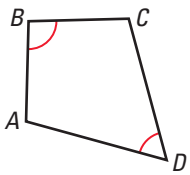
IDENTIFYING QUADRILATERALS **Identify the special quadrilateral. Use the most specific name.**

16.


17.


18.


IDENTIFYING QUADRILATERALS **What kinds of quadrilaterals meet the conditions shown? *ABCD* is not drawn to scale.**

19.


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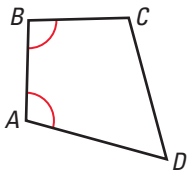

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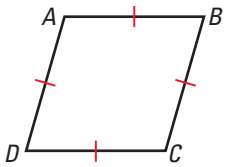

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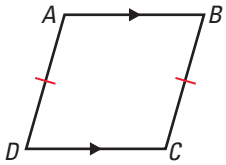
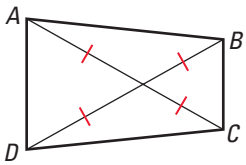
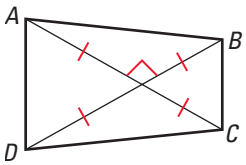


23.


24.


STUDENT HELP

Study Tip
See the summaries for parallelograms and rhombuses on pp. 340 and 365, and the list of postulates and theorems on pp. 828–837. You can refer to your summaries as you do the rest of the exercises.

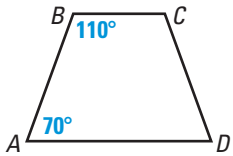
DESCRIBING METHODS OF PROOF **Summarize the ways you have learned to prove that a quadrilateral is the given special type of quadrilateral.**

25. kite **26.** square **27.** rectangle

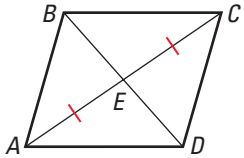
28. trapezoid **29.** isosceles trapezoid

DEVELOPING PROOF **Which two segments or angles must be congruent to enable you to prove *ABCD* is the given quadrilateral? Explain your reasoning. There may be more than one right answer.**

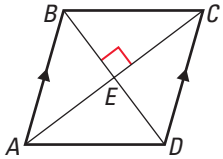
30. isosceles trapezoid



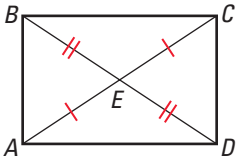
31. parallelogram



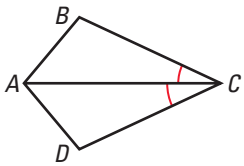
32. rhombus



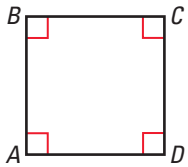
33. rectangle



34. kite



35. square



QUADRILATERALS **What kind of quadrilateral is *PQRS*? Justify your answer.**

36. $P(0, 0)$, $Q(0, 2)$, $R(5, 5)$, $S(2, 0)$ **37.** $P(1, 1)$, $Q(5, 1)$, $R(4, 8)$, $S(2, 8)$

38. $P(2, 1)$, $Q(7, 1)$, $R(7, 7)$, $S(2, 5)$ **39.** $P(0, 7)$, $Q(4, 8)$, $R(5, 2)$, $S(1, 1)$

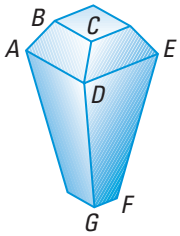
40. $P(1, 7)$, $Q(5, 9)$, $R(8, 3)$, $S(4, 1)$ **41.** $P(5, 1)$, $Q(9, 6)$, $R(5, 11)$, $S(1, 6)$

FOCUS ON CAREERS

REAL LIFE **GEMOLOGISTS** analyze the cut of a gem when determining its value.

INTERNET **CAREER LINK**
www.mcdougallittell.com

GEM CUTTING **In Exercises 42 and 43, use the following information.**
There are different ways of cutting gems to enhance the beauty of the jewel. One of the earliest shapes used for diamonds is called the *table cut*, as shown at the right. Each face of a cut gem is called a *facet*.



42. $\overline{BC} \parallel \overline{AD}$, $\overline{AB}$ and $\overline{DC}$ are not parallel. What shape is the facet labeled *ABCD*?

43. $\overline{DE} \parallel \overline{GF}$, $\overline{DG}$ and $\overline{EF}$ are congruent, but not parallel. What shape is the facet labeled *DEFG*?

44. **JUSTIFYING A CONSTRUCTION** Look back at the *Perpendicular to a Line* construction on page 130. Explain why this construction works.

DRAWING QUADRILATERALS **Draw $\overline{AC}$ and $\overline{BD}$ as described. What special type of quadrilateral is *ABCD*? Prove that your answer is correct.**

45. $\overline{AC}$ and $\overline{BD}$ bisect each other, but they are not perpendicular or congruent.

46. $\overline{AC}$ and $\overline{BD}$ bisect each other. $\overline{AC} \perp \overline{BD}$, $\overline{AC} \ncong \overline{BD}$

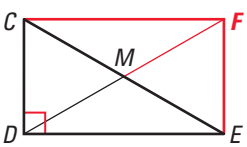
47. $\overline{AC} \perp \overline{BD}$, and $\overline{AC}$ bisects $\overline{BD}$. $\overline{BD}$ does not bisect $\overline{AC}$.

48. **LOGICAL REASONING** *EFGH*, *GHJK*, and *JKLM* are all parallelograms. If $\overline{EF}$ and $\overline{LM}$ are not collinear, what kind of quadrilateral is *EFLM*? Prove that your answer is correct.

STUDENT HELP

INTERNET **HOMEWORK HELP**
Visit our Web site www.mcdougallittell.com for help with Exs. 45–47.

49. **PROOF** Prove that the median of a right triangle is one half the length of the hypotenuse.



GIVEN ▶ $\angle CDE$ is a right angle. $\overline{CM} \cong \overline{EM}$

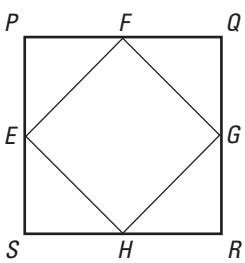
PROVE ▶ $\overline{DM} \cong \overline{CM}$

Plan for Proof First draw $\overline{CF}$ and $\overline{EF}$ so *CDEF* is a rectangle. (How?)

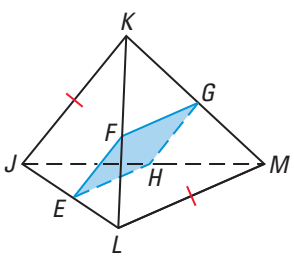
50. **PROOF** Use facts about angles to prove that the quadrilateral in Example 5 is a rectangle. (*Hint:* Let $x°$ be the measure of $\angle ABN$. Find the measures of the other angles in terms of x.)

PROOF **What special type of quadrilateral is *EFGH*? Prove that your answer is correct.**

51. **GIVEN** ▶ *PQRS* is a square. *E*, *F*, *G*, and *H* are midpoints of the sides of the square.

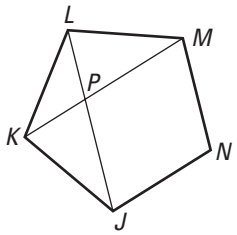


52. **GIVEN** ▶ $\overline{JK} \cong \overline{LM}$, *E*, *F*, *G*, and *H* are the midpoints of $\overline{JL}$, $\overline{KL}$, $\overline{KM}$, and $\overline{JM}$.



Test Preparation

53. MULTI-STEP PROBLEM Copy the diagram. *JKLMN* is a regular pentagon. You will identify *JPMN*.



a. What kind of triangle is $\triangle JKL$? Use $\triangle JKL$ to prove that $\angle LJN \cong \angle JLM$.

b. List everything you know about the interior angles of *JLMN*. Use these facts to prove that $\overline{JL} \parallel \overline{NM}$.

c. Reasoning similar to parts (a) and (b) shows that $\overline{KM} \parallel \overline{JN}$. Based on this and the result from part (b), what kind of shape is *JPMN*?

d. *Writing* Is *JPMN* a rhombus? Justify your answer.

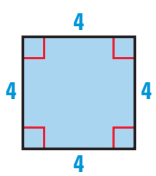
★ Challenge

54. PROOF $\overline{AC}$ and $\overline{BD}$ intersect each other at *N*. $\overline{AN} \cong \overline{BN}$ and $\overline{CN} \cong \overline{DN}$, but $\overline{AC}$ and $\overline{BD}$ do not bisect each other. Draw $\overline{AC}$ and $\overline{BD}$, and *ABCD*. What special type of quadrilateral is *ABCD*? Write a plan for a proof of your answer.

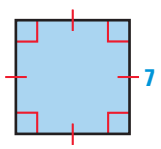
MIXED REVIEW

FINDING AREA Find the area of the figure. (Review 1.7 for 6.7)

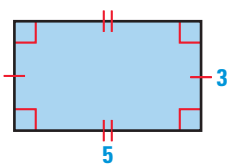
55.



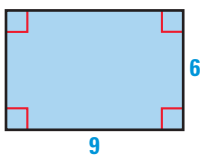
56.



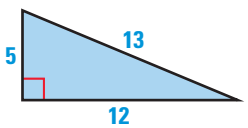
57.



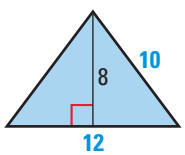
58.



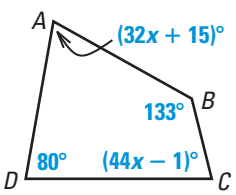
59.



60.



USING ALGEBRA In Exercises 61 and 62, use the diagram at the right. (Review 6.1)

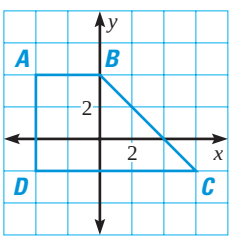


61. What is the value of x?

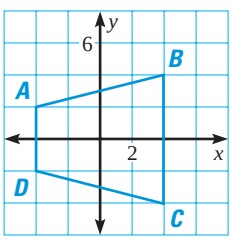
62. What is $m\angle A$? Use your result from Exercise 61.

FINDING THE MIDSEGMENT Find the length of the midsegment of the trapezoid. (Review 6.5 for 6.7)

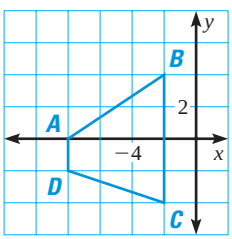
63.



64.



65.



6.7 Areas of Triangles and Quadrilaterals

What you should learn

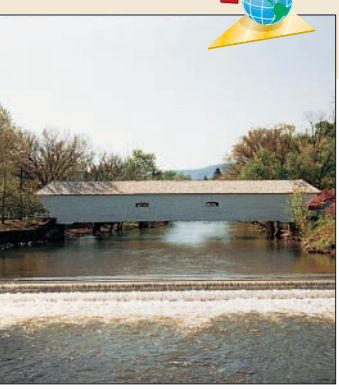
GOAL 1 Find the areas of squares, rectangles, parallelograms, and triangles.

GOAL 2 Find the areas of trapezoids, kites, and rhombuses, as applied in **Example 6**.

Why you should learn it

▼ To find areas of **real-life** surfaces, such as the roof of the covered bridge in **Exs. 48 and 49**.

REAL LIFE



GOAL 1 USING AREA FORMULAS

You can use the postulates below to prove several area theorems.

AREA POSTULATES

POSTULATE 22 ***Area of a Square Postulate***

The area of a square is the square of the length of its side, or $A = s^2$.

POSTULATE 23 ***Area Congruence Postulate***

If two polygons are congruent, then they have the same area.

POSTULATE 24 ***Area Addition Postulate***

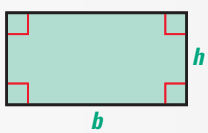
The area of a region is the sum of the areas of its nonoverlapping parts.

AREA THEOREMS

THEOREM 6.20 ***Area of a Rectangle***

The area of a rectangle is the product of its base and height.

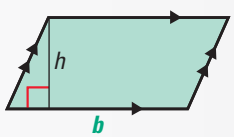
$A = bh$



THEOREM 6.21 ***Area of a Parallelogram***

The area of a parallelogram is the product of a base and its corresponding height.

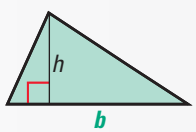
$A = bh$



THEOREM 6.22 ***Area of a Triangle***

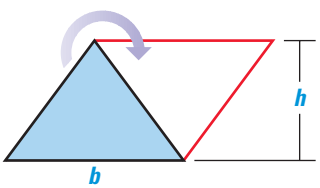
The area of a triangle is one half the product of a base and its corresponding height.

$A = \frac{1}{2}bh$



You can justify the area formulas for triangles and parallelograms as follows.

The area of a parallelogram is the area of a rectangle with the same base and height.



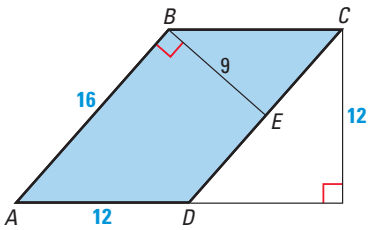
The area of a triangle is half the area of a parallelogram with the same base and height.

STUDENT HELP

Study Tip
To find the area of a parallelogram or triangle, you can use any side as the base. But be sure you measure the height of an altitude that is perpendicular to the base you have chosen.

EXAMPLE 1 *Using the Area Theorems*

Find the area of ▱$ABCD$.



SOLUTION

Method 1 Use $\overline{AB}$ as the base. So, $b = 16$ and $h = 9$.

Area $= bh = 16(9) = 144$ square units.

Method 2 Use $\overline{AD}$ as the base. So, $b = 12$ and $h = 12$.

Area $= bh = 12(12) = 144$ square units.

Notice that you get the same area with either base.

Using Algebra

EXAMPLE 2 *Finding the Height of a Triangle*

Rewrite the formula for the area of a triangle in terms of h. Then use your formula to find the height of a triangle that has an area of 12 and a base length of 6.

SOLUTION

Rewrite the area formula so h is alone on one side of the equation.

$A = \frac{1}{2}bh$	**Formula for the area of a triangle**
$2A = bh$	**Multiply both sides by 2.**
$\frac{2A}{b} = h$	**Divide both sides by *b*.**

Substitute **12** for A and **6** for b to find the height of the triangle.

$$h = \frac{2A}{b} = \frac{2(12)}{6} = 4$$

▶ The height of the triangle is 4.

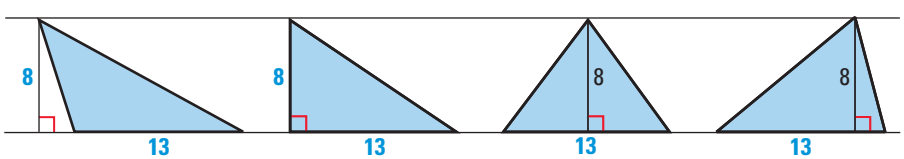
EXAMPLE 3 *Finding the Height of a Triangle*

A triangle has an area of 52 square feet and a base of 13 feet. Are all triangles with these dimensions congruent?

SOLUTION

Using the formula from Example 2, the height is $h = \frac{2(52)}{13} = 8$ feet.

There are many triangles with these dimensions. Some are shown below.



STUDENT HELP

Study Tip
Notice that the altitude of a triangle can be outside the triangle.

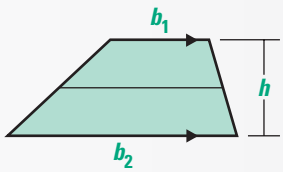
GOAL 2 AREAS OF TRAPEZOIDS, KITES, AND RHOMBUSES

THEOREMS

THEOREM 6.23 ***Area of a Trapezoid***

The area of a trapezoid is one half the product of the height and the sum of the bases.

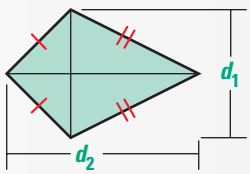
$$A = \frac{1}{2}h(b_1 + b_2)$$



THEOREM 6.24 ***Area of a Kite***

The area of a kite is one half the product of the lengths of its diagonals.

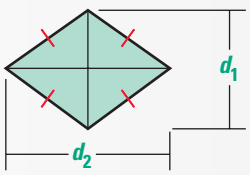
$$A = \frac{1}{2}d_1d_2$$



THEOREM 6.25 ***Area of a Rhombus***

The area of a rhombus is equal to one half the product of the lengths of the diagonals.

$$A = \frac{1}{2}d_1d_2$$

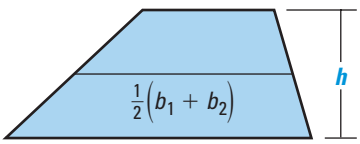


STUDENT HELP

Look Back
Remember that the length of the midsegment of a trapezoid is the average of the lengths of the bases. (p. 357)

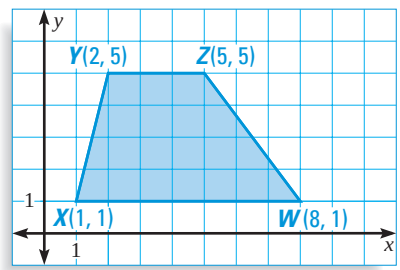
You will justify Theorem 6.23 in Exercises 58 and 59. You may find it easier to remember the theorem this way.

Area = **Length of Midsegment** · **Height**



EXAMPLE 4 *Finding the Area of a Trapezoid*

Find the area of trapezoid *WXYZ*.



SOLUTION

The height of *WXYZ* is $h = 5 - 1 = 4$.

Find the lengths of the bases.

$$b_1 = YZ = 5 - 2 = 3$$

$$b_2 = XW = 8 - 1 = 7$$

Substitute **4** for h, **3** for b_1, and **7** for b_2 to find the area of the trapezoid.

$$A = \frac{1}{2}h(b_1 + b_2) \quad \textbf{Formula for area of a trapezoid}$$

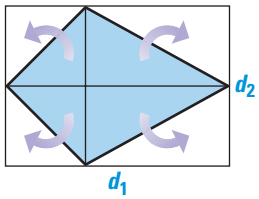
$$= \frac{1}{2}(4)(3 + 7) \quad \textbf{Substitute.}$$

$$= 20 \quad \textbf{Simplify.}$$

▶ The area of trapezoid *WXYZ* is 20 square units.

The diagram at the right justifies the formulas for the areas of kites and rhombuses.

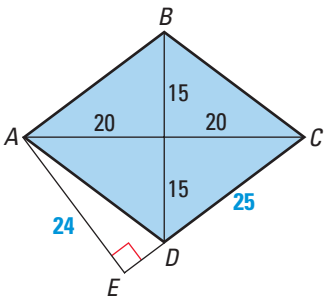
The diagram shows that the area of a kite is half the area of the rectangle whose length and width are the lengths of the diagonals of the kite. The same is true for a rhombus.



$A = \frac{1}{2}d_1d_2$

EXAMPLE 5 *Finding the Area of a Rhombus*

Use the information given in the diagram to find the area of rhombus *ABCD*.



SOLUTION

Method 1 Use the formula for the area of a rhombus. $d_1 = BD = 30$ and $d_2 = AC = 40$.

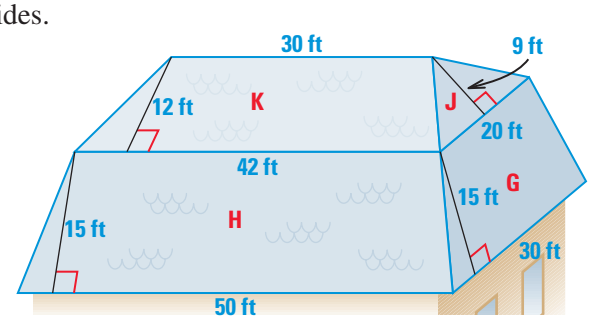
$$\begin{aligned} A &= \frac{1}{2}d_1d_2 \\ &= \frac{1}{2}(30)(40) \\ &= 600 \text{ square units} \end{aligned}$$

Method 2 Use the formula for the area of a parallelogram. $b = 25$ and $h = 24$.

$A = bh = 25(24) = 600$ square units

EXAMPLE 6 *Finding Areas*

REAL LIFE **ROOF** Find the area of the roof. *G*, *H*, and *K* are trapezoids and *J* is a triangle. The hidden back and left sides of the roof are the same as the front and right sides.



STUDENT HELP

Study Tip
To check that the answer is reasonable, approximate each trapezoid by a rectangle. The area of *H* should be less than 50 • 15, but more than 40 • 15.

SOLUTION

Area of $J = \frac{1}{2}(20)(9) = 90$ ft^2

Area of $H = \frac{1}{2}(15)(42 + 50) = 690$ ft^2

Area of $G = \frac{1}{2}(15)(20 + 30) = 375$ ft^2

Area of $K = \frac{1}{2}(12)(30 + 42) = 432$ ft^2

The roof has two congruent faces of each type.

Total Area $= 2(90 + 375 + 690 + 432) = 3174$

▶ The total area of the roof is 3174 square feet.

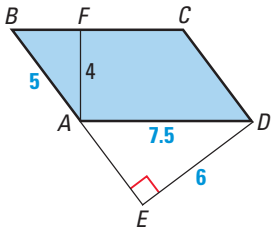
GUIDED PRACTICE

Vocabulary Check ✓

1. What is the *midsegment* of a trapezoid?

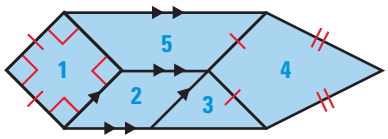
Concept Check ✓

2. If you use AB as the base to find the area of ▱$ABCD$ shown at the right, what should you use as the height?



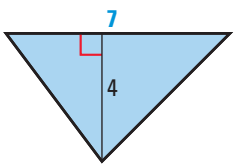
Skill Check ✓

Match the region with a formula for its area. Use each formula exactly once.



3. Region 1 — (A) $A = s^2$
4. Region 2 — (B) $A = \frac{1}{2}d_1d_2$
5. Region 3 — (C) $A = \frac{1}{2}bh$
6. Region 4 — (D) $A = \frac{1}{2}h(b_1 + b_2)$
7. Region 5 — (E) $A = bh$

Find the area of the polygon.

8.


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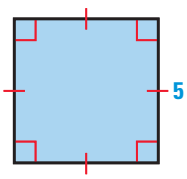

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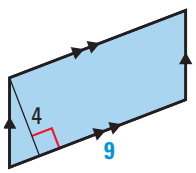
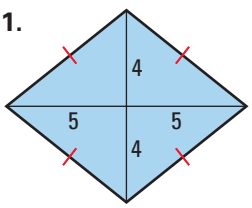


11.


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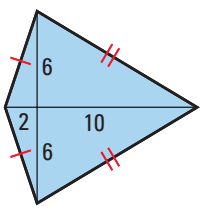
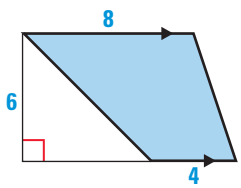


13.


PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 814.

STUDENT HELP

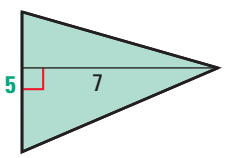
HOMEWORK HELP

Example 1: Exs. 14–19, 41–47

Example 2: Exs. 26–31

continued on p. 377

FINDING AREA **Find the area of the polygon.**

14.


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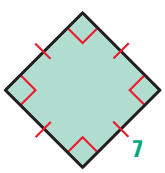

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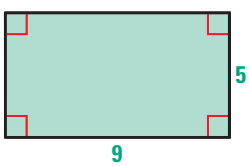

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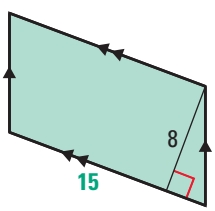

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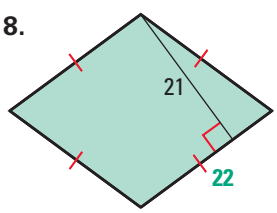

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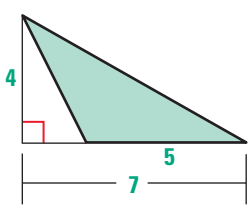


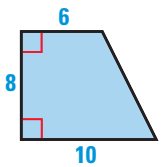
STUDENT HELP

HOMEWORK HELP
continued from p. 376

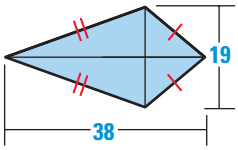
Example 3: Exs. 26–28, 39, 40
Example 4: Exs. 32–34
Example 5: Exs. 20–25, 44
Example 6: Exs. 35–38, 48–52

FINDING AREA **Find the area of the polygon.**

20.



21.



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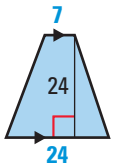
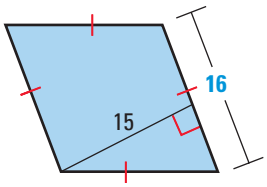


23.



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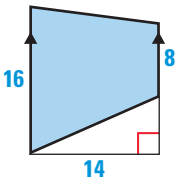
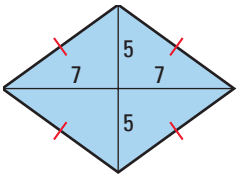
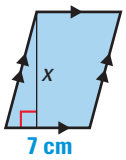


25.

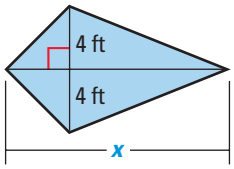


USING ALGEBRA **Find the value of *x*.**

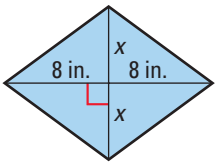
26. $A = 63\ \text{cm}^2$



27. $A = 48\ \text{ft}^2$



28. $A = 48\ \text{in.}^2$

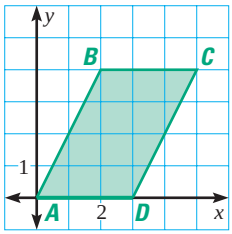


REWRITING FORMULAS **Rewrite the formula for the area of the polygon in terms of the given variable. Use the formulas on pages 372 and 374.**

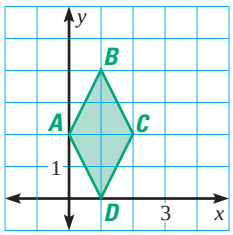
29. triangle, b **30.** kite, d_1 **31.** trapezoid, b_1

FINDING AREA **Find the area of quadrilateral *ABCD*.**

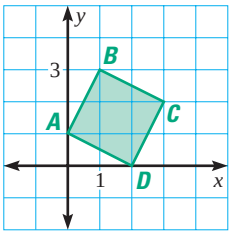
32.



33.

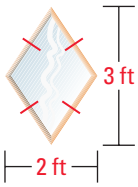


34.

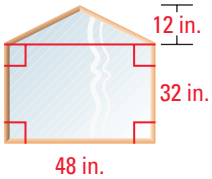


ENERGY CONSERVATION **The total area of a building's windows affects the cost of heating or cooling the building. Find the area of the window.**

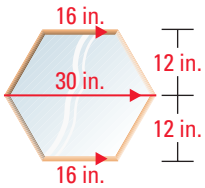
35.



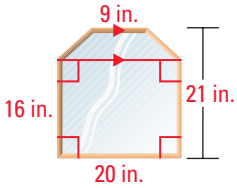
36.



37.



38.



FOCUS ON APPLICATIONS

REAL LIFE

INSULATION Insulation makes a building more energy efficient. The ability of a material to insulate is called its *R*-value. Many windows have an *R*-value of 1. Adobe has an *R*-value of 11.9.

INTERNET APPLICATION LINK
www.mcdougallittell.com

STUDENT HELP

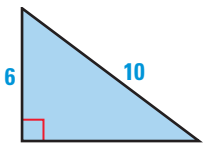
Study Tip
Remember that two polygons are congruent if their corresponding angles and sides are congruent.

39. **LOGICAL REASONING** Are all parallelograms with an area of 24 square feet and a base of 6 feet congruent? Explain.

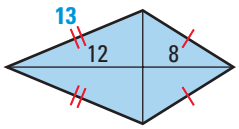
40. **LOGICAL REASONING** Are all rectangles with an area of 24 square feet and a base of 6 feet congruent? Explain.

USING THE PYTHAGOREAN THEOREM **Find the area of the polygon.**

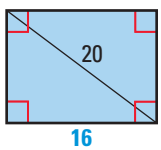
41.



42.



43.



44. **LOGICAL REASONING** What happens to the area of a kite if you double the length of one of the diagonals? if you double the length of both diagonals?

PARADE FLOATS **You are decorating a float for a parade. You estimate that, on average, a carnation will cover 3 square inches, a daisy will cover 2 square inches, and a chrysanthemum will cover 4 square inches. About how many flowers do you need to cover the shape on the float?**

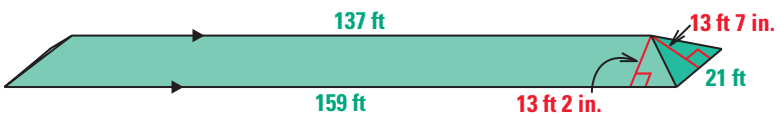
45. Carnations: 2 ft by 5 ft rectangle

46. Daisies: trapezoid ($b_1 = 5$ ft, $b_2 = 3$ ft, $h = 2$ ft)

47. Chrysanthemums: triangle ($b = 3$ ft, $h = 8$ ft)

BRIDGES **In Exercises 48 and 49, use the following information.**

The town of Elizabethton, Tennessee, restored the roof of this covered bridge with cedar shakes, a kind of rough wooden shingle. The shakes vary in width, but the average width is about 10 inches. So, on average, each shake protects a 10 inch by 10 inch square of roof.



48. In the diagram of the roof, the hidden back and left sides are the same as the front and right sides. What is the total area of the roof?

49. Estimate the number of shakes needed to cover the roof.

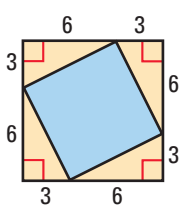
FOCUS ON PEOPLE

REAL LIFE

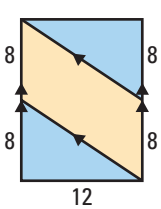
MARK CANDELARIA
When Mark Candelaria restored historic buildings in Scottsdale, Arizona, he calculated the areas of the walls and floors that needed to be replaced.

AREAS **Find the areas of the blue and yellow regions.**

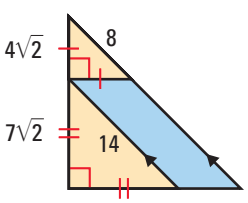
50.



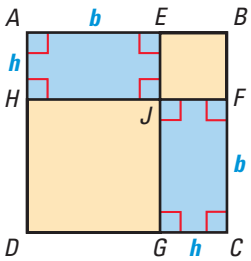
51.



52.



JUSTIFYING THEOREM 6.20 **In Exercises 53–57, you will justify the formula for the area of a rectangle. In the diagram, *AEJH* and *JFCG* are congruent rectangles with base length *b* and height *h*.**



53. What kind of shape is *EBFJ*? *HJGD*? Explain.

54. What kind of shape is *ABCD*? How do you know?

55. Write an expression for the length of a side of *ABCD*. Then write an expression for the area of *ABCD*.

56. Write expressions for the areas of *EBFJ* and *HJGD*.

STUDENT HELP

Look Back
For help with squaring binomial expressions, see p. 798.

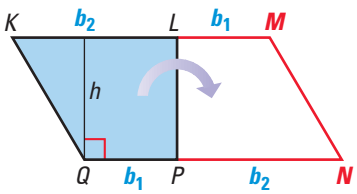
57. Substitute your answers from Exercises 55 and 56 into the following equation.

Let A = the area of *AEJH*. Solve the equation to find an expression for A.
Area of *ABCD* = Area of *HJGD* + Area of *EBFJ* + 2(Area of *AEJH*)

JUSTIFYING THEOREM 6.23 **Exercises 58 and 59 illustrate two ways to prove Theorem 6.23. Use the diagram to write a plan for a proof.**

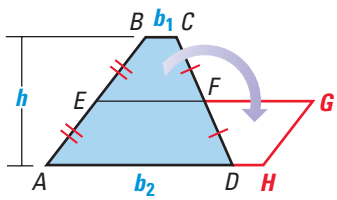
58. GIVEN ▶ *LPQK* is a trapezoid as shown. *LPQK* ≅ *PLMN*.

PROVE ▶ The area of *LPQK* is $\frac{1}{2}h(b_1 + b_2)$.



59. GIVEN ▶ *ABCD* is a trapezoid as shown. *EBCF* ≅ *GHDF*.

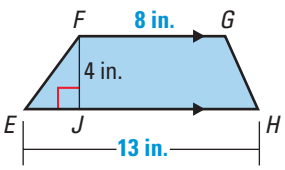
PROVE ▶ The area of *ABCD* is $\frac{1}{2}h(b_1 + b_2)$.



Test Preparation

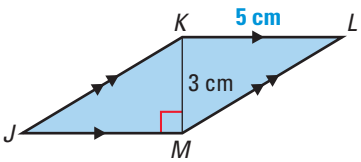
60. MULTIPLE CHOICE What is the area of trapezoid *EFGH*?

Ⓐ 25 in.^{2} Ⓑ 416 in.^{2}
Ⓒ 84 in.^{2} Ⓓ 42 in.^{2}
Ⓔ 68 in.^{2}



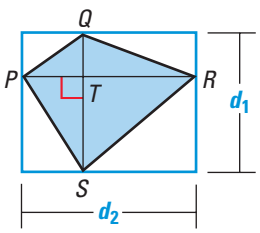
61. MULTIPLE CHOICE What is the area of parallelogram *JKLM*?

Ⓐ 12 cm^2 Ⓑ 15 cm^2
Ⓒ 18 cm^2 Ⓓ 30 cm^2
Ⓔ 40 cm^2



★ Challenge

62. *Writing* Explain why the area of *any* quadrilateral with perpendicular diagonals is $A = \frac{1}{2}d_1d_2$, where d_1 and d_2 are the lengths of the diagonals.

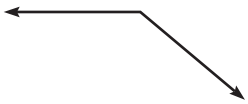


EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

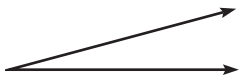
CLASSIFYING ANGLES **State whether the angle appears to be *acute, right,* or *obtuse*. Then estimate its measure.** (Review 1.4 for 7.1)

63.



64.

65.



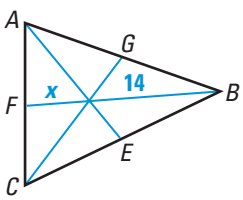
PLACING FIGURES IN A COORDINATE PLANE **Place the triangle in a coordinate plane and label the coordinates of the vertices.** (Review 4.7 for 7.1)

66. A triangle has a base length of 3 units and a height of 4 units.

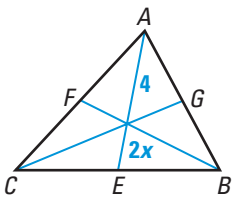
67. An isosceles triangle has a base length of 10 units and a height of 5 units.

USING ALGEBRA **In Exercises 68–70, $\overline{AE}$, $\overline{BF}$, and $\overline{CG}$ are medians. Find the value of *x*.** (Review 5.3)

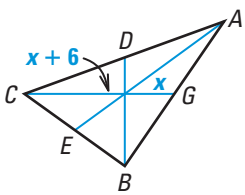
68.



69.



70.

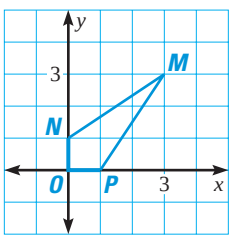


QUIZ 3

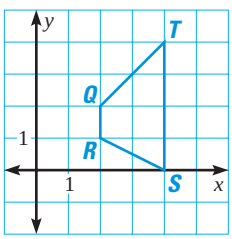
Self-Test for Lessons 6.6 and 6.7

What special type of quadrilateral is shown? Give the most specific name, and justify your answer. (Lesson 6.6)

1.



2.



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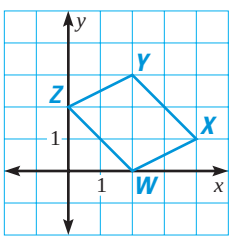
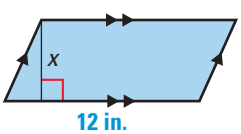
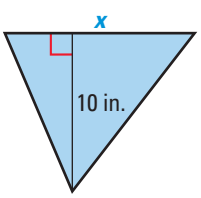


The shape has an area of 60 square inches. Find the value of *x*. (Lesson 6.7)

4.



5.



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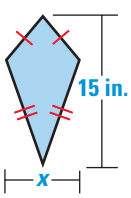


7. **GOLD BULLION** Gold bullion is molded into blocks with cross sections that are isosceles trapezoids. A cross section of a 25 kilogram block has a height of 5.4 centimeters and bases of 8.3 centimeters and 11 centimeters. What is the area of the cross section? (Lesson 6.7)

CHAPTER 6

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Identify, name, and describe polygons. (6.1)	Lay the foundation for work with polygons.
Use the sum of the measures of the interior angles of a quadrilateral. (6.1)	Find an unknown measure of an angle of a quadrilateral. (p. 324)
Use properties of parallelograms. (6.2)	Solve problems in areas such as furniture design. (p. 333)
Prove that a quadrilateral is a parallelogram. (6.3)	Explore real-life tools, such as a bicycle derailleur. (p. 343)
Use coordinate geometry with parallelograms. (6.3)	Use coordinates to prove theorems. (p. 344)
Use properties of rhombuses, rectangles, and squares, including properties of diagonals. (6.4)	Simplify real-life tasks, such as building a rectangular frame. (p. 350)
Use properties of trapezoids and kites. (6.5)	Reach conclusions about geometric figures and real-life objects, such as a wedding cake. (p. 357)
Identify special types of quadrilaterals based on limited information. (6.6)	Describe real-world shapes, such as tents. (p. 367)
Prove that a quadrilateral is a special type of quadrilateral. (6.6)	Use alternate methods of proof. (p. 365)
Find the areas of rectangles, kites, parallelograms, squares, triangles, trapezoids, and rhombuses. (6.7)	Find areas of real-life surfaces, such as the roof of a covered bridge. (p. 378)

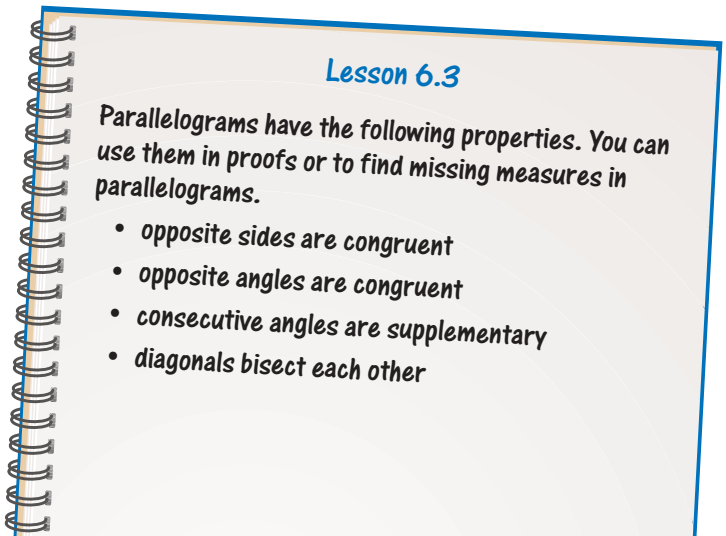
How does Chapter 6 fit into the BIGGER PICTURE of geometry?

In this chapter, you studied properties of polygons, focusing on properties of quadrilaterals. You learned in Chapter 4 that a triangle is a rigid structure. Polygons with more than three sides do not form rigid structures. For instance, on page 336, you learned that a scissors lift can be raised and lowered because its beams form parallelograms, which are nonrigid figures. Quadrilaterals occur in many natural and manufactured structures. Understanding properties of special quadrilaterals will help you analyze real-life problems in areas such as architecture, design, and construction.

STUDY STRATEGY

How did your study group help you learn?

The notes you made, following the **Study Strategy** on page 320, may resemble this one about order of operations.



CHAPTER 6

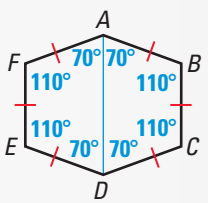
Chapter Review

- polygon, p. 322
- sides of a polygon, p. 322
- vertex, vertices, p. 322
- convex, p. 323
- nonconvex, concave, p. 323
- equilateral polygon, p. 323
- equiangular polygon, p. 323
- regular polygon, p. 323
- diagonal of a polygon, p. 324
- parallelogram, p. 330
- rhombus, p. 347
- rectangle, p. 347
- square, p. 347
- trapezoid, p. 356
- bases of a trapezoid, p. 356
- base angles of a trapezoid, p. 356
- legs of a trapezoid, p. 356
- isosceles trapezoid, p. 356
- midsegment of a trapezoid, p. 357
- kite, p. 358

6.1 POLYGONS

Examples on pp. 322–324

EXAMPLES Hexagon *ABCDEF* is convex and equilateral. It is not regular because it is not both equilateral and equiangular. $\overline{AD}$ is a diagonal of *ABCDEF*. The sum of the measures of the interior angles of quadrilateral *ABCD* is 360°.



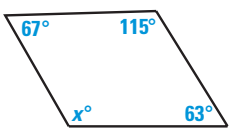
Draw a figure that fits the description.

1. a regular pentagon

2. a concave octagon

Find the value of *x*.

3.



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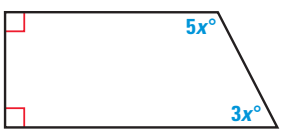

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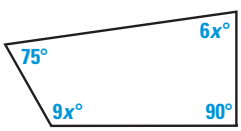
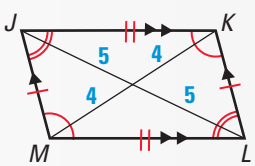


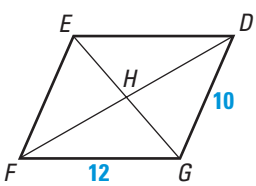
6.2 PROPERTIES OF PARALLELOGRAMS

Examples on pp. 330–333

EXAMPLES Quadrilateral *JKLM* is a parallelogram. Opposite sides are parallel and congruent. Opposite angles are congruent. Consecutive angles are supplementary. The diagonals bisect each other.



Use parallelogram *DEFG* at the right.



6. If $DH = 9.5$, find FH and DF.

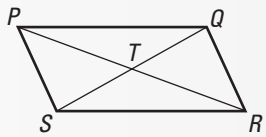
7. If $m\angle GDE = 65°$, find $m\angle EFG$ and $m\angle DEF$.

8. Find the perimeter of ▱*DEFG*.

6.3 PROVING QUADRILATERALS ARE PARALLELOGRAMS

Examples on pp. 338–341

EXAMPLES You are given that $\overline{PQ} \cong \overline{RS}$ and $\overline{PS} \cong \overline{RQ}$. Since both pairs of opposite sides are congruent, *PQRS* must be a parallelogram.



Is *PQRS* a parallelogram? Explain.

9. $PQ = QR$, $RS = SP$

10. $\angle SPQ \cong \angle QRS$, $\angle PQR \cong \angle RSP$

11. $\overline{PS} \cong \overline{RQ}$, $\overline{PQ} \parallel \overline{RS}$

12. $m\angle PSR + m\angle SRQ = 180°$, $\angle PSR \cong \angle RQP$

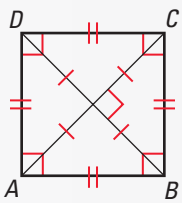
6.4 RHOMBUSES, RECTANGLES, AND SQUARES

Examples on pp. 347–350

EXAMPLES *ABCD* is a rhombus since it has 4 congruent sides. The diagonals of a rhombus are perpendicular and each one bisects a pair of opposite angles.

ABCD is a rectangle since it has 4 right angles. The diagonals of a rectangle are congruent.

ABCD is a square since it has 4 congruent sides and 4 right angles.



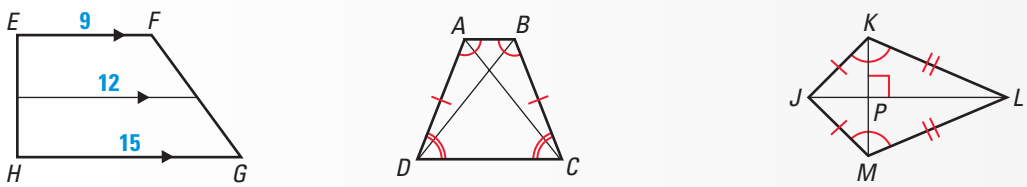
List each special quadrilateral for which the statement is always true. Consider parallelograms, rectangles, rhombuses, and squares.

13. Diagonals are perpendicular. **14.** Opposite sides are parallel. **15.** It is equilateral.

6.5 TRAPEZOIDS AND KITES

Examples on pp. 356–358

EXAMPLES *EFGH* is a trapezoid. *ABCD* is an isosceles trapezoid. Its base angles and diagonals are congruent. *JKLM* is a kite. Its diagonals are perpendicular, and one pair of opposite angles are congruent.



Use the diagram of isosceles trapezoid *ABCD*.

16. If $AB = 6$ and $CD = 16$, find the length of the midsegment.

17. If $m\angle DAB = 112°$, find the measures of the other angles of *ABCD*.

18. Explain how you could use congruent triangles to show that $\angle ACD \cong \angle BDC$.

6.6 SPECIAL QUADRILATERALS

Examples on pp. 364–366

EXAMPLES To prove that a quadrilateral is a rhombus, you can use any one of the following methods.

- Show that it has four congruent sides.
- Show that it is a parallelogram whose diagonals are perpendicular.
- Show that each diagonal bisects a pair of opposite angles.

What special type of quadrilateral is *PQRS*? Give the most specific name, and justify your answer.

19. $P(0, 3)$, $Q(5, 6)$, $R(2, 11)$, $S(-3, 8)$

20. $P(0, 0)$, $Q(6, 8)$, $R(8, 5)$, $S(4, -6)$

21. $P(2, -1)$, $Q(4, -5)$, $R(0, -3)$, $S(-2, 1)$

22. $P(-5, 0)$, $Q(-3, 6)$, $R(1, 6)$, $S(1, 2)$

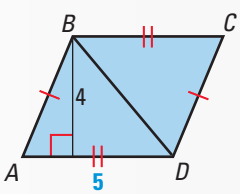
6.7 AREAS OF TRIANGLES AND QUADRILATERALS

Examples on pp. 372–375

EXAMPLES

Area of ▱$ABCD = bh = 5 \cdot 4 = 20$

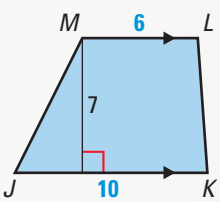
Area of $\triangle ABD = \frac{1}{2}bh = \frac{1}{2} \cdot 5 \cdot 4 = 10$



Area of trapezoid $JKLM = \frac{1}{2}h(b_1 + b_2)$

$= \frac{1}{2} \cdot 7 \cdot (10 + 6)$

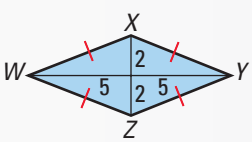
$= 56$



Area of rhombus $WXYZ = \frac{1}{2}d_1d_2$

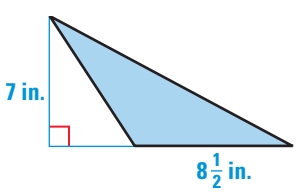
$= \frac{1}{2} \cdot 10 \cdot 4$

$= 20$



Find the area of the triangle or quadrilateral.

23.



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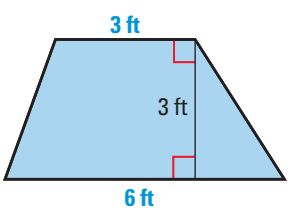

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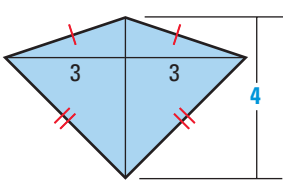


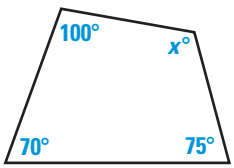
CHAPTER 6

Chapter Test

1. Sketch a concave pentagon.

Find the value of each variable.

2.



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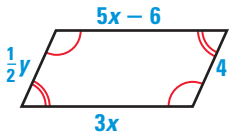

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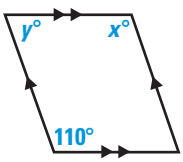

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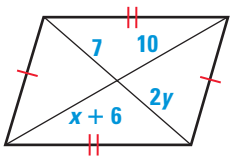


Decide if you are given enough information to prove that the quadrilateral is a parallelogram.

6. Diagonals are congruent.

7. Consecutive angles are supplementary.

8. Two pairs of consecutive angles are congruent.

9. The diagonals have the same midpoint.

Decide whether the statement is *always, sometimes,* or *never* true.

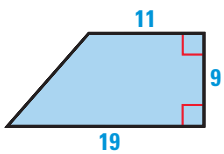
10. A rectangle is a square.

11. A parallelogram is a trapezoid.

12. A rhombus is a parallelogram.

What special type of quadrilateral is shown? Justify your answer.

13.



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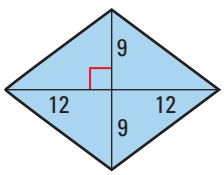

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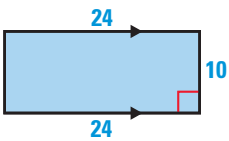

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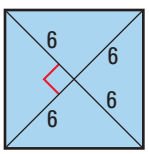
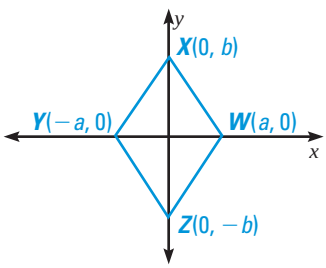
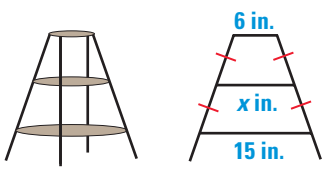


17. Refer to the coordinate diagram at the right. Use the Distance Formula to prove that *WXYZ* is a rhombus. Then explain how the diagram can be used to show that the diagonals of a rhombus bisect each other and are perpendicular.

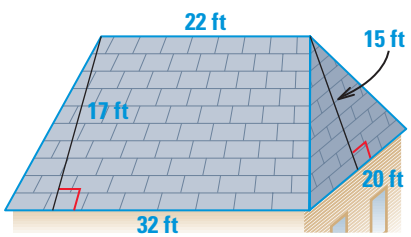


18. Sketch a kite and label it *ABCD*. Mark all congruent sides and angles of the kite. State what you know about the diagonals $\overline{AC}$ and $\overline{BD}$ and justify your answer.

19. **PLANT STAND** You want to build a plant stand with three equally spaced circular shelves. You want the top shelf to have a diameter of 6 inches and the bottom shelf to have a diameter of 15 inches. The diagram at the right shows a vertical cross section of the plant stand. What is the diameter of the middle shelf?



20. **HIP ROOF** The sides of a *hip roof* form two trapezoids and two triangles, as shown. The two sides not shown are congruent to the corresponding sides that are shown. Find the total area of the sides of the roof.



7.1 Rigid Motion in a Plane

What you should learn

GOAL 1 Identify the three basic rigid transformations.

GOAL 2 Use transformations in **real-life** situations, such as building a kayak in **Example 5**.

Why you should learn it

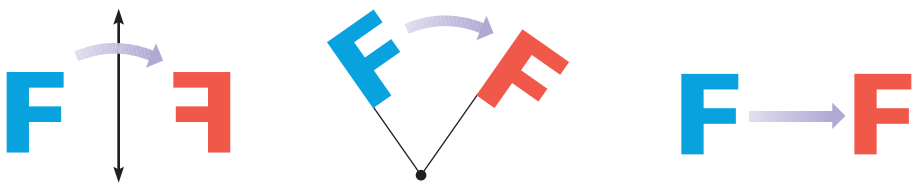
▼ Transformations help you when planning a stenciled design, such as on the wall below and the stencil in **Ex. 41**.

REAL LIFE

GOAL 1 IDENTIFYING TRANSFORMATIONS

Figures in a plane can be reflected, rotated, or translated to produce new figures. The new figure is called the **image**, and the original figure is called the **preimage**. The operation that *maps*, or moves, the preimage onto the image is called a **transformation**.

In this chapter, you will learn about three basic transformations—*reflections*, *rotations*, and *translations*—and combinations of these. For each of the three transformations below, the blue figure is the preimage and the red figure is the image. This color convention will be used throughout this book.



Reflection in a line **Rotation about a point** **Translation**

Some transformations involve labels. When you name an image, take the corresponding point of the preimage and add a prime symbol. For instance, if the preimage is A, then the image is A', read as "*A prime*."

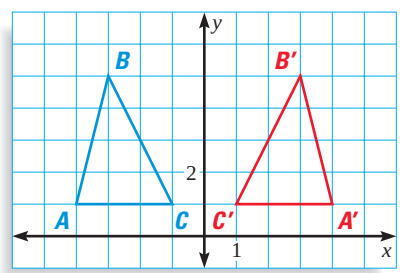
EXAMPLE 1 *Naming Transformations*

Use the graph of the transformation at the right.

a. Name and describe the transformation.

b. Name the coordinates of the vertices of the image.

c. Is $\triangle ABC$ congruent to its image?



SOLUTION

a. The transformation is a reflection in the y-axis. You can imagine that the image was obtained by flipping $\triangle ABC$ over the y-axis.

b. The coordinates of the vertices of the image, $\triangle A'B'C'$, are $A'(4, 1)$, $B'(3, 5)$, and $C'(1, 1)$.

c. Yes, $\triangle ABC$ is congruent to its image $\triangle A'B'C'$. One way to show this would be to use the Distance Formula to find the lengths of the sides of both triangles. Then use the SSS Congruence Postulate.

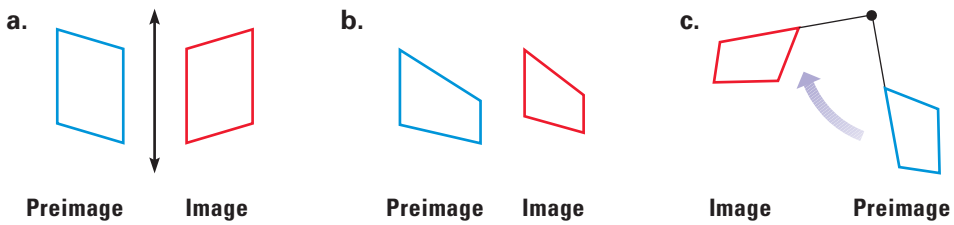
STUDENT HELP

Study Tip
The term **isometry** comes from the Greek phrase *isos metrom*, meaning *equal measure*.

An **isometry** is a transformation that preserves lengths. Isometries also preserve angle measures, parallel lines, and distances between points. Transformations that are isometries are called *rigid transformations*.

EXAMPLE 2 *Identifying Isometries*

Which of the following transformations appear to be isometries?



SOLUTION

a. This transformation appears to be an isometry. The blue parallelogram is reflected in a line to produce a congruent red parallelogram.

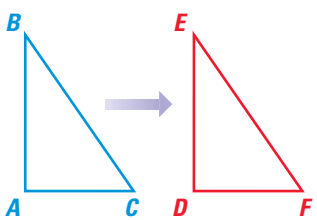
b. This transformation is not an isometry. The image is not congruent to the preimage.

c. This transformation appears to be an isometry. The blue parallelogram is rotated about a point to produce a congruent red parallelogram.

MAPPINGS You can describe the transformation in the diagram by writing "$\triangle ABC$ is *mapped onto* $\triangle DEF$." You can also use arrow notation as follows:

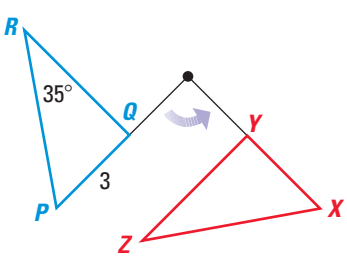
$$\triangle ABC \rightarrow \triangle DEF$$

The order in which the vertices are listed specifies the correspondence. Either of the descriptions implies that $A \rightarrow D$, $B \rightarrow E$, and $C \rightarrow F$.



EXAMPLE 3 *Preserving Length and Angle Measure*

In the diagram, $\triangle PQR$ is mapped onto $\triangle XYZ$. The mapping is a rotation. Given that $\triangle PQR \rightarrow \triangle XYZ$ is an isometry, find the length of $\overline{XY}$ and the measure of $\angle Z$.



SOLUTION

The statement "$\triangle PQR$ is mapped onto $\triangle XYZ$" implies that $P \rightarrow X$, $Q \rightarrow Y$, and $R \rightarrow Z$. Because the transformation is an isometry, the two triangles are congruent.

▶ So, $XY = PQ = 3$ and $m\angle Z = m\angle R = 35°$.

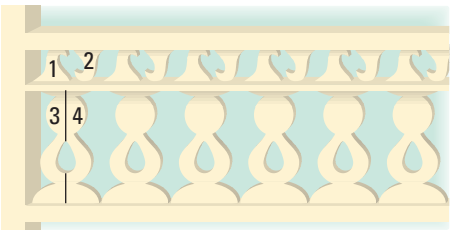
FOCUS ON APPLICATIONS

CARPENTER GOTHIC The woodwork of carpenter gothic houses contains decorative patterns. Notice the translations in the patterns of the carpenter gothic house above.

GOAL 2 USING TRANSFORMATIONS IN REAL LIFE

EXAMPLE 4 *Identifying Transformations*

CARPENTRY You are assembling pieces of wood to complete a railing for your porch. The finished railing should resemble the one below.



a. How are pieces 1 and 2 related? pieces 3 and 4?

b. In order to assemble the rail as shown, explain why you need to know how the pieces are related.

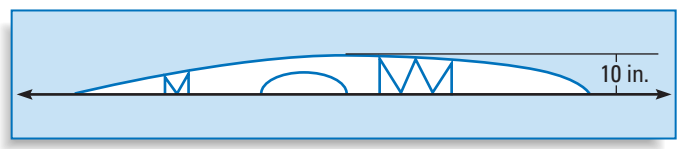
SOLUTION

a. Pieces 1 and 2 are related by a rotation. Pieces 3 and 4 are related by a reflection.

b. Knowing how the pieces are related helps you manipulate the pieces to create the desired pattern.

EXAMPLE 5 *Using Transformations*

BUILDING A KAYAK Many building plans for kayaks show the layout and dimensions for only half of the kayak. A plan of the top view of a kayak is shown below.



a. What type of transformation can a builder use to visualize plans for the entire body of the kayak?

b. Using the plan above, what is the maximum width of the entire kayak?

SOLUTION

a. The builder can use a reflection to visualize the entire kayak. For instance, when one half of the kayak is reflected in a line through its center, you obtain the other half of the kayak.

b. The two halves of the finished kayak are congruent, so the width of the entire kayak will be 2(10), or 20 inches.

GUIDED PRACTICE

Vocabulary Check ✓

1. An operation that maps a preimage onto an image is called a __?__.

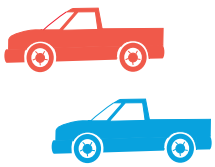
Concept Check ✓

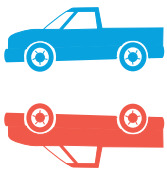
Complete the statement with *always, sometimes,* or *never*.

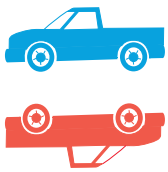
2. The preimage and the image of a transformation are __?__ congruent.
3. A transformation that is an isometry __?__ preserves length.
4. An isometry __?__ maps an acute triangle onto an obtuse triangle.

Skill Check ✓

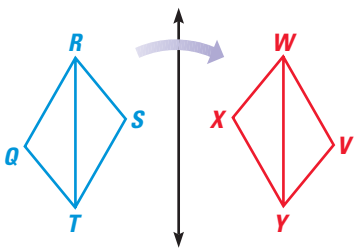
Name the transformation that maps the blue pickup truck (preimage) onto the red pickup (image).

5.


6.


7.


Use the figure shown, where figure *QRST* is mapped onto figure *VWXY*.



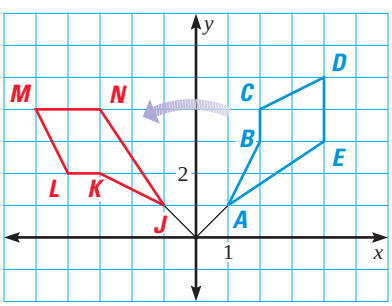
8. Name the preimage of $\overline{XY}$.
9. Name the image of $\overline{QR}$.
10. Name two angles that have the same measure.
11. Name a triangle that appears to be congruent to $\triangle RST$.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 815.

NAMING TRANSFORMATIONS **Use the graph of the transformation below.**



12. Figure $ABCDE \rightarrow$ Figure __?__
13. Name and describe the transformation.
14. Name two sides with the same length.
15. Name two angles with the same measure.
16. Name the coordinates of the preimage of point L.
17. Show two corresponding sides have the same length, using the Distance Formula.

STUDENT HELP

HOMEWORK HELP

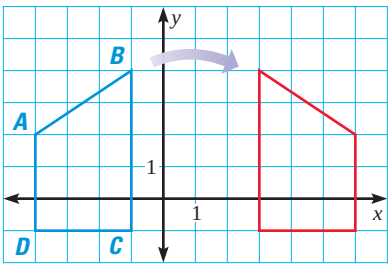
Example 1: Exs. 12–22
Example 2: Exs. 23–25
Example 3: Exs. 26–31
Example 4: Exs. 36–39
Example 5: Ex. 41

ANALYZING STATEMENTS **Is the statement *true* or *false*?**

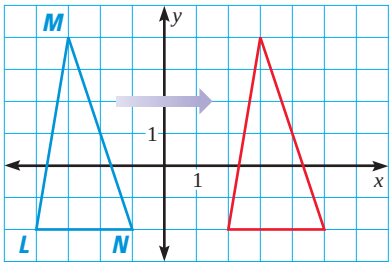
18. Isometries preserve angle measures and parallel lines.
19. Transformations that are *not* isometries are called rigid transformations.
20. A reflection in a line is a type of transformation.

DESCRIBING TRANSFORMATIONS **Name and describe the transformation. Then name the coordinates of the vertices of the image.**

21.

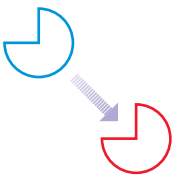


22.

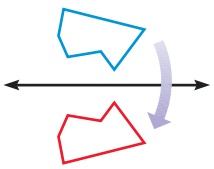


ISOMETRIES **Does the transformation appear to be an isometry? Explain.**

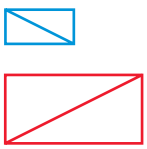
23.



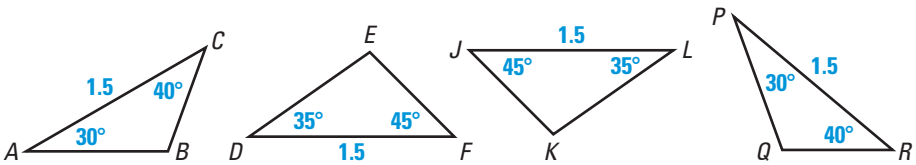
24.



25.



COMPLETING STATEMENTS **Use the diagrams to complete the statement.**



26. $\triangle ABC \rightarrow \triangle$ _?_

27. $\triangle DEF \rightarrow \triangle$ _?_

28. $\triangle$ _?_ $\rightarrow \triangle EFD$

29. $\triangle$ _?_ $\rightarrow \triangle ACB$

30. $\triangle LJK \rightarrow \triangle$ _?_

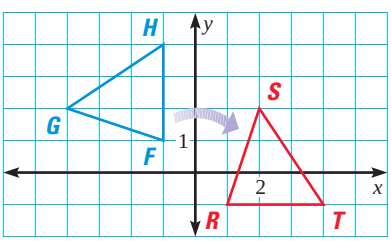
31. $\triangle$ _?_ $\rightarrow \triangle CBA$

STUDENT HELP

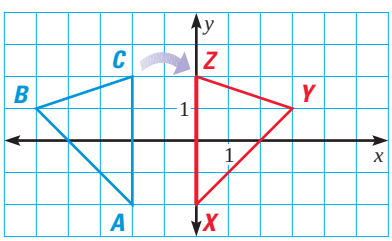
HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 32 and 33.

SHOWING AN ISOMETRY **Show that the transformation is an isometry by using the Distance Formula to compare the side lengths of the triangles.**

32. $\triangle FGH \rightarrow \triangle RST$

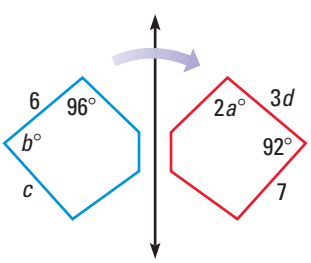


33. $\triangle ABC \rightarrow \triangle XYZ$

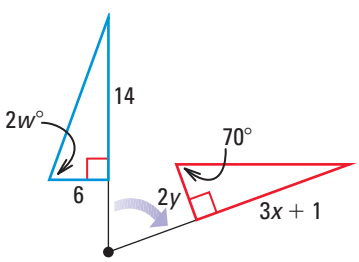


USING ALGEBRA **Find the value of each variable, given that the transformation is an isometry.**

34.



35.



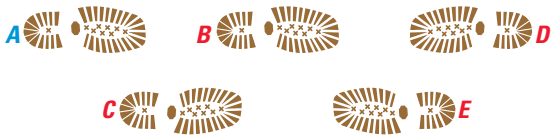
FOOTPRINTS **In Exercises 36–39, name the transformation that will map footprint *A* onto the indicated footprint.**

36. Footprint *B*

37. Footprint *C*

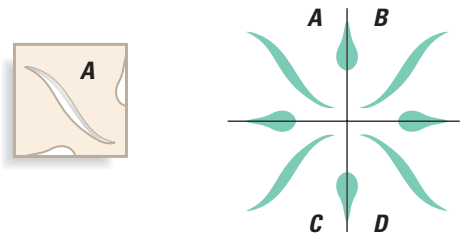
38. Footprint *D*

39. Footprint *E*



40. *Writing* Can a point or a line segment be its own preimage? Explain and illustrate your answer.

41. **STENCILING** You are stenciling the living room of your home. You want to use the stencil pattern below on the left to create the design shown. What type of transformation will you use to manipulate the stencil from *A* to *B*? from *A* to *C*? from *A* to *D*?



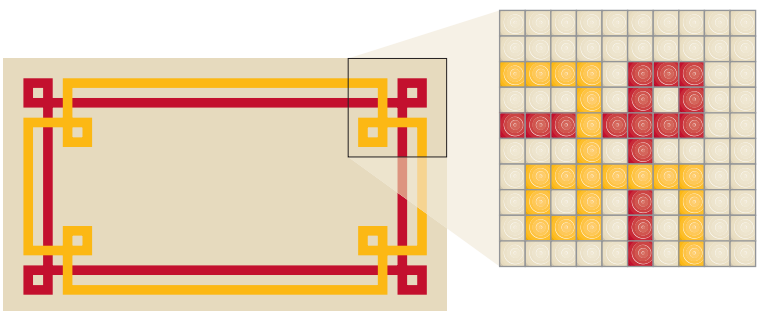
FOCUS ON APPLICATIONS

REAL LIFE EMBROIDERY Before machines, all stitching was done by hand. Completing samplers, such as the one above, served as practice for those learning how to stitch.

42. **MACHINE EMBROIDERY** Computerized embroidery machines are used to sew letters and designs on fabric. A computerized embroidery machine can use the same symbol to create several different letters. Which of the letters below are rigid transformations of other letters? Explain how a computerized embroidery machine can create these letters from one symbol.

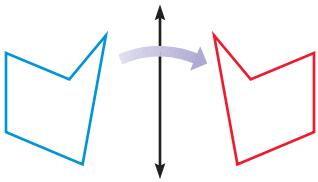
abcdefghijklm
nopqrstuvwxyz

43. **TILING A FLOOR** You are tiling a kitchen floor using the design shown below. You use a plan to lay the tile for the upper right corner of the floor design. Describe how you can use the plan to complete the other three corners of the floor.



Test Preparation

44. MULTIPLE CHOICE What type of transformation is shown?



Ⓐ slide Ⓑ reflection

Ⓒ translation Ⓓ rotation

45. MULTIPLE CHOICE Which of the following is *not* a rotation of the figure at right?

Ⓐ 

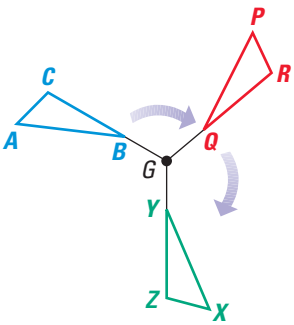
Ⓑ 

Ⓒ 

Ⓓ 

★ Challenge

46. TWO-COLUMN PROOF Write a two-column proof using the given information and the diagram.



GIVEN ▶ $\triangle ABC \rightarrow \triangle PQR$ and $\triangle PQR \rightarrow \triangle XYZ$ are isometries.

PROVE ▶ $\triangle ABC \rightarrow \triangle XYZ$ is an isometry.

Plan for Proof Show that $\overline{AB} \cong \overline{XY}$, $\overline{BC} \cong \overline{YZ}$, and $\overline{AC} \cong \overline{XZ}$.

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

USING THE DISTANCE FORMULA **Find the distance between the two points.** (Review 1.3 for 7.2)

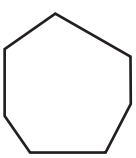
47. $A(3, 10)$, $B(-2, -2)$

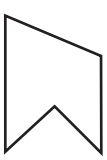
48. $C(5, -7)$, $D(-11, 6)$

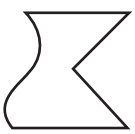
49. $E(0, 8)$, $F(-8, 3)$

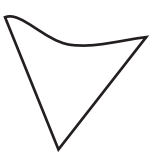
50. $G(0, -7)$, $H(6, 3)$

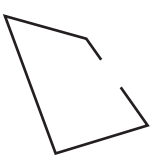
IDENTIFYING POLYGONS **Determine whether the figure is a polygon. If it is not, explain why not.** (Review 6.1 for 7.2)

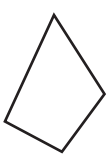
51. 

52. 

53. 

54. 

55. 

56. 

USING COORDINATE GEOMETRY **Use two different methods to show that the points represent the vertices of a parallelogram.** (Review 6.3)

57. $P(0, 4)$, $Q(7, 6)$, $R(8, -2)$, $S(1, -4)$

58. $W(1, 5)$, $X(9, 5)$, $Y(6, -1)$, $Z(-2, -1)$

7.2 Reflections

What you should learn

GOAL 1 Identify and use reflections in a plane.

GOAL 2 Identify relationships between reflections and line symmetry.

Why you should learn it

▼ Reflections and line symmetry can help you understand how mirrors in a kaleidoscope create interesting patterns, as in **Example 5**.

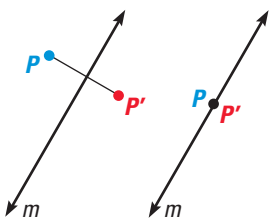
REAL LIFE

GOAL 1 USING REFLECTIONS IN A PLANE

One type of transformation uses a line that acts like a mirror, with an image reflected in the line. This transformation is a **reflection** and the mirror line is the **line of reflection**.

A reflection in a line m is a transformation that maps every point P in the plane to a point P', so that the following properties are true:

1. If P is not on m, then m is the perpendicular bisector of $\overline{PP'}$.
2. If P is on m, then $P = P'$.

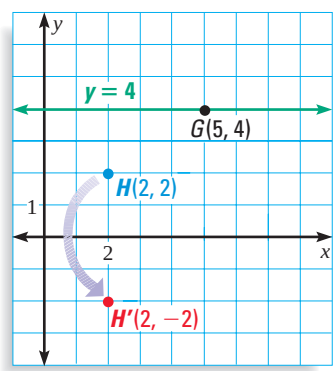


EXAMPLE 1 *Reflections in a Coordinate Plane*

Graph the given reflection.

a. $H(2, 2)$ in the x-axis

b. $G(5, 4)$ in the line $y = 4$



SOLUTION

a. Since H is two units above the x-axis, its reflection, H', is two units below the x-axis.

b. Start by graphing $y = 4$ and G. From the graph, you can see that G is on the line. This implies that $G = G'$.

Reflections in the coordinate axes have the following properties:

1. If (x, y) is reflected in the x-axis, its image is the point $(x, -y)$.
2. If (x, y) is reflected in the y-axis, its image is the point $(-x, y)$.

In Lesson 7.1, you learned that an isometry preserves lengths. Theorem 7.1 relates isometries and reflections.

THEOREM

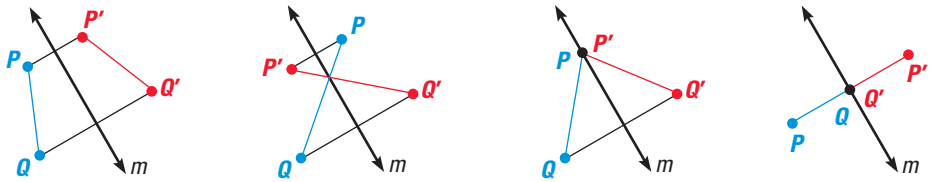
THEOREM 7.1 ***Reflection Theorem***

A reflection is an isometry.

STUDENT HELP

Study Tip
Some theorems have more than one case, such as the Reflection Theorem. To fully prove this type of theorem, all of the cases must be proven.

To prove the Reflection Theorem, you need to show that a reflection preserves the length of a segment. Consider a segment $\overline{PQ}$ that is reflected in a line m to produce $\overline{P'Q'}$. The four cases to consider are shown below.



Case 1
P and Q are on the same side of m.

Case 2
P and Q are on opposite sides of m.

Case 3
One point lies on m and $\overline{PQ}$ is not perpendicular to m.

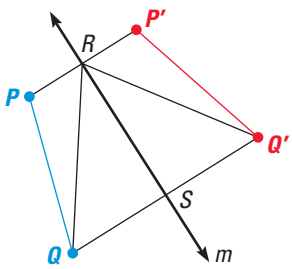
Case 4
Q lies on m and $\overline{PQ} \perp m$.

EXAMPLE 2 *Proof of Case 1 of Theorem 7.1*

Proof

GIVEN ▶ A reflection in m maps P onto P' and Q onto Q'.

PROVE ▶ $PQ = P'Q'$



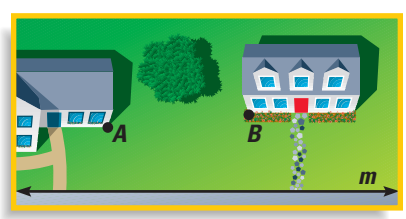
Paragraph Proof For this case, P and Q are on the same side of line m. Draw $\overline{PP'}$ and $\overline{QQ'}$, intersecting line m at R and S. Draw $\overline{RQ}$ and $\overline{RQ'}$.

By the definition of a reflection, $m \perp \overline{QQ'}$ and $\overline{QS} \cong \overline{Q'S}$. It follows that $\triangle RSQ \cong \triangle RSQ'$ using the SAS Congruence Postulate. This implies $\overline{RQ} \cong \overline{RQ'}$ and $\angle QRS \cong \angle Q'RS$. Because $\overleftrightarrow{RS}$ is a perpendicular bisector of $\overline{PP'}$, you have enough information to apply SAS to conclude that $\triangle RQP \cong \triangle RQ'P'$. Because corresponding parts of congruent triangles are congruent, $PQ = P'Q'$.

EXAMPLE 3 *Finding a Minimum Distance*

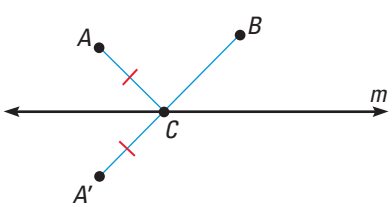
REAL LIFE

SURVEYING Two houses are located on a rural road m, as shown at the right. You want to place a telephone pole on the road at point C so that the length of the telephone cable, $AC + BC$, is a minimum. Where should you locate C?



SOLUTION

Reflect A in line m to obtain A'. Then, draw $\overline{A'B}$. Label the point at which this segment intersects m as C. Because $\overline{A'B}$ represents the shortest distance between A' and B, and $AC = A'C$, you can conclude that at point C a minimum length of telephone cable is used.



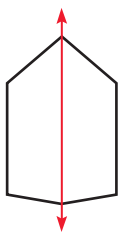
GOAL 2 REFLECTIONS AND LINE SYMMETRY

A figure in the plane has a **line of symmetry** if the figure can be mapped onto itself by a reflection in the line.

EXAMPLE 4 *Finding Lines of Symmetry*

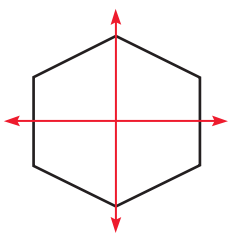
Hexagons can have different lines of symmetry depending on their shape.

a.



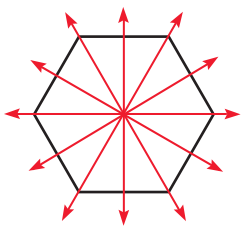
This hexagon has only one line of symmetry.

b.



This hexagon has two lines of symmetry.

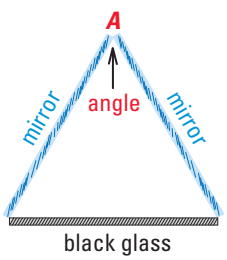
c.



This hexagon has six lines of symmetry.

EXAMPLE 5 *Identifying Reflections*

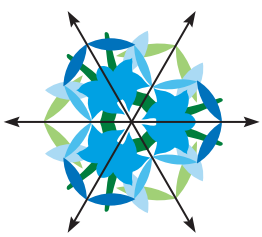
KALEIDOSCOPES Inside a kaleidoscope, two mirrors are placed next to each other to form a V, as shown at the right. The angle between the mirrors determines the number of lines of symmetry in the image. The formula below can be used to calculate the angle between the mirrors, A, or the number of lines of symmetry in the image, n.



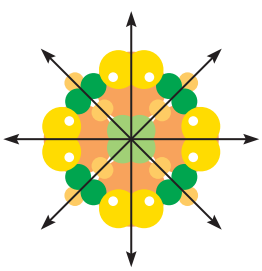
$$n(m\angle A) = 180°$$

Use the formula to find the angle that the mirrors must be placed for the image of a kaleidoscope to resemble the design.

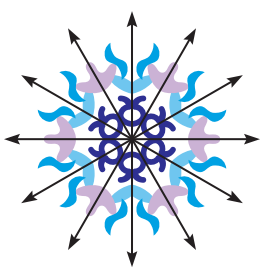
a.



b.



c.



SOLUTION

a. There are 3 lines of symmetry. So, you can write $3(m\angle A) = 180°$. The solution is $m\angle A = 60°$.

b. There are 4 lines of symmetry. So, you can write $4(m\angle A) = 180°$. The solution is $m\angle A = 45°$.

c. There are 6 lines of symmetry. So, you can write $6(m\angle A) = 180°$. The solution is $m\angle A = 30°$.

FOCUS ON PEOPLE

KALEIDOSCOPES Sue and Bob Rioux design and make kaleidoscopes. The kaleidoscope in front of Sue is called *Sea Angel*.

APPLICATION LINK
www.mcdougallittell.com

GUIDED PRACTICE

Vocabulary Check ✓

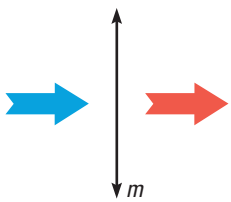
1. Describe what a *line of symmetry* is.

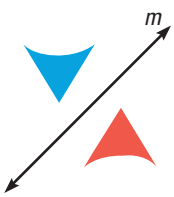
Concept Check ✓

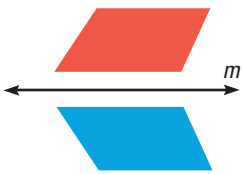
2. When a point is reflected in the x-axis, how are the coordinates of the image related to the coordinates of the preimage?

Skill Check ✓

Determine whether the blue figure maps onto the red figure by a reflection in line *m*.

3.


4.


5.


Use the diagram at the right to complete the statement.

6. $\overline{AB} \rightarrow$ __?__

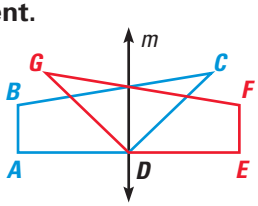
7. __?__ $\rightarrow \angle DEF$

8. $C \rightarrow$ __?__

9. $D \rightarrow$ __?__

10. __?__ $\rightarrow \angle GFE$

11. __?__ $\rightarrow \overline{DG}$



FLOWERS **Determine the number of lines of symmetry in the flower.**

12.


13.

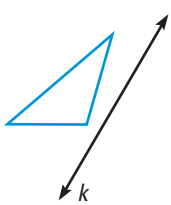

14.

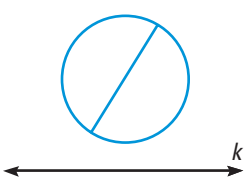

PRACTICE AND APPLICATIONS

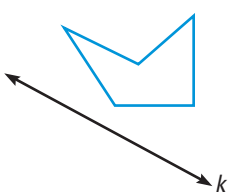
STUDENT HELP

Extra Practice to help you master skills is on pp. 815 and 816.

DRAWING REFLECTIONS **Trace the figure and draw its reflection in line *k*.**

15.


16.


17.


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 15–30
Example 2: Exs. 33–35
Example 3: Exs. 36–40
Example 4: Exs. 31, 32
Example 5: Exs. 44–46

ANALYZING STATEMENTS **Decide whether the conclusion is *true* or *false*. Explain your reasoning.**

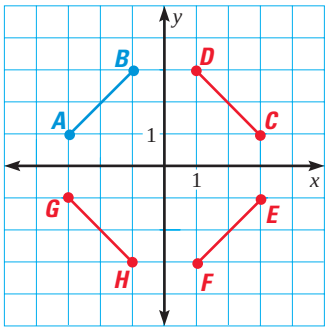
18. If $N(2, 4)$ is reflected in the line $y = 2$, then N' is $(2, 0)$.

19. If $M(6, -2)$ is reflected in the line $x = 3$, then M' is $(0, -2)$.

20. If $W(-6, -3)$ is reflected in the line $y = -2$, then W' is $(-6, 1)$.

21. If $U(5, 3)$ is reflected in the line $x = 1$, then U' is $(-3, 3)$.

REFLECTIONS IN A COORDINATE PLANE **Use the diagram at the right to name the image of $\overline{AB}$ after the reflection.**



22. Reflection in the x-axis

23. Reflection in the y-axis

24. Reflection in the line $y = x$

25. Reflection in the y-axis, followed by a reflection in the x-axis.

REFLECTIONS **In Exercises 26–29, find the coordinates of the reflection without using a coordinate plane. Then check your answer by plotting the image and preimage on a coordinate plane.**

26. $S(0, 2)$ reflected in the x-axis

27. $T(3, 8)$ reflected in the x-axis

28. $Q(-3, -3)$ reflected in the y-axis

29. $R(7, -2)$ reflected in the y-axis

30. **CRITICAL THINKING** Draw a triangle on the coordinate plane and label its vertices. Then reflect the triangle in the line $y = x$. What do you notice about the coordinates of the vertices of the preimage and the image?

LINES OF SYMMETRY **Sketch the figure, if possible.**

31. An octagon with exactly two lines of symmetry

32. A quadrilateral with exactly four lines of symmetry

PARAGRAPH PROOF **In Exercises 33–35, write a paragraph proof for each case of Theorem 7.1. (Refer to the diagrams on page 405.)**

33. In Case 2, it is given that a reflection in m maps P onto P' and Q onto Q'. Also, $\overline{PQ}$ intersects m at point R.

PROVE ▶ $PQ = P'Q'$

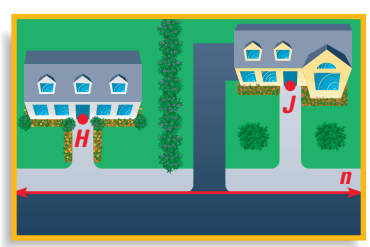
34. In Case 3, it is given that a reflection in m maps P onto P' and Q onto Q'. Also, P lies on line m and $\overline{PQ}$ is not perpendicular to m.

PROVE ▶ $PQ = P'Q'$

35. In Case 4, it is given that a reflection in m maps P onto P' and Q onto Q'. Also, Q lies on line m and $\overline{PQ}$ is perpendicular to line m.

PROVE ▶ $PQ = P'Q'$

36. **DELIVERING PIZZA** You park your car at some point K on line n. You deliver a pizza to house H, go back to your car, and deliver a pizza to house J. Assuming that you cut across both lawns, explain how to estimate K so the distance that you travel is as small as possible.



STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 37–40.

MINIMUM DISTANCE **Find point C on the x-axis so $AC + BC$ is a minimum.**

37. $A(1, 5)$, $B(7, 1)$

38. $A(2, -2)$, $B(11, -4)$

39. $A(-1, 4)$, $B(6, 3)$

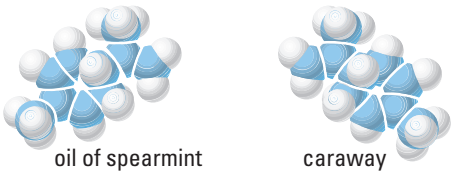
40. $A(-4, 6)$, $B(3.5, 9)$

FOCUS ON CAREERS

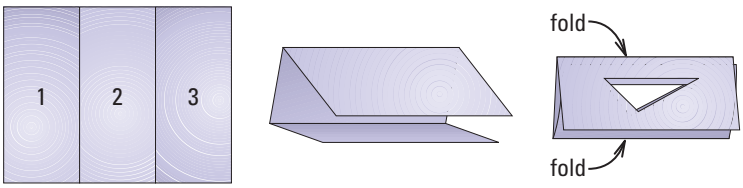
REAL LIFE **CHEMIST** Some chemists study the molecular structure of living things. The research done by these chemists has led to important discoveries in the field of medicine.

INTERNET **CAREER LINK**
www.mcdougallittell.com

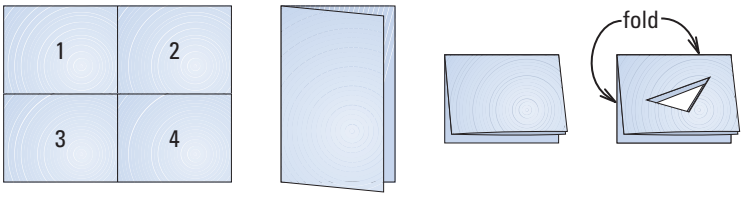
41. **CHEMISTRY CONNECTION** The figures at the right show two versions of the carvone molecule. One version is oil of spearmint and the other is caraway. How are the structures of these two molecules related?



42. **PAPER FOLDING** Fold a piece of paper and label it as shown. Cut a scalene triangle out of the folded paper and unfold the paper. How are triangle 2 and triangle 3 related to triangle 1?

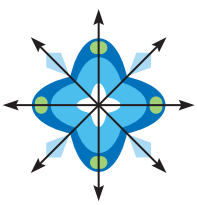


43. **PAPER FOLDING** Fold a piece of paper and label it as shown. Cut a scalene triangle out of the folded paper and unfold the paper. How are triangles 2, 3, and 4 related to triangle 1?

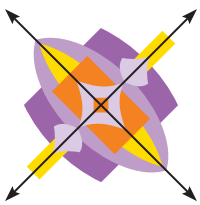


KALEIDOSCOPES In Exercises 44–46, calculate the angle at which the mirrors must be placed for the image of a kaleidoscope to resemble the given design. (Use the formula in Example 5 on page 406.)

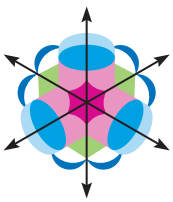
44.



45.



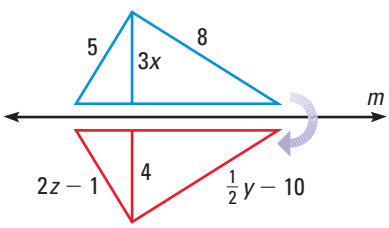
46.



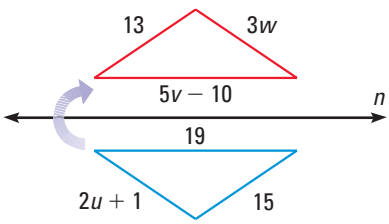
47. **TECHNOLOGY** Use geometry software to draw a polygon reflected in line m. Connect the corresponding vertices of the preimage and image. Measure the distance between each vertex and line m. What do you notice about these measures?

USING ALGEBRA Find the value of each variable, given that the diagram shows a reflection in a line.

48.

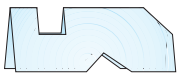


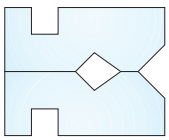
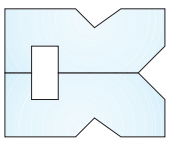
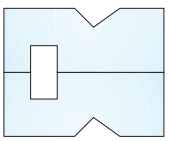
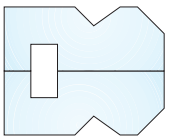
49.



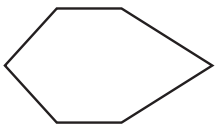
Test Preparation

50. MULTIPLE CHOICE A piece of paper is folded in half and some cuts are made, as shown. Which figure represents the piece of paper unfolded?



Ⓐ 
Ⓑ 
Ⓒ 
Ⓓ 

51. MULTIPLE CHOICE How many lines of symmetry does the figure at the right have?



Ⓐ 0 Ⓑ 1 Ⓒ 2
Ⓓ 3 Ⓔ 6

★ Challenge

WRITING AN EQUATION Follow the steps to write an equation for the line of reflection.

52. Graph $R(2, 1)$ and $R'(-2, -1)$. Draw a segment connecting the two points.

53. Find the midpoint of $\overline{RR'}$ and name it Q.

54. Find the slope of $\overline{RR'}$. Then write the slope of a line perpendicular to $\overline{RR'}$.

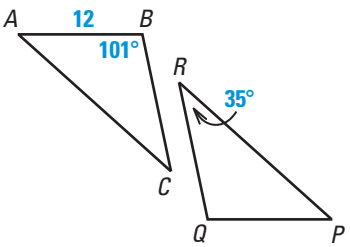
55. Write an equation of the line that is perpendicular to $\overline{RR'}$ and passes through Q.

56. Repeat Exercises 52–55 using $R(-2, 3)$ and $R'(3, -2)$.

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

CONGRUENT TRIANGLES Use the diagram, in which $\triangle ABC \cong \triangle PQR$, to complete the statement. (Review 4.2 for 7.3)



57. $\angle A \cong$ ___?___ **58.** $PQ =$ ___?___

59. $\overline{QR} \cong$ ___?___ **60.** $m\angle C =$ ___?___

61. $m\angle Q =$ ___?___ **62.** $\angle R \cong$ ___?___

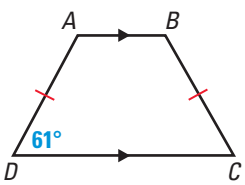
FINDING SIDE LENGTHS OF A TRIANGLE Two side lengths of a triangle are given. Describe the length of the third side, *c*, with an inequality. (Review 5.5)

63. $a = 7, b = 17$ **64.** $a = 9, b = 21$ **65.** $a = 12, b = 33$

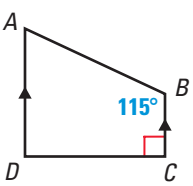
66. $a = 26, b = 6$ **67.** $a = 41.2, b = 15.5$ **68.** $a = 7.1, b = 11.9$

FINDING ANGLE MEASURES Find the angle measures of *ABCD*. (Review 6.5)

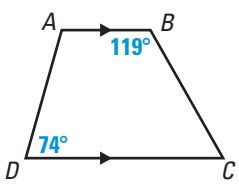
69.



70.



71.



7.3 Rotations

What you should learn

GOAL 1 Identify rotations in a plane.

GOAL 2 Use rotational symmetry in **real-life** situations, such as the logo designs in **Example 5**.

Why you should learn it

▼ Rotations and rotational symmetry can be used to create a design, as in the wheel hubs below and in **Exs. 36–38**.

REAL LIFE

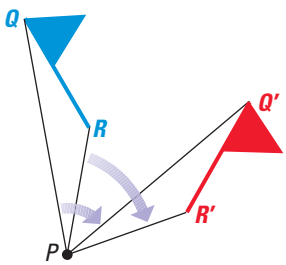
GOAL 1 USING ROTATIONS

A **rotation** is a transformation in which a figure is turned about a fixed point. The fixed point is the **center of rotation**. Rays drawn from the center of rotation to a point and its image form an angle called the **angle of rotation**.

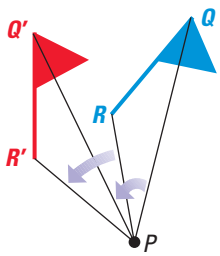
A rotation about a point P through x degrees ($x°$) is a transformation that maps every point Q in the plane to a point Q', so that the following properties are true:

1. If Q is not point P, then $QP = Q'P$ and $m\angle QPQ' = x°$.
2. If Q is point P, then $Q = Q'$.

Rotations can be clockwise or counterclockwise, as shown below.



Clockwise rotation of 60°



Counterclockwise rotation of 40°

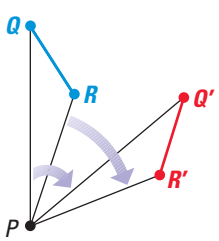
THEOREM

THEOREM 7.2 ***Rotation Theorem***

A rotation is an isometry.

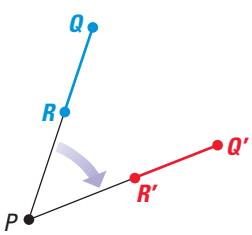
To prove the Rotation Theorem, you need to show that a rotation preserves the length of a segment. Consider a segment $\overline{QR}$ that is rotated about a point P to produce $\overline{Q'R'}$. The three cases are shown below. The first case is proved in Example 1.

CASE 1



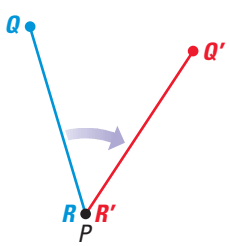
***R, Q,* and *P* are noncollinear.**

CASE 2



***R, Q,* and *P* are collinear.**

CASE 3

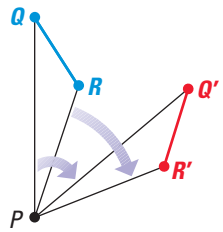


***P* and *R* are the same point.**

EXAMPLE 1 *Proof of Theorem 7.2*

Proof

Write a paragraph proof for Case 1 of the Rotation Theorem.



GIVEN ▶ A rotation about P maps Q onto Q' and R onto R'.

PROVE ▶ $\overline{QR} \cong \overline{Q'R'}$

SOLUTION

Paragraph Proof By the definition of a rotation, $PQ = PQ'$ and $PR = PR'$. Also, by the definition of a rotation, $m\angle QPQ' = m\angle RPR'$.

You can use the Angle Addition Postulate and the subtraction property of equality to conclude that $m\angle QPR = m\angle Q'PR'$. This allows you to use the SAS Congruence Postulate to conclude that $\triangle QPR \cong \triangle Q'PR'$. Because corresponding parts of congruent triangles are congruent, $\overline{QR} \cong \overline{Q'R'}$.

. . . . . .

STUDENT HELP

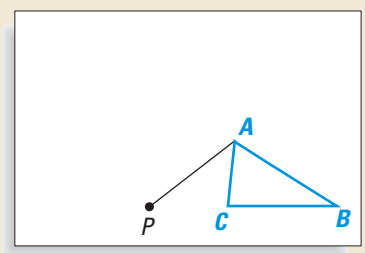
Look Back
For help with using a protractor, see p. 27.

You can use a compass and a protractor to help you find the images of a polygon after a rotation. The following construction shows you how.

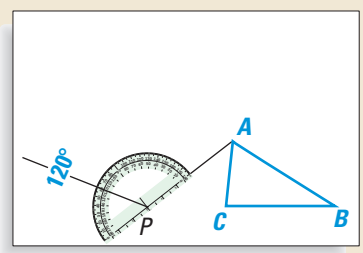
ACTIVITY Construction

Rotating a Figure

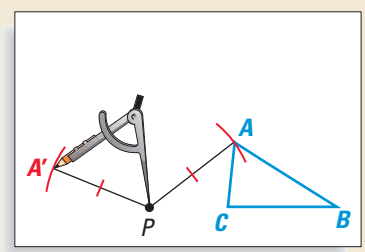
Use the following steps to draw the image of $\triangle ABC$ after a 120° counterclockwise rotation about point P.



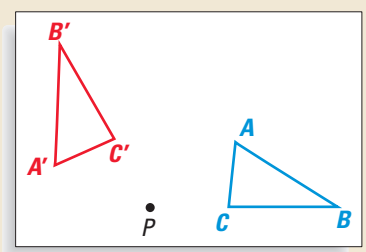
1. Draw a segment connecting vertex A and the center of rotation point P.



2. Use a protractor to measure a 120° angle counterclockwise and draw a ray.



3. Place the point of the compass at P and draw an arc from A to locate A'.



4. Repeat **Steps 1–3** for each vertex. Connect the vertices to form the image.

EXAMPLE 2 *Rotations in a Coordinate Plane*

STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

In a coordinate plane, sketch the quadrilateral whose vertices are $A(2, -2)$, $B(4, 1)$, $C(5, 1)$, and $D(5, -1)$. Then, rotate $ABCD$ 90° counterclockwise about the origin and name the coordinates of the new vertices. Describe any patterns you see in the coordinates.

SOLUTION

Plot the points, as shown in blue. Use a protractor, a compass, and a straightedge to find the rotated vertices. The coordinates of the preimage and image are listed below.

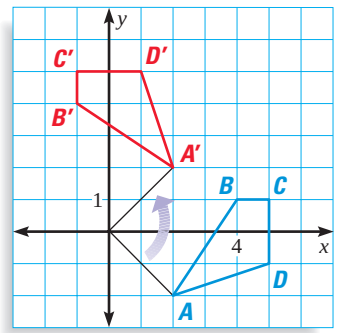


Figure *ABCD*	Figure *A′B′C′D′*
$A(2, -2)$	$A'(2, 2)$
$B(4, 1)$	$B'(-1, 4)$
$C(5, 1)$	$C'(-1, 5)$
$D(5, -1)$	$D'(1, 5)$

In the list above, the x-coordinate of the image is the opposite of the y-coordinate of the preimage. The y-coordinate of the image is the x-coordinate of the preimage.

▶ This transformation can be described as $(x, y) \rightarrow (-y, x)$.

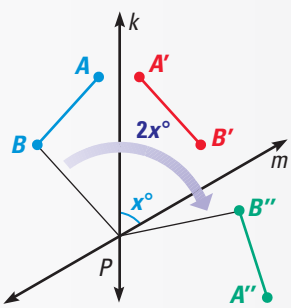
THEOREM

THEOREM 7.3

If lines k and m intersect at point P, then a reflection in k followed by a reflection in m is a rotation about point P.

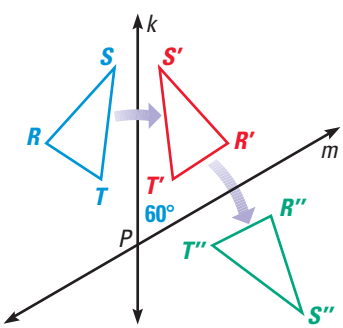
The angle of rotation is $2x°$, where $x°$ is the measure of the acute or right angle formed by k and m.

$$m\angle BPB'' = 2x°$$



EXAMPLE 3 *Using Theorem 7.3*

In the diagram, $\triangle RST$ is reflected in line k to produce $\triangle R'S'T'$. This triangle is then reflected in line m to produce $\triangle R''S''T''$. Describe the transformation that maps $\triangle RST$ to $\triangle R''S''T''$.



SOLUTION

The acute angle between lines k and m has a measure of 60°. Applying Theorem 7.3 you can conclude that the transformation that maps $\triangle RST$ to $\triangle R''S''T''$ is a clockwise rotation of 120° about point P.

GOAL 2 ROTATIONS AND ROTATIONAL SYMMETRY

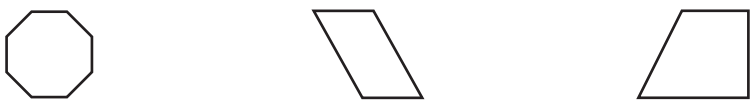
A figure in the plane has **rotational symmetry** if the figure can be mapped onto itself by a rotation of 180° or less. For instance, a square has rotational symmetry because it maps onto itself by a rotation of 90°.

0° rotation 45° rotation 90° rotation

EXAMPLE 4 *Identifying Rotational Symmetry*

Which figures have rotational symmetry? For those that do, describe the rotations that map the figure onto itself.

a. Regular octagon **b.** Parallelogram **c.** Trapezoid



SOLUTION

a. This octagon has rotational symmetry. It can be mapped onto itself by a clockwise or counterclockwise rotation of 45°, 90°, 135°, or 180° about its center.

b. This parallelogram has rotational symmetry. It can be mapped onto itself by a clockwise or counterclockwise rotation of 180° about its center.

c. The trapezoid does not have rotational symmetry.

FOCUS ON CAREERS

LOGO DESIGNERS create symbols that represent the name of a company or organization. The logos appear on packaging, letterheads, and Web sites.

CAREER LINK
www.mcdougallittell.com

EXAMPLE 5 *Using Rotational Symmetry*

LOGO DESIGN A music store called Ozone is running a contest for a store logo. The winning logo will be displayed on signs throughout the store and in the store's advertisements. The only requirement is that the logo include the store's name. Two of the entries are shown below. What do you notice about them?

a.

b.

SOLUTION

a. This design has rotational symmetry about its center. It can be mapped onto itself by a clockwise or counterclockwise rotation of 180°.

b. This design also has rotational symmetry about its center. It can be mapped onto itself by a clockwise or counterclockwise rotation of 90° or 180°.

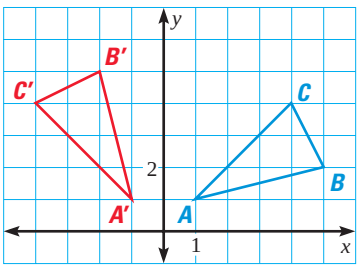
GUIDED PRACTICE

Vocabulary Check ✓

1. What is a *center of rotation*?

Concept Check ✓

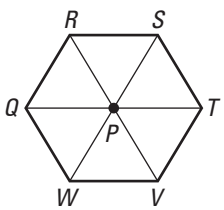
Use the diagram, in which $\triangle ABC$ is mapped onto $\triangle A'B'C'$ by a rotation of 90° about the origin.



2. Is the rotation clockwise or counterclockwise?
3. Does $AB = A'B'$? Explain.
4. Does $AA' = BB'$? Explain.
5. If the rotation of $\triangle ABC$ onto $\triangle A'B'C'$ was obtained by a reflection of $\triangle ABC$ in some line k followed by a reflection in some line m, what would be the measure of the acute angle between lines k and m? Explain.

Skill Check ✓

The diagonals of the regular hexagon below form six equilateral triangles. Use the diagram to complete the sentence.



6. A clockwise rotation of 60° about P maps R onto __?__.
7. A counterclockwise rotation of 60° about __?__ maps R onto Q.
8. A clockwise rotation of 120° about Q maps R onto __?__.
9. A counterclockwise rotation of 180° about P maps V onto __?__.

Determine whether the figure has rotational symmetry. If so, describe the rotations that map the figure onto itself.

10. 

11. 

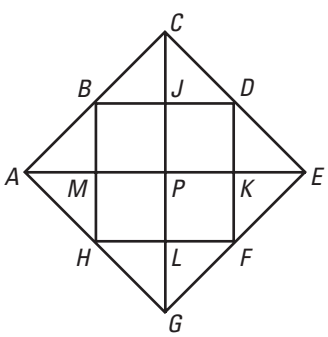
12. 

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 816.

DESCRIBING AN IMAGE **State the segment or triangle that represents the image. You can use tracing paper to help you visualize the rotation.**



13. 90° clockwise rotation of $\overline{AB}$ about P
14. 90° clockwise rotation of $\overline{KF}$ about P
15. 90° counterclockwise rotation of $\overline{CE}$ about E
16. 90° counterclockwise rotation of $\overline{FL}$ about H
17. 180° rotation of $\triangle KEF$ about P
18. 180° rotation of $\triangle BCJ$ about P
19. 90° clockwise rotation of $\triangle APG$ about P

STUDENT HELP

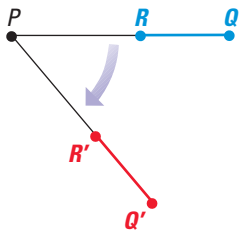
HOMEWORK HELP

Example 1: Exs. 13–21
Example 2: Exs. 22–29
Example 3: Exs. 30–33
Example 4: Exs. 36–38
Example 5: Exs. 39–42

PARAGRAPH PROOF **Write a paragraph proof for the case of Theorem 7.2.**

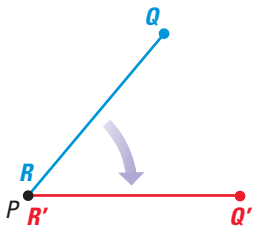
20. GIVEN ▶ A rotation about P maps Q onto Q' and R onto R'.

PROVE ▶ $\overline{QR} \cong \overline{Q'R'}$



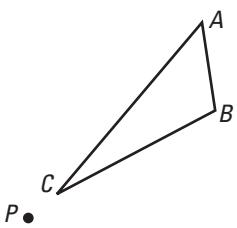
21. GIVEN ▶ A rotation about P maps Q onto Q' and R onto R'. P and R are the same point.

PROVE ▶ $\overline{QR} \cong \overline{Q'R'}$

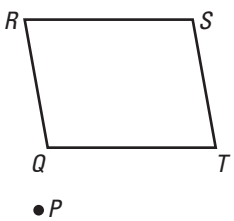


ROTATING A FIGURE **Trace the polygon and point *P* on paper. Then, use a straightedge, compass, and protractor to rotate the polygon clockwise the given number of degrees about *P*.**

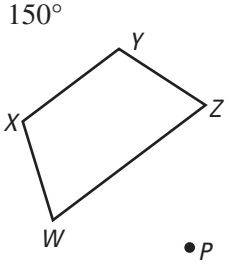
22. 60°



23. 135°

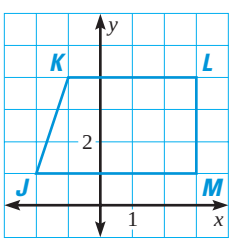


24. 150°

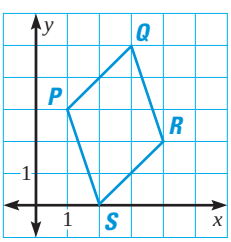


ROTATIONS IN A COORDINATE PLANE **Name the coordinates of the vertices of the image after a clockwise rotation of the given number of degrees about the origin.**

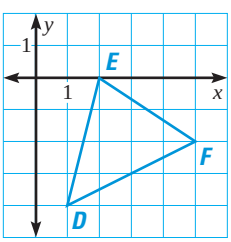
25. 90°



26. 180°

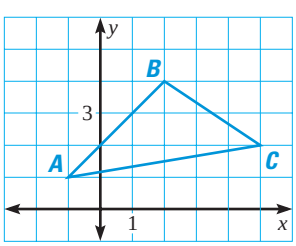


27. 270°

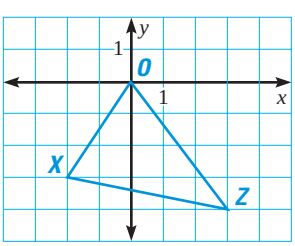


FINDING A PATTERN **Use the given information to rotate the triangle. Name the vertices of the image and compare with the vertices of the preimage. Describe any patterns you see.**

28. 90° clockwise about origin

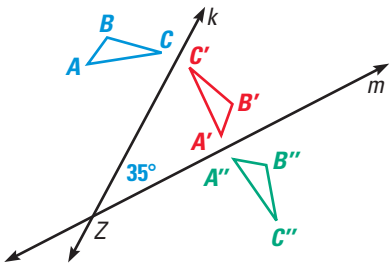


29. 180° clockwise about origin

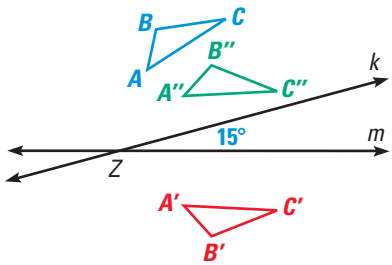


USING THEOREM 7.3 **Find the angle of rotation that maps $\triangle ABC$ onto $\triangle A''B''C''$.**

30.



31.



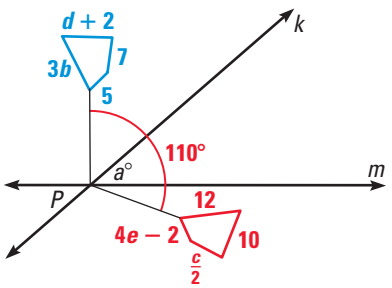
LOGICAL REASONING **Lines *m* and *n* intersect at point *D*. Consider a reflection of $\triangle ABC$ in line *m* followed by a reflection in line *n*.**

32. What is the angle of rotation about D, when the measure of the acute angle between lines m and n is 36°?

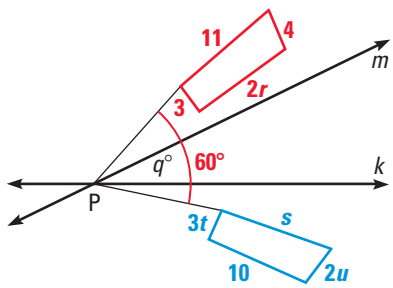
33. What is the measure of the acute angle between lines m and n, when the angle of rotation about D is 162°?

USING ALGEBRA **Find the value of each variable in the rotation of the polygon about point *P*.**

34.



35.



WHEEL HUBS **Describe the rotational symmetry of the wheel hub.**

36.

37.

38.

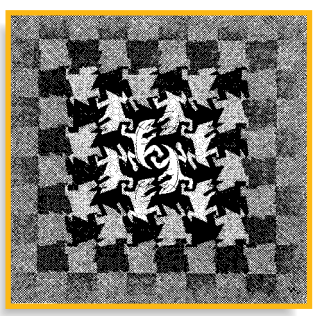
ROTATIONS IN ART **In Exercises 39–42, refer to the image below by M.C. Escher. The piece is called *Development I* and was completed in 1937.**

39. Does the piece have rotational symmetry? If so, describe the rotations that map the image onto itself.

40. Would your answer to Exercise 39 change if you disregard the shading of the figures? Explain your reasoning.

41. Describe the center of rotation.

42. Is it possible that this piece could be hung upside down? Explain.



M.C. Escher's *"Development I"* © 1999 Cordon Art B.V. - Baarn - Holland. All rights reserved.

FOCUS ON PEOPLE

M.C. ESCHER is a Dutch graphic artist whose works include optical illusions and geometric patterns. (M.C. Escher's *"Hand with Reflecting Sphere"* © 1999 Cordon Art B.V. - Baarn - Holland. All rights reserved.)

APPLICATION LINK
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Test Preparation

43. MULTI-STEP PROBLEM Follow the steps below.

a. Graph $\triangle RST$ whose vertices are $R(1, 1)$, $S(4, 3)$, and $T(5, 1)$.

b. Reflect $\triangle RST$ in the y-axis to obtain $\triangle R'S'T'$. Name the coordinates of the vertices of the reflection.

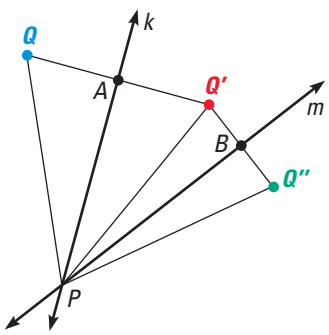
c. Reflect $\triangle R'S'T'$ in the line $y = -x$ to obtain $\triangle R''S''T''$. Name the coordinates of the vertices of the reflection.

d. Describe a single transformation that maps $\triangle RST$ onto $\triangle R''S''T''$.

e. *Writing* Explain how to show a 90° counterclockwise rotation of any polygon about the origin using two reflections of the figure.

★ Challenge

44. PROOF Use the diagram and the given information to write a paragraph proof for Theorem 7.3.



GIVEN ▶ Lines k and m intersect at point P, Q is any point not on k or m.

PROVE ▶ **a.** If you reflect point Q in k, and then reflect its image Q' in m, Q'' is the image of Q after a rotation about point P.

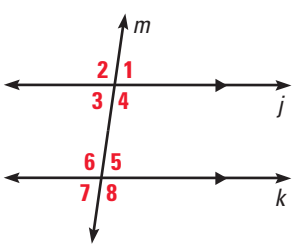
b. $m\angle QPQ'' = 2(m\angle APB)$.

Plan for Proof First show $k \perp \overline{QQ'}$ and $\overline{QA} \cong \overline{Q'A}$. Then show $\triangle QAP \cong \triangle Q'AP$. Use a similar argument to show $\triangle Q'BP \cong \triangle Q''BP$. Use the congruent triangles and substitution to show that $\overline{QP} \cong \overline{Q''P}$. That proves part (a) by the definition of a rotation. You can use the congruent triangles to prove part (b).

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

PARALLEL LINES Find the measure of the angle using the diagram, in which $j \parallel k$ and $m\angle 1 = 82°$. (Review 3.3 for 7.4)



45. $m\angle 5$ **46.** $m\angle 7$

47. $m\angle 3$ **48.** $m\angle 6$

49. $m\angle 4$ **50.** $m\angle 8$

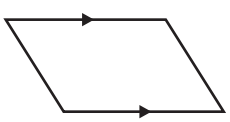
DRAWING TRIANGLES In Exercises 51–53, draw the triangle. (Review 5.2)

51. Draw a triangle whose circumcenter lies outside the triangle.

52. Draw a triangle whose circumcenter lies on the triangle.

53. Draw a triangle whose circumcenter lies inside the triangle.

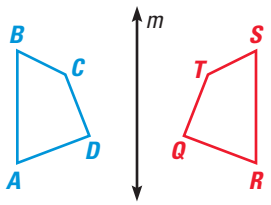
54. PARALLELOGRAMS Can it be proven that the figure at the right is a parallelogram? If not, explain why not. (Review 6.2)



QUIZ 1

Self-Test for Lessons 7.1–7.3

Use the transformation at the right. (Lesson 7.1)

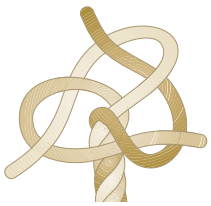


1. Figure $ABCD \rightarrow$ Figure _?_
2. Name and describe the transformation.
3. Is the transformation an isometry? Explain.

In Exercises 4–7, find the coordinates of the reflection without using a coordinate plane. (Lesson 7.2)

4. $L(2, 3)$ reflected in the x-axis
5. $M(-2, -4)$ reflected in the y-axis
6. $N(-4, 0)$ reflected in the x-axis
7. $P(8.2, -3)$ reflected in the y-axis

8. **KNOTS** The knot at the right is a *wall knot*, which is generally used to prevent the end of a rope from running through a pulley. Describe the rotations that map the knot onto itself and describe the center of rotation. (Lesson 7.3)



MATH & History — History of Decorative Patterns

APPLICATION LINK
www.mcdougallittell.com

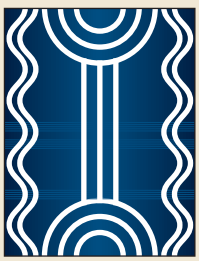
THEN

FOR THOUSANDS OF YEARS, people have adorned their buildings, pottery, clothing, and jewelry with decorative patterns. Simple patterns were created by using a transformation of a shape.

NOW

TODAY, you are likely to find computer generated patterns decorating your clothes, CD covers, sports equipment, computer desktop, and even textbooks.

1. The design at the right is based on a piece of pottery by Marsha Gomez. How many lines of symmetry does the design have?
2. Does the design have rotational symmetry? If so, describe the rotation that maps the pattern onto itself.



c. 1300 B.C.

Egyptian jewelry is decorated with patterns.

Tiles are arranged in symmetric patterns in the Alhambra in Spain.

c. 1300

1899

Painted textile pattern called 'Bulow Birds'

Marsha Gomez decorates pottery with symmetrical patterns.

1990s

7.4 Translations and Vectors

What you should learn

GOAL 1 Identify and use translations in the plane.

GOAL 2 Use vectors in **real-life** situations, such as navigating a sailboat in **Example 6**.

Why you should learn it

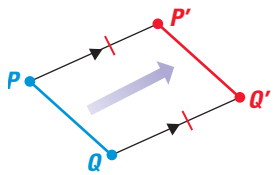
▼ You can use translations and vectors to describe the path of an aircraft, such as the hot-air balloon in **Exs. 53–55**.

REAL LIFE

GOAL 1 USING PROPERTIES OF TRANSLATIONS

A **translation** is a transformation that maps every two points P and Q in the plane to points P' and Q', so that the following properties are true:

1. $PP' = QQ'$
2. $\overline{PP'} \parallel \overline{QQ'}$, or $\overline{PP'}$ and $\overline{QQ'}$ are collinear.



THEOREM

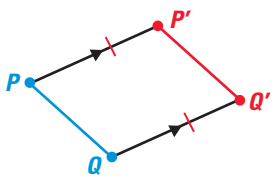
THEOREM 7.4 ***Translation Theorem***

A translation is an isometry.

Theorem 7.4 can be proven as follows.

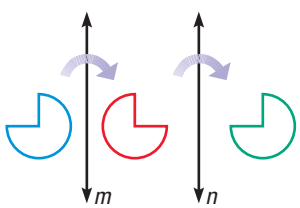
GIVEN ▶ $PP' = QQ'$, $\overline{PP'} \parallel \overline{QQ'}$

PROVE ▶ $PQ = P'Q'$



Paragraph Proof The quadrilateral $PP'Q'Q$ has a pair of opposite sides that are congruent and parallel, which implies $PP'Q'Q$ is a parallelogram. From this you can conclude $PQ = P'Q'$. (Exercise 43 asks for a coordinate proof of Theorem 7.4, which covers the case where $\overline{PQ}$ and $\overline{P'Q'}$ are collinear.)

You can find the image of a translation by gliding a figure in the plane. Another way to find the image of a translation is to complete one reflection after another in two parallel lines, as shown. The properties of this type of translation are stated below.

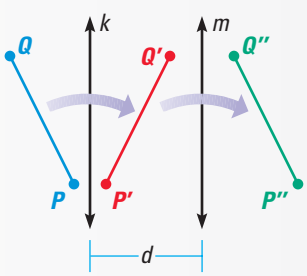


THEOREM

THEOREM 7.5

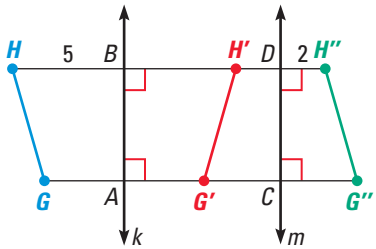
If lines k and m are parallel, then a reflection in line k followed by a reflection in line m is a translation. If P'' is the image of P, then the following is true:

1. $\overleftrightarrow{PP''}$ is perpendicular to k and m.
2. $PP'' = 2d$, where d is the distance between k and m.



EXAMPLE 1 *Using Theorem 7.5*

In the diagram, a reflection in line k maps $\overline{GH}$ to $\overline{G'H'}$, a reflection in line m maps $\overline{G'H'}$ to $\overline{G''H''}$, $k \parallel m$, $HB = 5$, and $DH'' = 2$.



a. Name some congruent segments.

b. Does $AC = BD$? Explain.

c. What is the length of $\overline{GG''}$?

STUDENT HELP

Study Tip

In Lesson 7.2, you learned that the line of reflection is the perpendicular bisector of the segment connecting a point and its image. In Example 1, you can use this property to conclude that figure *ABDC* is a rectangle.

SOLUTION

a. Here are some sets of congruent segments: $\overline{GH}$, $\overline{G'H'}$, and $\overline{G''H''}$; $\overline{HB}$ and $\overline{H'B}$; $\overline{H'D}$ and $\overline{H''D}$.

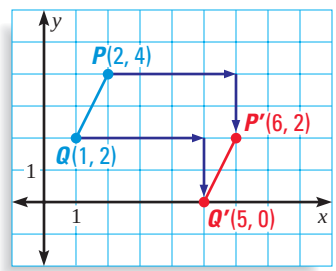
b. Yes, $AC = BD$ because $\overline{AC}$ and $\overline{BD}$ are opposite sides of a rectangle.

c. Because $GG'' = HH''$, the length of $\overline{GG''}$ is $5 + 5 + 2 + 2$, or 14 units.

Translations in a coordinate plane can be described by the following coordinate notation:

$$(x, y) \rightarrow (x + a, y + b)$$

where a and b are constants. Each point shifts a units horizontally and b units vertically. For instance, in the coordinate plane at the right, the translation $(x, y) \rightarrow (x + 4, y - 2)$ shifts each point 4 units to the right and 2 units down.



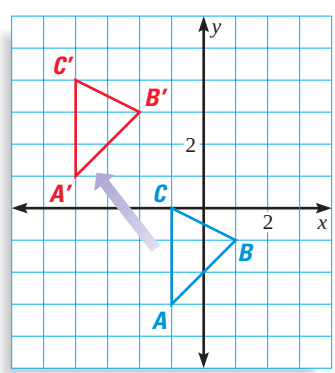
EXAMPLE 2 *Translations in a Coordinate Plane*

Sketch a triangle with vertices $A(-1, -3)$, $B(1, -1)$, and $C(-1, 0)$. Then sketch the image of the triangle after the translation $(x, y) \rightarrow (x - 3, y + 4)$.

SOLUTION

Plot the points as shown. Shift each point 3 units to the left and 4 units up to find the translated vertices. The coordinates of the vertices of the preimage and image are listed below.

$\triangle ABC$	$\triangle A'B'C'$
$A(-1, -3)$	$A'(-4, 1)$
$B(1, -1)$	$B'(-2, 3)$
$C(-1, 0)$	$C'(-4, 4)$



Notice that each x-coordinate of the image is 3 units less than the x-coordinate of the preimage and each y-coordinate of the image is 4 units more than the y-coordinate of the preimage.

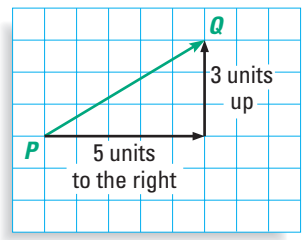
STUDENT HELP

Study Tip
When writing a vector in component form, use the correct brackets. The brackets used to write the component form of a vector are different than the parentheses used to write an ordered pair.

GOAL 2 TRANSLATIONS USING VECTORS

Another way to describe a translation is by using a vector. A **vector** is a quantity that has both direction and *magnitude*, or size, and is represented by an arrow drawn between two points.

The diagram shows a vector. The **initial point**, or starting point, of the vector is P and the **terminal point**, or ending point, is Q. The vector is named $\overrightarrow{PQ}$, which is read as "vector PQ." The *horizontal component* of $\overrightarrow{PQ}$ is 5 and the *vertical component* is 3.

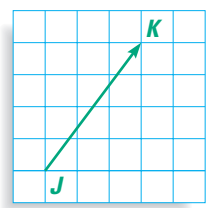


The **component form** of a vector combines the horizontal and vertical components. So, the component form of $\overrightarrow{PQ}$ is $\langle 5, 3\rangle$.

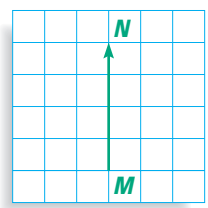
EXAMPLE 3 *Identifying Vector Components*

In the diagram, name each vector and write its component form.

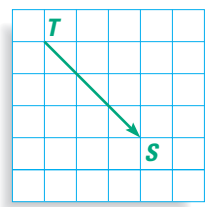
a.



b.



c.



SOLUTION

a. The vector is $\overrightarrow{JK}$. To move from the initial point J to the terminal point K, you move 3 units to the right and 4 units up. So, the component form is $\langle 3, 4\rangle$.

b. The vector is $\overrightarrow{MN} = \langle 0, 4\rangle$.

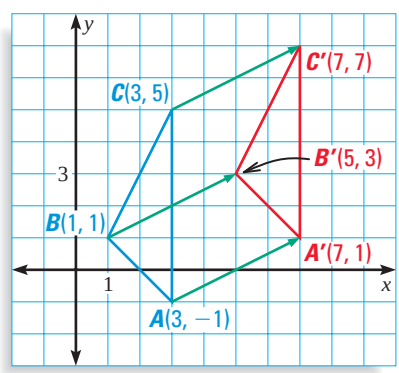
c. The vector is $\overrightarrow{TS} = \langle 3, -3\rangle$.

EXAMPLE 4 *Translation Using Vectors*

The component form of $\overrightarrow{GH}$ is $\langle 4, 2\rangle$. Use $\overrightarrow{GH}$ to translate the triangle whose vertices are $A(3, -1)$, $B(1, 1)$, and $C(3, 5)$.

SOLUTION

First graph $\triangle ABC$. The component form of $\overrightarrow{GH}$ is $\langle 4, 2\rangle$, so the image vertices should all be 4 units to the right and 2 units up from the preimage vertices. Label the image vertices as $A'(7, 1)$, $B'(5, 3)$, and $C'(7, 7)$. Then, using a straightedge, draw $\triangle A'B'C'$. Notice that the vectors drawn from preimage to image vertices are parallel.

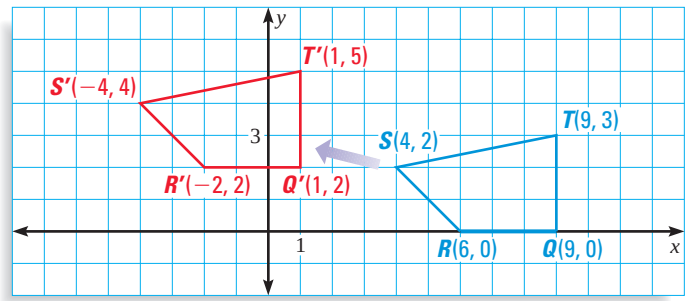


STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

EXAMPLE 5 *Finding Vectors*

In the diagram, $QRST$ maps onto $Q'R'S'T'$ by a translation. Write the component form of the vector that can be used to describe the translation.



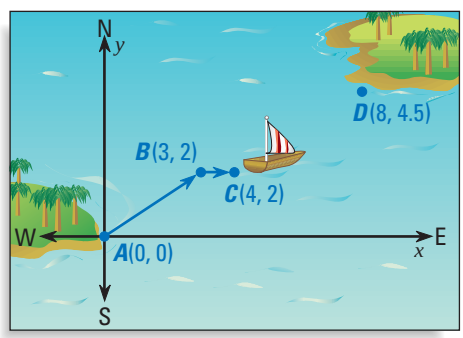
SOLUTION

Choose any vertex and its image, say R and R'. To move from R to R', you move 8 units to the left and 2 units up. The component form of the vector is $\langle -8, 2\rangle$.

✓**CHECK** To check the solution, you can start any where on the preimage and move 8 units to the left and 2 units up. You should end on the corresponding point of the image.

EXAMPLE 6 *Using Vectors*

REAL LIFE **NAVIGATION** A boat travels a straight path between two islands, A and D. When the boat is 3 miles east and 2 miles north of its starting point it encounters a storm at point B. The storm pushes the boat off course to point C, as shown.



a. Write the component forms of the two vectors shown in the diagram.

b. The final destination is 8 miles east and 4.5 miles north of the starting point. Write the component form of the vector that describes the path the boat can follow to arrive at its destination.

SOLUTION

a. The component form of the vector from $A(0, 0)$ to $B(3, 2)$ is

$$\overrightarrow{AB} = \langle 3 - 0, 2 - 0\rangle = \langle 3, 2\rangle.$$

The component form of the vector from $B(3, 2)$ to $C(4, 2)$ is

$$\overrightarrow{BC} = \langle 4 - 3, 2 - 2\rangle = \langle 1, 0\rangle.$$

b. The boat needs to travel from its current position, point C, to the island, point D. To find the component form of the vector from $C(4, 2)$ to $D(8, 4.5)$, subtract the corresponding coordinates:

$$\overrightarrow{CD} = \langle 8 - 4, 4.5 - 2\rangle = \langle 4, 2.5\rangle.$$

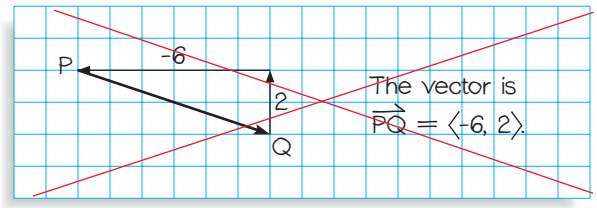
GUIDED PRACTICE

Vocabulary Check ✓

1. A ___?___ is a quantity that has both ___?___ and magnitude.

Concept Check ✓

2. ERROR ANALYSIS Describe Jerome's error.



Skill Check ✓

Use coordinate notation to describe the translation.

3. 6 units to the right and 2 units down

4. 3 units up and 4 units to the right

5. 7 units to the left and 1 unit up

6. 8 units down and 5 units to the left

Complete the statement using the description of the translation. In the description, points (0, 2) and (8, 5) are two vertices of a pentagon.

7. If (0, 2) maps onto (0, 0), then (8, 5) maps onto (___?___, ___?___).

8. If (0, 2) maps onto (___?___, ___?___), then (8, 5) maps onto (3, 7).

9. If (0, 2) maps onto (−3, −5), then (8, 5) maps onto (___?___, ___?___).

10. If (0, 2) maps onto (___?___, ___?___), then (8, 5) maps onto (0, 0).

Draw three vectors that can be described by the given component form.

11. $\langle 3, 5 \rangle$ **12.** $\langle 0, 4 \rangle$ **13.** $\langle -6, 0 \rangle$ **14.** $\langle -5, -1 \rangle$

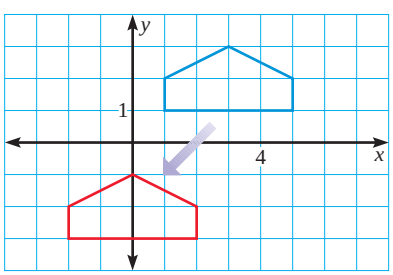
PRACTICE AND APPLICATIONS

STUDENT HELP

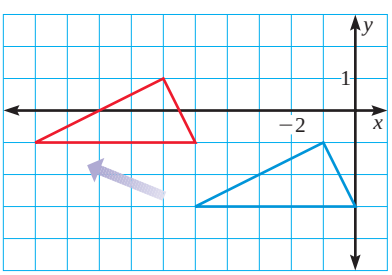
Extra Practice to help you master skills is on p. 816.

DESCRIBING TRANSLATIONS Describe the translation using (a) coordinate notation and (b) a vector in component form.

15.



16.



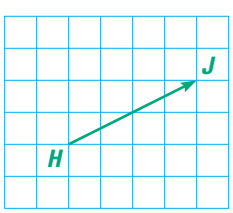
STUDENT HELP

HOMEWORK HELP

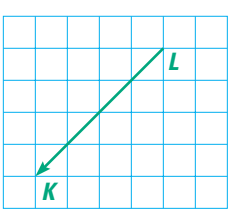
Example 1: Exs. 20–24
Example 2: Exs. 15, 16, 25–34
Example 3: Exs. 15–19
Example 4: Exs. 39–42
Example 5: Exs. 44–47
Example 6: Exs. 53–55

IDENTIFYING VECTORS Name the vector and write its component form.

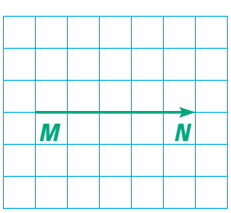
17.



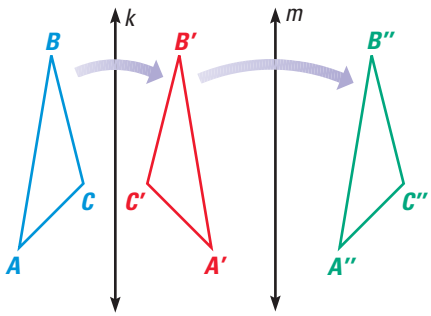
18.



19.



USING THEOREM 7.5 **In the diagram, $k \parallel m$, $\triangle ABC$ is reflected in line k, and $\triangle A'B'C'$ is reflected in line m.**



20. A translation maps $\triangle ABC$ onto which triangle?

21. Which lines are perpendicular to $\overleftrightarrow{AA''}$?

22. Name two segments parallel to $\overline{BB''}$.

23. If the distance between k and m is 1.4 inches, what is the length of $\overline{CC''}$?

24. Is the distance from B' to m the same as the distance from B'' to m? Explain.

IMAGE AND PREIMAGE **Consider the translation that is defined by the coordinate notation $(x, y) \rightarrow (x + 12, y - 7)$.**

25. What is the image of (5, 3)?

26. What is the image of (−1, −2)?

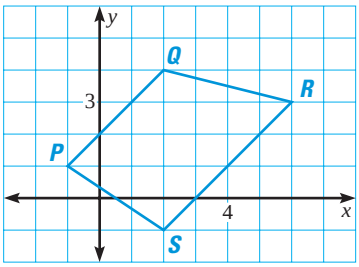
27. What is the preimage of (−2, 1)?

28. What is the preimage of (0, −6)?

29. What is the image of (0.5, 2.5)?

30. What is the preimage of (−5.5, −5.5)?

DRAWING AN IMAGE **Copy figure *PQRS* and draw its image after the translation.**



31. $(x, y) \rightarrow (x + 1, y - 4)$

32. $(x, y) \rightarrow (x - 6, y + 7)$

33. $(x, y) \rightarrow (x + 5, y - 2)$

34. $(x, y) \rightarrow (x - 1, y - 3)$

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 35–38.

LOGICAL REASONING **Use a straightedge and graph paper to help determine whether the statement is true.**

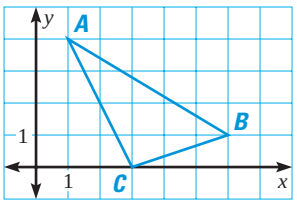
35. If line p is a translation of a different line q, then p is parallel to q.

36. It is possible for a translation to map a line p onto a perpendicular line q.

37. If a translation maps $\triangle ABC$ onto $\triangle DEF$ and a translation maps $\triangle DEF$ onto $\triangle GHK$, then a translation maps $\triangle ABC$ onto $\triangle GHK$.

38. If a translation maps $\triangle ABC$ onto $\triangle DEF$, then $AD = BE = CF$.

TRANSLATING A TRIANGLE **In Exercises 39–42, use a straightedge and graph paper to translate $\triangle ABC$ by the given vector.**



39. $\langle 2, 4 \rangle$

40. $\langle 3, -2 \rangle$

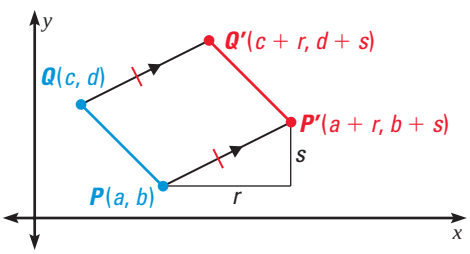
41. $\langle -1, -5 \rangle$

42. $\langle -4, 1 \rangle$

43. **PROOF** Use coordinate geometry and the Distance Formula to write a paragraph proof of Theorem 7.4.

GIVEN ▶ $PP' = QQ'$ and $\overline{PP'} \parallel \overline{QQ'}$

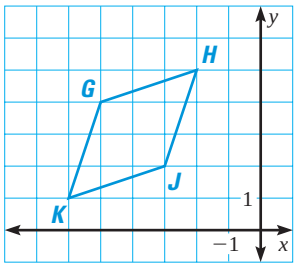
PROVE ▶ $PQ = P'Q'$



VECTORS **The vertices of the image of *GHJK* after a translation are given. Choose the vector that describes the translation.**

A. $\overrightarrow{PQ} = \langle 1, -3 \rangle$ **B.** $\overrightarrow{PQ} = \langle 0, 1 \rangle$

C. $\overrightarrow{PQ} = \langle -1, -3 \rangle$ **D.** $\overrightarrow{PQ} = \langle 6, -1 \rangle$



44. $G'(-6, 1), H'(-3, 2), J'(-4, -1), K'(-7, -2)$

45. $G'(1, 3), H'(4, 4), J'(3, 1), K'(0, 0)$

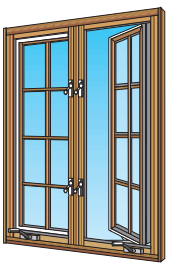
46. $G'(-4, 1), H'(-1, 2), J'(-2, -1), K'(-5, -2)$

47. $G'(-5, 5), H'(-2, 6), J'(-3, 3), K'(-6, 2)$

WINDOW FRAMES **In Exercises 48–50, decide whether "opening the window" is a translation of the moving part.**

48. Double hung

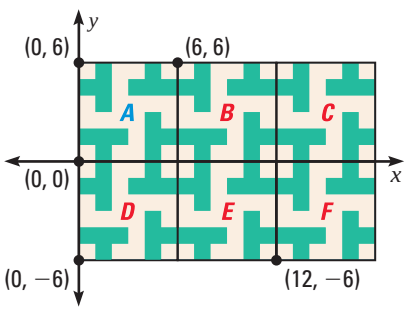
49. Casement



50. Sliding

51. **DATA COLLECTION** Look through some newspapers and magazines to find patterns containing translations.

52. **COMPUTER-AIDED DESIGN** Mosaic floors can be designed on a computer. An example is shown at the right. On the computer, the design in square A is copied to cover an entire floor. The translation $(x, y) \rightarrow (x + 6, y)$ maps square A onto square B. Use coordinate notation to describe the translations that map square A onto squares C, D, E, and F.



FOCUS ON APPLICATIONS

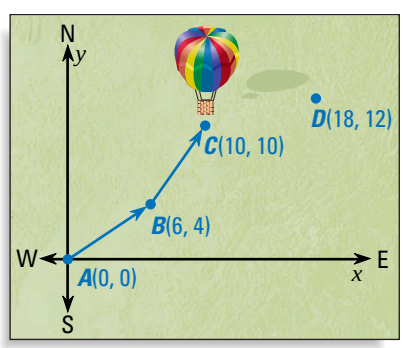
REAL LIFE **HOT-AIR BALLOONS**
Bertrand Piccard and Brian Jones journeyed around the world in their hot-air balloon in 19 days.

NAVIGATION **A hot-air balloon is flying from town *A* to town *D*. After the balloon leaves town *A* and travels 6 miles east and 4 miles north, it runs into some heavy winds at point *B*. The balloon is blown off course as shown in the diagram.**

53. Write the component forms of the two vectors in the diagram.

54. Write the component form of the vector that describes the path the balloon can take to arrive in town D.

55. Suppose the balloon was not blown off course. Write the component form of the vector that describes this journey from town A to town D.



Test Preparation

QUANTITATIVE COMPARISON In Exercises 56–59, choose the statement that is true about the given quantities.

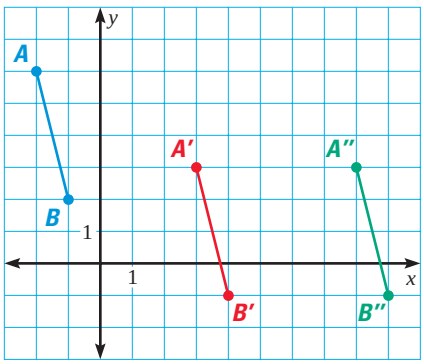
Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

The translation $(x, y) \rightarrow (x + 5, y - 3)$ maps $\overline{AB}$ to $\overline{A'B'}$, and the translation $(x, y) \rightarrow (x + 5, y)$ maps $\overline{A'B'}$ to $\overline{A''B''}$.



	Column A	Column B
56.	AB	$A'B'$
57.	AB	AA'
58.	BB'	$A'A''$
59.	$A'B''$	$A''B'$

★ Challenge

USING ALGEBRA A translation of $\overline{AB}$ is described by $\overrightarrow{PQ}$. Find the value of each variable.

60. $\overrightarrow{PQ} = \langle 4, 1 \rangle$

$A(-1, w), A'(2x + 1, 4)$

$B(8y - 1, 1), B'(3, 3z)$

61. $\overrightarrow{PQ} = \langle 3, -6 \rangle$

$A(r - 1, 8), A'(3, s + 1)$

$B(2t - 2, u), B'(5, -2u)$

EXTRA CHALLENGE
www.mcdougallittell.com

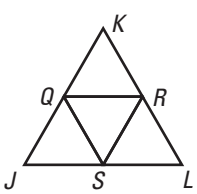
MIXED REVIEW

FINDING SLOPE Find the slope of the line that passes through the given points. (Review 3.6)

62. $A(0, -2), B(-7, -8)$ **63.** $C(2, 3), D(-1, 18)$ **64.** $E(-10, 1), F(-1, 1)$

65. $G(-2, 12), H(-1, 6)$ **66.** $J(-6, 0), K(0, 10)$ **67.** $M(-3, -3), N(9, 6)$

COMPLETING THE STATEMENT In $\triangle JKL$, points Q, R, and S are midpoints of the sides. (Review 5.4)



68. If $JK = 12$, then $SR =$ ___?___.

69. If $QR = 6$, then $JL =$ ___?___.

70. If $RL = 6$, then $QS =$ ___?___.

REFLECTIONS IN A COORDINATE PLANE Decide whether the statement is *true* or *false*. (Review 7.2 for 7.5)

71. If $N(3, 4)$ is reflected in the line $y = -1$, then N' is $(3, -6)$.

72. If $M(-5, 3)$ is reflected in the line $x = -2$, then M' is $(3, 1)$.

73. If $W(4, 3)$ is reflected in the line $y = 2$, then W' is $(1, 4)$.

7.5 Glide Reflections and Compositions

What you should learn

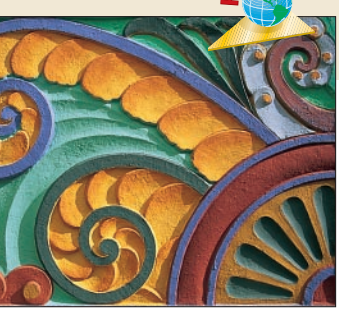
GOAL 1 Identify glide reflections in a plane.

GOAL 2 Represent transformations as compositions of simpler transformations.

Why you should learn it

▼ Compositions of transformations can help when creating patterns in **real life**, such as the decorative pattern below and in **Exs. 35–37**.

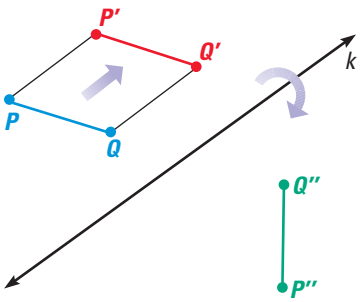
REAL LIFE



GOAL 1 USING GLIDE REFLECTIONS

A translation, or glide, and a reflection can be performed one after the other to produce a transformation known as a *glide reflection*. A **glide reflection** is a transformation in which every point P is mapped onto a point P'' by the following steps:

1. A translation maps P onto P'.
2. A reflection in a line k parallel to the direction of the translation maps P' onto P''.



As long as the line of reflection is parallel to the direction of the translation, it does not matter whether you glide first and then reflect, or reflect first and then glide.

EXAMPLE 1 *Finding the Image of a Glide Reflection*

Use the information below to sketch the image of $\triangle ABC$ after a glide reflection.

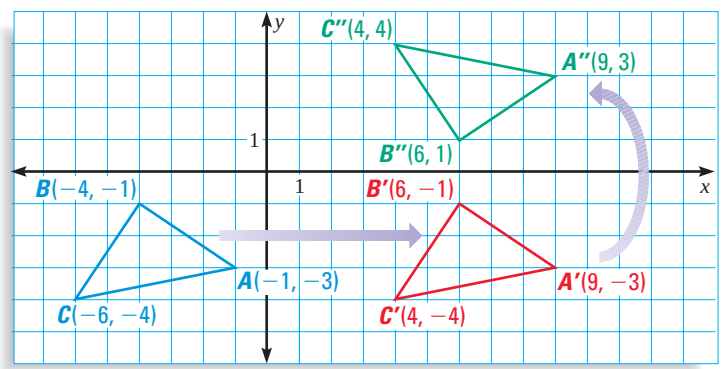
$A(-1, -3)$, $B(-4, -1)$, $C(-6, -4)$

Translation: $(x, y) \rightarrow (x + 10, y)$

Reflection: in the x-axis

SOLUTION

Begin by graphing $\triangle ABC$. Then, shift the triangle 10 units to the right to produce $\triangle A'B'C'$. Finally, reflect the triangle in the x-axis to produce $\triangle A''B''C''$.



In Example 1, try reversing the order of the transformations. Notice that the resulting image will have the same coordinates as $\triangle A''B''C''$ above. This is true because the line of reflection is parallel to the direction of the translation.

GOAL 2 USING COMPOSITIONS

When two or more transformations are combined to produce a single transformation, the result is called a **composition** of the transformations.

THEOREM

THEOREM 7.6 ***Composition Theorem***

The composition of two (or more) isometries is an isometry.

Because a glide reflection is a composition of a translation and a reflection, this theorem implies that glide reflections are isometries. In a glide reflection, the order in which the transformations are performed does not affect the final image. For other compositions of transformations, the order may affect the final image.

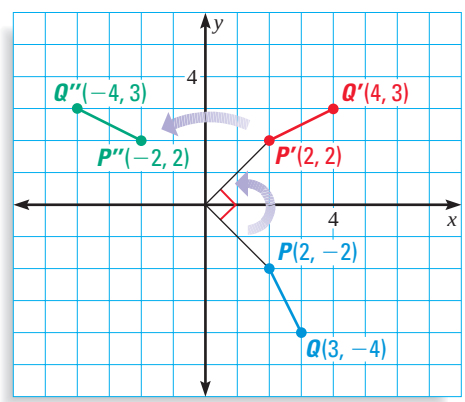
EXAMPLE 2 *Finding the Image of a Composition*

Sketch the image of $\overline{PQ}$ after a composition of the given rotation and reflection.

$P(2, -2)$, $Q(3, -4)$

Rotation: 90° counterclockwise about the origin

Reflection: in the y-axis



SOLUTION

Begin by graphing $\overline{PQ}$. Then rotate the segment 90° counterclockwise about the origin to produce $\overline{P'Q'}$. Finally, reflect the segment in the y-axis to produce $\overline{P''Q''}$.

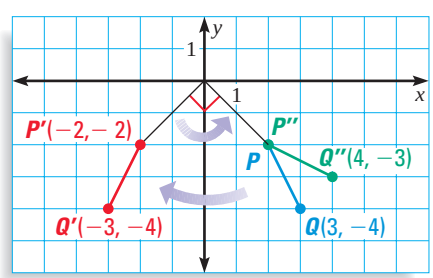
EXAMPLE 3 *Comparing Orders of Compositions*

Repeat Example 2, but switch the order of the composition by performing the reflection first and the rotation second. What do you notice?

SOLUTION

Graph $\overline{PQ}$. Then reflect the segment in the y-axis to obtain $\overline{P'Q'}$. Rotate $\overline{P'Q'}$ 90° counterclockwise about the origin to obtain $\overline{P''Q''}$. Instead of being in Quadrant II, as in Example 2, the image is in Quadrant IV.

▶ The order which the transformations are performed affects the final image.



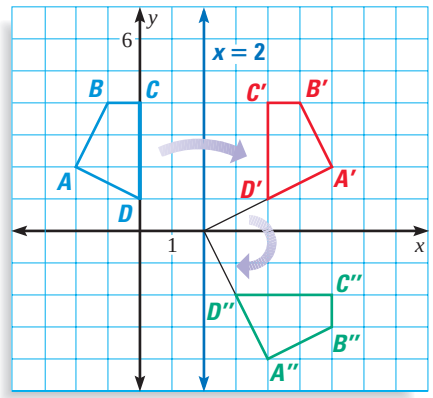
STUDENT HELP

Study Tip

Unlike the addition or multiplication of real numbers, the composition of transformations is not generally commutative.

EXAMPLE 4 *Describing a Composition*

Describe the composition of transformations in the diagram.



SOLUTION

Two transformations are shown. First, figure $ABCD$ is reflected in the line $x = 2$ to produce figure $A'B'C'D'$. Then, figure $A'B'C'D'$ is rotated 90° clockwise about the point (2, 0) to produce figure $A''B''C''D''$.

EXAMPLE 5 *Describing a Composition*

PUZZLES The mathematical game pentominoes is a tiling game that uses twelve different types of tiles, each composed of five squares. The tiles are referred to by the letters they resemble. The object of the game is to pick up and arrange the tiles to create a given shape. Use compositions of transformations to describe how the tiles below will complete the 6 × 5 rectangle.

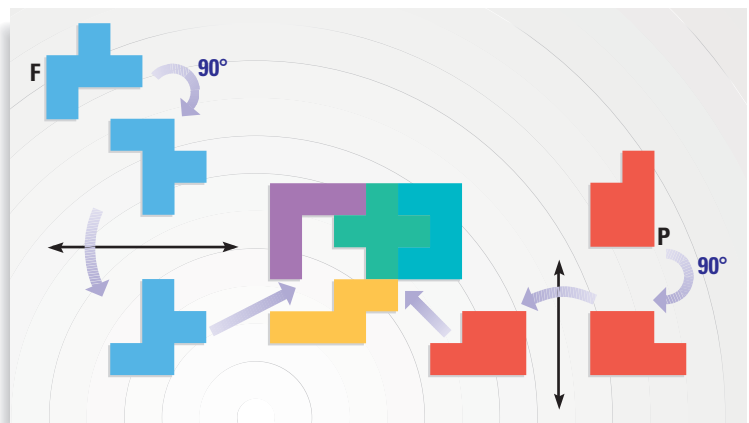
STUDENT HELP

Study Tip
You can make your own pentomino tiles by cutting the shapes out of graph paper.

SOLUTION

To complete part of the rectangle, rotate the F tile 90° clockwise, reflect the tile over a horizontal line, and translate it into place.

To complete the rest of the rectangle, rotate the P tile 90° clockwise, reflect the tile over a vertical line, and translate it into place.



GUIDED PRACTICE

Vocabulary Check ✓

1. In a glide reflection, the direction of the __?__ must be parallel to the line of __?__.

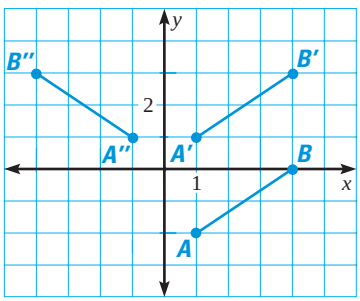
Concept Check ✓

Complete the statement with *always, sometimes,* or *never*.

2. The order in which two transformations are performed __?__ affects the resulting image.
3. In a glide reflection, the order in which the two transformations are performed __?__ matters.
4. A composition of isometries is __?__ an isometry.

Skill Check ✓

In the diagram, $\overline{AB}$ is the preimage of a glide reflection.



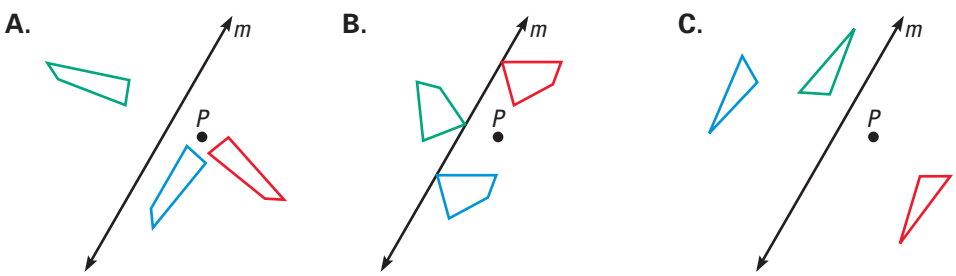
5. Which segment is a translation of $\overline{AB}$?
6. Which segment is a reflection of $\overline{A'B'}$?
7. Name the line of reflection.
8. Use coordinate notation to describe the translation.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 816.

LOGICAL REASONING Match the composition with the diagram, in which the blue figure is the preimage of the red figure and the red figure is the preimage of the green figure.



9. Rotate about point P, then reflect in line m.
10. Reflect in line m, then rotate about point P.
11. Translate parallel to line m, then rotate about point P.

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 9–15
Example 2: Exs. 16–19
Example 3: Exs. 20, 21
Example 4: Exs. 22–25
Example 5: Ex. 38

FINDING AN IMAGE Sketch the image of $A(-3, 5)$ after the described glide reflection.

12. **Translation:** $(x, y) \rightarrow (x, y - 4)$
Reflection: in the y-axis
13. **Translation:** $(x, y) \rightarrow (x + 4, y + 1)$
Reflection: in $y = -2$
14. **Translation:** $(x, y) \rightarrow (x - 6, y - 1)$
Reflection: in $x = -1$
15. **Translation:** $(x, y) \rightarrow (x - 3, y - 3)$
Reflection: in $y = x$

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 16–19.

SKETCHING COMPOSITIONS **Sketch the image of $\triangle PQR$ after a composition using the given transformations in the order they appear.**

16. $P(4, 2)$, $Q(7, 0)$, $R(9, 3)$
Translation: $(x, y) \rightarrow (x - 2, y + 3)$
Rotation: $90°$ clockwise about $T(0, 3)$

17. $P(4, 5)$, $Q(7, 1)$, $R(8, 8)$
Translation: $(x, y) \rightarrow (x, y - 7)$
Reflection: in the y-axis

18. $P(-9, -2)$, $Q(-9, -5)$, $R(-5, -4)$
Translation: $(x, y) \rightarrow (x + 14, y + 1)$
Translation: $(x, y) \rightarrow (x - 3, y + 8)$

19. $P(-7, 2)$, $Q(-6, 7)$, $R(-2, -1)$
Reflection: in the x-axis
Rotation: $90°$ clockwise about origin

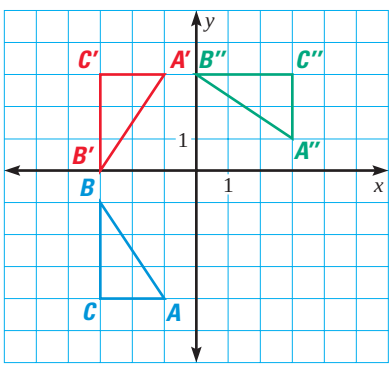
REVERSING ORDERS **Sketch the image of $\overline{FG}$ after a composition using the given transformations in the order they appear. Then, perform the transformations in reverse order. Does the order affect the final image?**

20. $F(4, -4)$, $G(1, -2)$
Rotation: $90°$ clockwise about origin
Reflection: in the y-axis

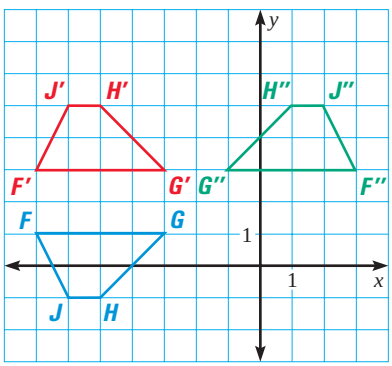
21. $F(-1, -3)$, $G(-4, -2)$
Reflection: in the line $x = 1$
Translation: $(x, y) \rightarrow (x + 2, y + 10)$

DESCRIBING COMPOSITIONS **In Exercises 22–25, describe the composition of the transformations.**

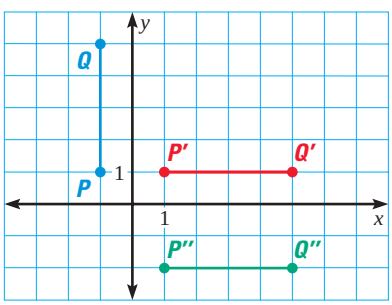
22.



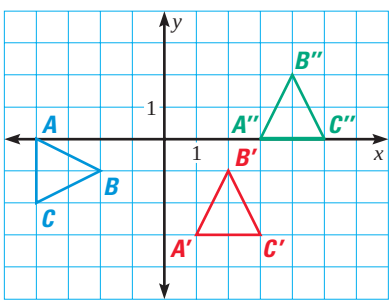
23.



24.



25.



26. *Writing* Explain why a glide reflection is an isometry.

27. **LOGICAL REASONING** Which are preserved by a glide reflection?

A. distance **B.** angle measure **C.** parallel lines

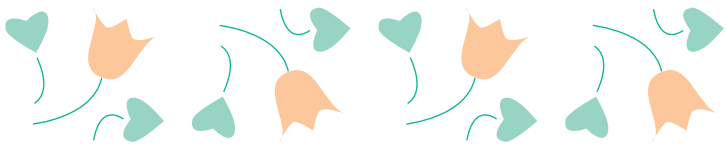
28. **TECHNOLOGY** Use geometry software to draw a polygon. Show that if you reflect the polygon and then translate it in a direction that is *not* parallel to the line of reflection, then the final image is *different* from the final image if you perform the translation first and the reflection second.

CRITICAL THINKING **In Exercises 29 and 30, the first translation maps J to J' and the second maps J' to J''. Find the translation that maps J to J''.**

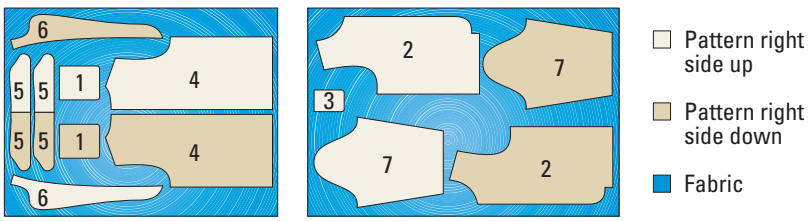
29. **Translation 1:** $(x, y) \rightarrow (x + 7, y - 2)$
Translation 2: $(x, y) \rightarrow (x - 1, y + 3)$
Translation: $(x, y) \rightarrow (\_\_?\_\_, \_\_?\_\_)$

30. **Translation 1:** $(x, y) \rightarrow (x + 9, y + 4)$
Translation 2: $(x, y) \rightarrow (x + 6, y - 4)$
Translation: $(x, y) \rightarrow (\_\_?\_\_, \_\_?\_\_)$

31. **STENCILING A BORDER** The border pattern below was made with a stencil. Describe how the border was created using one stencil four times.



CLOTHING PATTERNS **The diagram shows the pattern pieces for a jacket arranged on some blue fabric.**



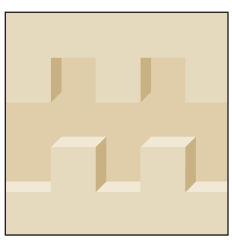
32. Which pattern pieces are translated?

33. Which pattern pieces are reflected?

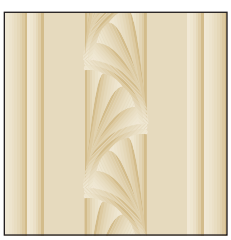
34. Which pattern pieces are glide reflected?

ARCHITECTURE **In Exercises 35–37, describe the transformations that are combined to create the pattern in the architectural element.**

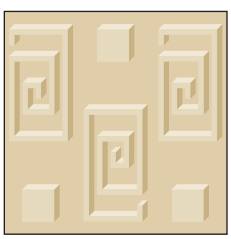
35.



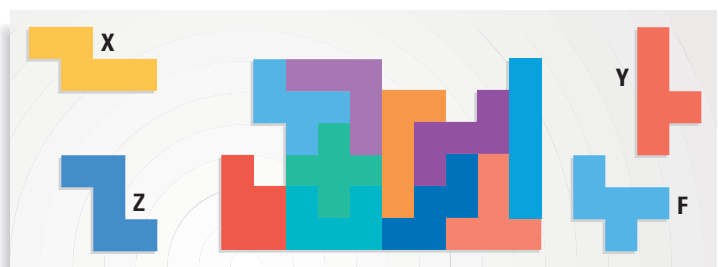
36.



37.



38. **PENTOMINOES** Use compositions of transformations to describe how to pick up and arrange the tiles to complete the 6×10 rectangle.



FOCUS ON CAREERS

ARCHITECTURAL HISTORIANS play an important role in the renovation of an old building. Their expertise is used to help restore a building to its original splendor.

CAREER LINK
www.mcdougallittell.com

Test Preparation

39. MULTI-STEP PROBLEM Follow the steps below.

a. On a coordinate plane, draw a point and its image after a glide reflection that uses the x-axis as the line of reflection.

b. Connect the point and its image. Make a conjecture about the midpoint of the segment.

c. Use the coordinates from part (a) to prove your conjecture.

d. CRITICAL THINKING Can you extend your conjecture to include glide reflections that do not use the x-axis as the line of reflection?

★ Challenge

40. USING ALGEBRA Solve for the variables in the glide reflection of $\triangle JKL$ described below.

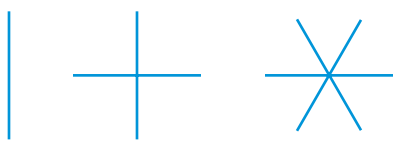
$J(-2, -1)$	Translate $(x, y) \rightarrow (x + 3, y)$	$J'(c + 1, -1)$	Reflect in x-axis	$J''(1, -f)$
$K(-4, 2a)$		$K'(5d - 11, 4)$		$K''(-1, 3g + 5)$
$L(b - 6, 6)$		$L'(2, 4e)$		$L''(h + 4, -6)$

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

ANALYZING PATTERNS Sketch the next figure in the pattern. (Review 1.1 for 7.6)

41.



42.

43.

44.

COORDINATE GEOMETRY In Exercises 45–47, decide whether ▱*PQRS* is a *rhombus*, a *rectangle*, or a *square*. Explain your reasoning. (Review 6.4)

45. $P(1, -2)$, $Q(5, -1)$, $R(6, -5)$, $S(2, -6)$

46. $P(10, 7)$, $Q(15, 7)$, $R(15, 1)$, $S(10, 1)$

47. $P(8, -4)$, $Q(10, -7)$, $R(8, -10)$, $S(6, -7)$

48. ROTATIONS A segment has endpoints $(3, -8)$ and $(7, -1)$. If the segment is rotated 90° counterclockwise about the origin, what are the endpoints of its image? (Review 7.3)

STUDYING TRANSLATIONS Sketch $\triangle ABC$ with vertices $A(-9, 7)$, $B(-9, 1)$, and $C(-5, 6)$. Then translate the triangle by the given vector and name the vertices of the image. (Review 7.4)

49. $\langle 3, 2\rangle$ **50.** $\langle -1, -5\rangle$ **51.** $\langle 6, 0\rangle$

52. $\langle -4, -4\rangle$ **53.** $\langle 0, 2.5\rangle$ **54.** $\langle 1.5, -4.5\rangle$

7.6 Frieze Patterns

What you should learn

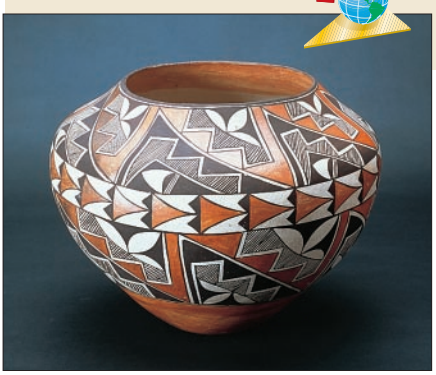
GOAL 1 Use transformations to classify frieze patterns.

GOAL 2 Use frieze patterns to design border patterns in **real life**, such as the tiling pattern in **Example 4**.

Why you should learn it

▼ You can use frieze patterns to create decorative borders for **real-life** objects, such as the pottery below and the pottery in **Exs. 35–37**.

REAL LIFE



GOAL 1 CLASSIFYING FRIEZE PATTERNS

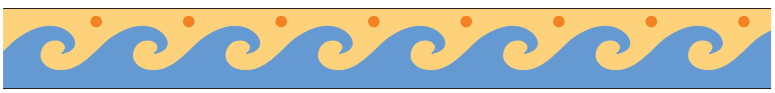
A **frieze pattern** or **border pattern** is a pattern that extends to the left and right in such a way that the pattern can be mapped onto itself by a horizontal translation. In addition to being mapped onto itself by a horizontal translation, some frieze patterns can be mapped onto themselves by other transformations.

1. Translation	T
2. 180° rotation	R
3. Reflection in a horizontal line	H
4. Reflection in a vertical line	V
5. Horizontal glide reflection	G

EXAMPLE 1 *Describing Frieze Patterns*

Describe the transformations that will map each frieze pattern onto itself.

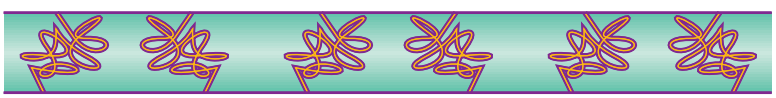
a.



b.

c.

d.



SOLUTION

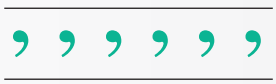
a. This frieze pattern can be mapped onto itself by a horizontal translation (T).

b. This frieze pattern can be mapped onto itself by a horizontal translation (T) or by a 180° rotation (R).

c. This frieze pattern can be mapped onto itself by a horizontal translation (T) or by a horizontal glide reflection (G).

d. This frieze pattern can be mapped onto itself by a horizontal translation (T) or by a reflection in a vertical line (V).

CONCEPT SUMMARY **CLASSIFICATIONS OF FRIEZE PATTERNS**

T	**Translation**	
TR	**Translation and 180° rotation**	
TG	**Translation and horizontal glide reflection**	
TV	**Translation and vertical line reflection**	
THG	**Translation, horizontal line reflection, and horizontal glide reflection**	
TRVG	**Translation, 180° rotation, vertical line reflection, and horizontal glide reflection**	
TRHVG	**Translation, 180° rotation, horizontal line reflection, vertical line reflection, and horizontal glide reflection**	

STUDENT HELP

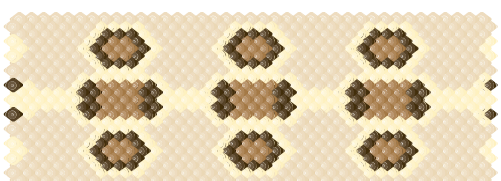
Study Tip
To help classify a frieze pattern, you can use a process of elimination. This process is described at the right and in the tree diagram in **Ex. 53**.

To classify a frieze pattern into one of the seven categories, you first decide whether the pattern has 180° rotation. If it does, then there are three possible classifications: TR, TRVG, and TRHVG.

If the frieze pattern does not have 180° rotation, then there are four possible classifications: T, TV, TG, and THG. Decide whether the pattern has a line of reflection. By a process of elimination, you will reach the correct classification.

EXAMPLE 2 *Classifying a Frieze Pattern*

SNAKES Categorize the snakeskin pattern of the mountain adder.



SOLUTION

This pattern is a TRHVG. The pattern can be mapped onto itself by a translation, a 180° rotation, a reflection in a horizontal line, a reflection in a vertical line, and a horizontal glide reflection.

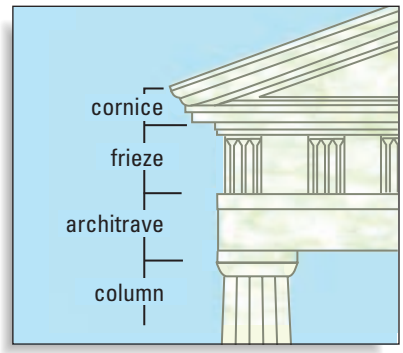
FOCUS ON APPLICATIONS

ARCHITECTURE Features of classical architecture from Greece and Rome are seen in "neo-classical" buildings today, such as the Supreme Court building shown.

GOAL 2 USING FRIEZE PATTERNS IN REAL LIFE

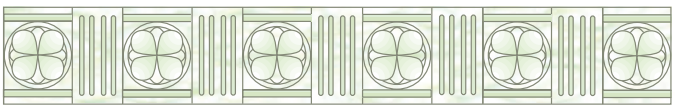
EXAMPLE 3 *Identifying Frieze Patterns*

ARCHITECTURE The frieze patterns of ancient Doric buildings are located between the cornice and the architrave, as shown at the right. The frieze patterns consist of alternating sections. Some sections contain a person or a symmetric design. Other sections have simple patterns of three or four vertical lines.



Portions of two frieze patterns are shown below. Classify the patterns.

a.



b.

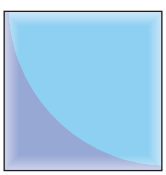
SOLUTION

a. Following the diagrams on the previous page, you can see that this frieze pattern has rotational symmetry, line symmetry about a horizontal line and a vertical line, and that the pattern can be mapped onto itself by a glide reflection. So, the pattern can be classified as TRHVG.

b. The only transformation that maps this pattern onto itself is a translation. So, the pattern can be classified as T.

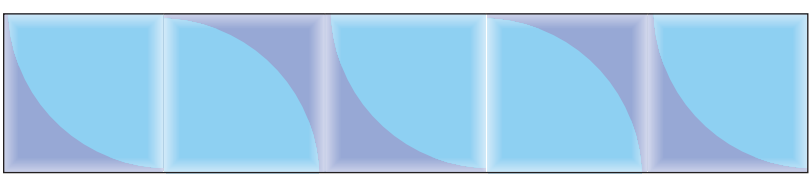
EXAMPLE 4 *Drawing a Frieze Pattern*

TILING A border on a bathroom wall is created using the decorative tile at the right. The border pattern is classified as TR. Draw one such pattern.



SOLUTION

Begin by rotating the given tile 180°. Use this tile and the original tile to create a pattern that has rotational symmetry. Then translate the pattern several times to create the frieze pattern.



GUIDED PRACTICE

Vocabulary Check ✓

1. Describe the term *frieze pattern* in your own words.

Concept Check ✓

2. **ERROR ANALYSIS** Describe Lucy's error below.

Skill Check ✓

In Exercises 3–6, describe the transformations that map the frieze pattern onto itself.

3. 

4. 

5. 

6. 

7. List the five possible transformations, along with their letter abbreviations, that can be found in a frieze pattern.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 816.

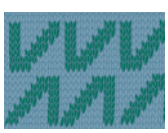
SWEATER PATTERN **Each row of the sweater is a frieze pattern. Match the row with its classification.**

A. TRHVG **B.** TR **C.** TRVG **D.** THG

8. 

9. 

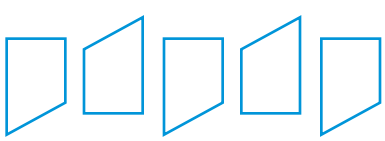
10. 

11. 

CLASSIFYING PATTERNS **Name the isometries that map the frieze pattern onto itself.**

12. 

13. 

14. 

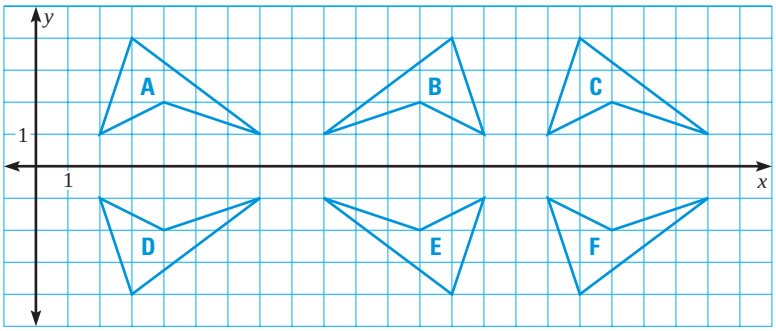
15. 

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 8–15
Example 2: Exs. 16–23
Example 3: Exs. 32–39
Example 4: Exs. 40–43

DESCRIBING TRANSFORMATIONS **Use the diagram of the frieze pattern.**



16. Is there a reflection in a vertical line? If so, describe the reflection(s).

17. Is there a reflection in a horizontal line? If so, describe the reflection(s).

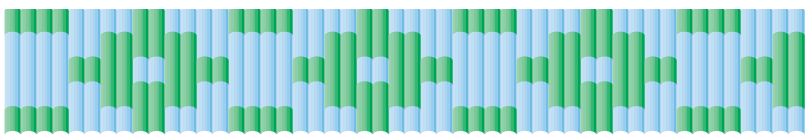
18. Name and describe the transformation that maps A onto F.

19. Name and describe the transformation that maps D onto B.

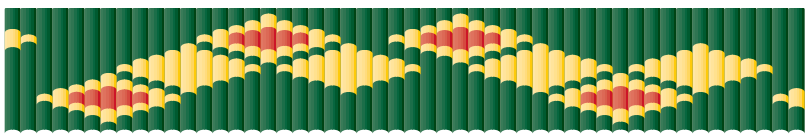
20. Classify the frieze pattern.

PET COLLARS **In Exercises 21–23, use the chart on page 438 to classify the frieze pattern on the pet collars.**

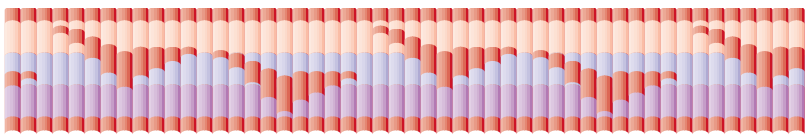
21.



22.



23.



24. **TECHNOLOGY** Pick one of the seven classifications of patterns and use geometry software to create a frieze pattern of that classification. Print and color your frieze pattern.

25. **DATA COLLECTION** Use a library, magazines, or some other reference source to find examples of frieze patterns. How many of the seven classifications of patterns can you find?

CREATING A FRIEZE PATTERN **Use the design below to create a frieze pattern with the given classification.**

26. TR

27. TV

28. TG

29. THG

30. TRVG

31. TRHVG

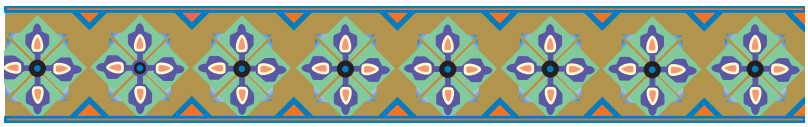
FOCUS ON APPLICATIONS

NIKKO MEMORIAL
The building shown is a memorial to Tokugawa Ieyasu (1543–1616), the founder of the Tokugawa Shogunate.

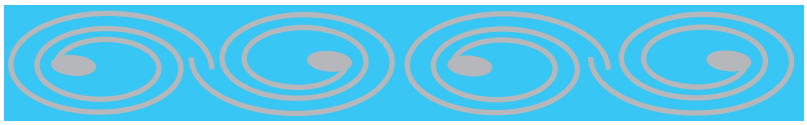
JAPANESE PATTERNS **The patterns shown were used in Japan during the Tokugawa Shogunate. Classify the frieze patterns.**

32.

33.



34.



POTTERY **In Exercises 35–37, use the pottery shown below. This pottery was created by the Acoma Indians. The Acoma pueblo is America's oldest continually inhabited city.**

35. Identify any frieze patterns on the pottery.

36. Classify the frieze pattern(s) you found in Exercise 35.

37. Create your own frieze pattern similar to the patterns shown on the pottery.

38. Look back to the southwestern pottery on page 437. Describe and classify one of the frieze patterns on the pottery.

39. **LOGICAL REASONING** You are decorating a large circular vase and decide to place a frieze pattern around its base. You want the pattern to consist of ten repetitions of a design. If the diameter of the base is about 9.5 inches, how wide should each design be?

TILING **In Exercises 40–42, use the tile to create a border pattern with the given classification. Your border should consist of one row of tiles.**

40. TR

41. TRVG

42. TRHVG

43. *Writing* Explain how the design of the tiles in Exercises 40–42 is a factor in the classification of the patterns. For instance, could the tile in Exercise 40 be used to create a single row of tiles classified as THG?

CRITICAL THINKING **Explain why the combination is not a category for frieze pattern classification.**

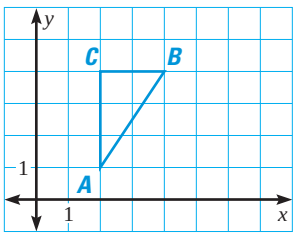
44. TVG **45.** THV **46.** TRG

STUDENT HELP

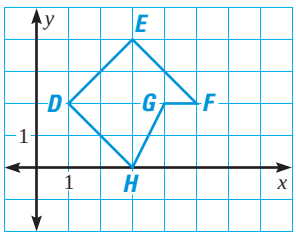
INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 47 and 48.

USING THE COORDINATE PLANE **The figure shown in the coordinate plane is part of a frieze pattern with the given classification. Copy the graph and draw the figures needed to complete the pattern.**

47. TR



48. TRVG

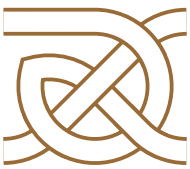


Test Preparation

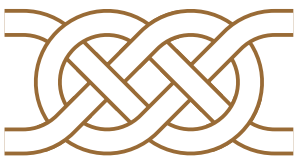
MULTI-STEP PROBLEM **In Exercises 49–52, use the following information.**

In Celtic art and design, border patterns are used quite frequently, especially in jewelry. Three different designs are shown.

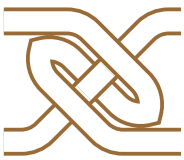
A.



B.



C.



49. Use translations to create a frieze pattern of each design.

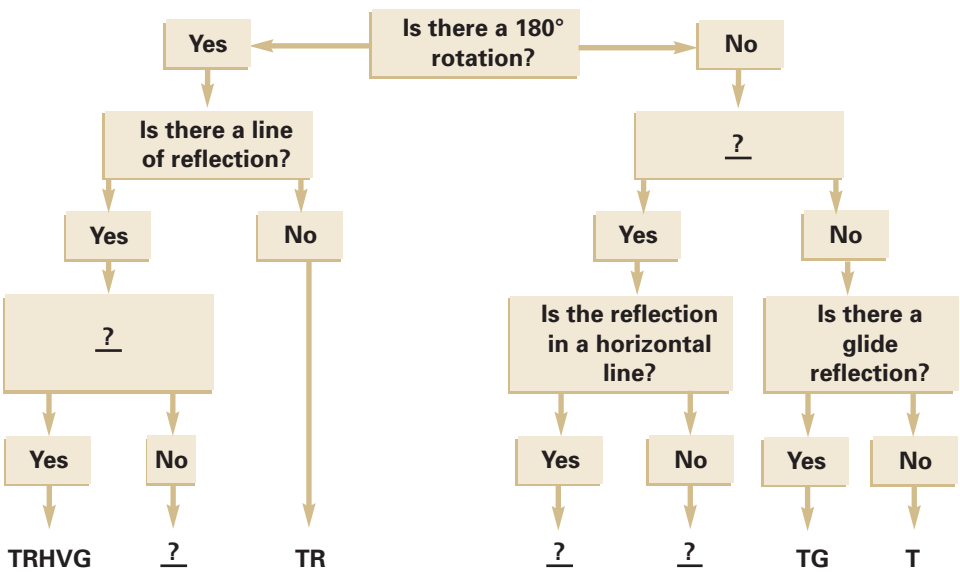
50. Classify each frieze pattern that you created.

51. Which design does not have rotational symmetry? Use rotations to create a new frieze pattern of this design.

52. *Writing* If a design has 180° rotational symmetry, it cannot be used to create a frieze pattern with classification *T*. Explain why not.

★ Challenge

53. **TREE DIAGRAM** The following tree diagram can help classify frieze patterns. Copy the tree diagram and fill in the missing parts.



Mixed Review

Ratios **Find the ratio of girls to boys in a class, given the number of boys and the total number of students.** (Skills Review for 8.1)

54. 12 boys, 23 students

55. 8 boys, 21 students

56. 3 boys, 13 students

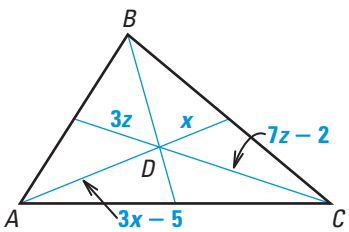
57. 19 boys, 35 students

58. 11 boys, 18 students

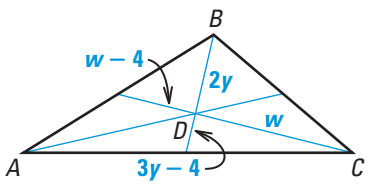
59. 10 boys, 20 students

Properties of Medians **Given that *D* is the centroid of $\triangle ABC$, find the value of each variable.** (Review 5.3)

60.

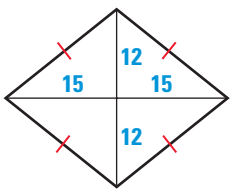


61.

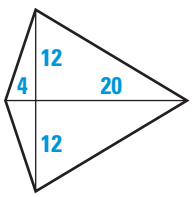


Finding Area **Find the area of the quadrilateral.** (Review 6.7)

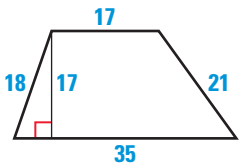
62.



63.



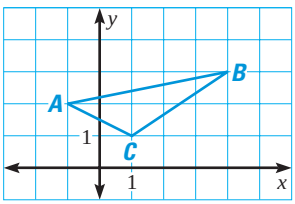
64.



Quiz 2

Self-Test for Lessons 7.4–7.6

Write the coordinates of the vertices *A′*, *B′*, and *C′* after $\triangle ABC$ is translated by the given vector. (Lesson 7.4)



1. $\langle 1, 3\rangle$

2. $\langle -3, 4\rangle$

3. $\langle -2, -4\rangle$

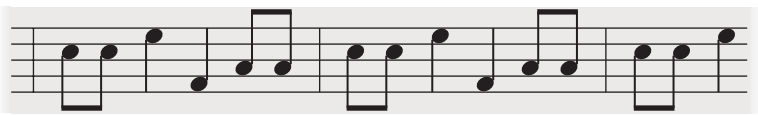
4. $\langle 5, 2\rangle$

In Exercises 5 and 6, sketch the image of $\triangle PQR$ after a composition using the given transformations in the order they appear. (Lesson 7.5)

5. $P(5, 1)$, $Q(3, 4)$, $R(0, 1)$
Translation: $(x, y) \rightarrow (x - 2, y - 4)$
Reflection: in the y-axis

6. $P(7, 2)$, $Q(3, 1)$, $R(6, -1)$
Translation: $(x, y) \rightarrow (x - 4, y + 3)$
Rotation: 90° clockwise about origin

7. **Musical Notes** Do the notes shown form a frieze pattern? If so, classify the frieze pattern. (Lesson 7.6)



CHAPTER 7

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Identify types of rigid transformations. (7.1)	Plan a stencil pattern, using one design repeated many times. (p. 401)
Use properties of reflections. (7.2)	Choose the location of a telephone pole so that the length of the cable is a minimum. (p. 405)
Relate reflections and line symmetry. (7.2)	Understand the construction of the mirrors in a kaleidoscope. (p. 406)
Relate rotations and rotational symmetry. (7.3)	Use rotational symmetry to design a logo. (p. 415)
Use properties of translations. (7.4)	Use vectors to describe the path of a hot-air balloon. (p. 427)
Use properties of glide reflections. (7.5)	Describe the transformations in patterns in architecture. (p. 435)
Classify frieze patterns. (7.6)	Identify the frieze patterns in pottery. (p. 442)

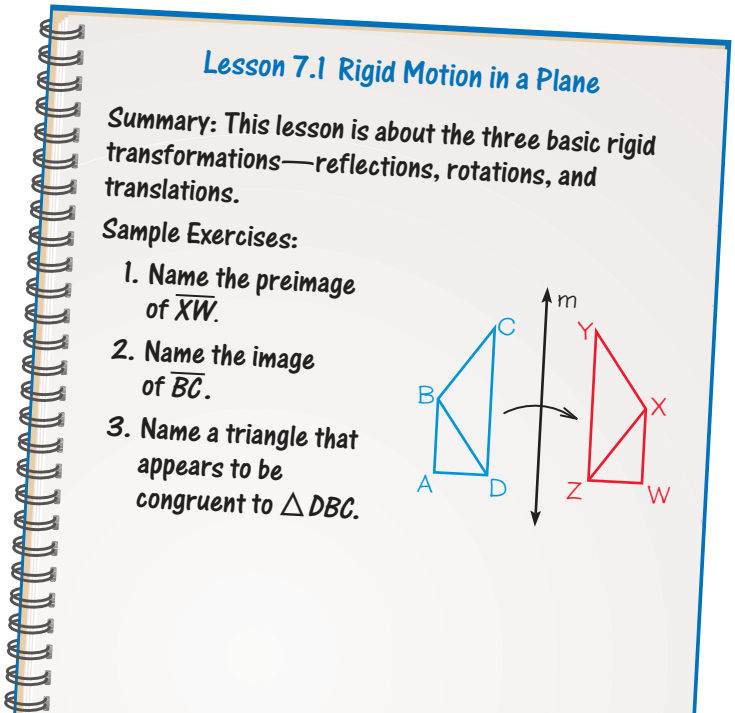
How does Chapter 7 fit into the BIGGER PICTURE of geometry?

In this chapter, you learned that the basic rigid transformations in the plane are reflections, rotations, translations, and glide reflections. Rigid transformations are closely connected to the concept of congruence. That is, two plane figures are congruent if and only if one can be mapped onto the other by exactly one rigid transformation or by a composition of rigid transformations. In the next chapter, you will study transformations that are not rigid. You will learn that some nonrigid transformations are closely connected to the concept of similarity.

STUDY STRATEGY

How did making sample exercises help you?

Some sample exercises you made, following the **Study Strategy** on p. 394, may resemble these.



Lesson 7.1 Rigid Motion in a Plane

Summary: This lesson is about the three basic rigid transformations—reflections, rotations, and translations.

Sample Exercises:

1. Name the preimage of $\overline{XW}$.
2. Name the image of $\overline{BC}$.
3. Name a triangle that appears to be congruent to $\triangle DBC$.

CHAPTER 7

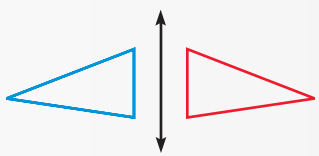
Chapter Review

- image, p. 396
- preimage, p. 396
- transformation, p. 396
- isometry, p. 397
- reflection, p. 404
- line of reflection, p. 404
- line of symmetry, p. 406
- rotation, p. 412
- center of rotation, p. 412
- angle of rotation, p. 412
- rotational symmetry, p. 415
- translation, p. 421
- vector, p. 423
- initial point, p. 423
- terminal point, p. 423
- component form, p. 423
- glide reflection, p. 430
- composition, p. 431
- frieze pattern, or border pattern, p. 437

7.1 Rigid Motion in a Plane

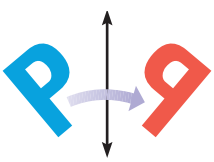
Examples on pp. 396–398

EXAMPLE The blue triangle is reflected to produce the congruent red triangle, so the transformation is an isometry.

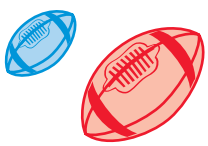


Does the transformation appear to be an isometry? Explain.

1.



2.

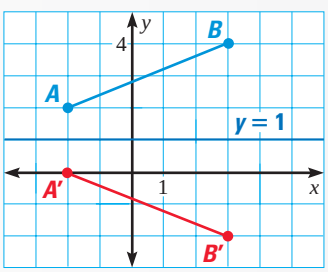


3.

7.2 Reflections

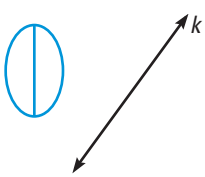
Examples on pp. 404–406

EXAMPLE In the diagram, $\overline{AB}$ is reflected in the line $y = 1$, so $\overline{A'B'}$ has endpoints $A'(-2, 0)$ and $B'(3, -2)$.

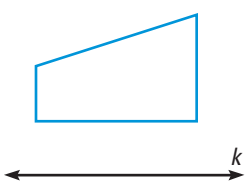


Copy the figure and draw its reflection in line *k*.

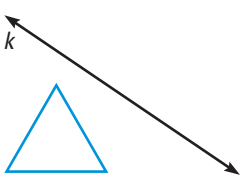
4.



5.



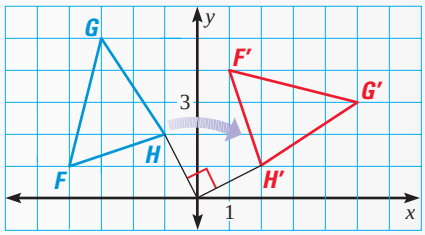
6.



7.3 ROTATIONS

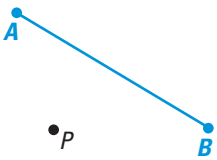
Examples on pp. 412–415

EXAMPLE In the diagram, $\triangle FGH$ is rotated 90° clockwise about the origin.

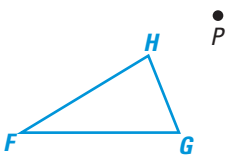


Copy the figure and point *P*. Then, use a straightedge, a compass, and a protractor to rotate the figure 60° counterclockwise about *P*.

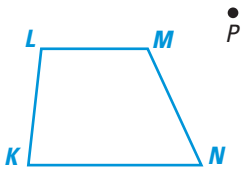
7.



8.



9.

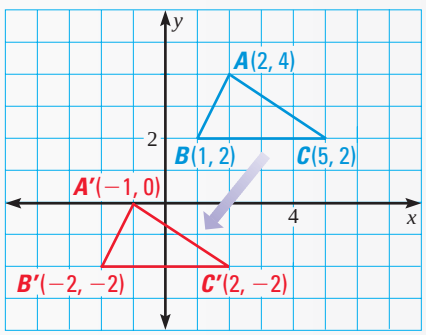


7.4 TRANSLATIONS AND VECTORS

Examples on pp. 421–424

EXAMPLE Using the vector $\langle -3, -4\rangle$, $\triangle ABC$ can be translated to $\triangle A'B'C'$.

$A(2, 4)$	$A'(-1, 0)$
$B(1, 2)$	$B'(-2, -2)$
$C(5, 2)$	$C'(2, -2)$



The vertices of the image of $\triangle LMN$ after a translation are given. Choose the vector that describes the translation.

10. $L'(-1, -3), M'(4, -2), N'(6, 2)$

11. $L'(-5, 1), M'(0, 2), N'(2, 6)$

12. $L'(-3, 2), M'(2, 3), N'(4, 7)$

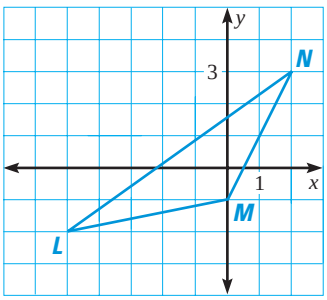
13. $L'(-7, 3), M'(-2, 4), N'(0, 8)$

A. $\overrightarrow{PQ} = \langle 0, 3\rangle$

B. $\overrightarrow{PQ} = \langle -2, 5\rangle$

C. $\overrightarrow{PQ} = \langle 4, -1\rangle$

D. $\overrightarrow{PQ} = \langle 2, 4\rangle$



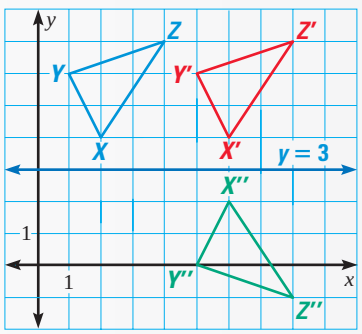
7.5 GLIDE REFLECTIONS AND COMPOSITIONS

Examples on pp. 430–432

EXAMPLE The diagram shows the image of $\triangle XYZ$ after a glide reflection.

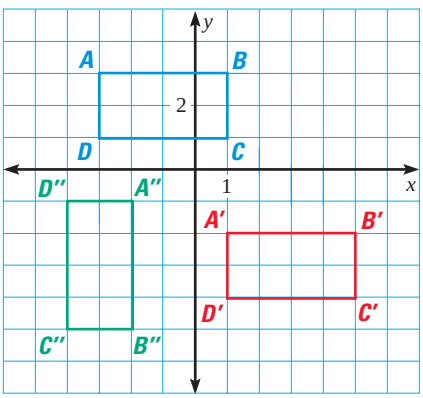
Translation: $(x, y) \rightarrow (x + 4, y)$

Reflection: in the line $y = 3$

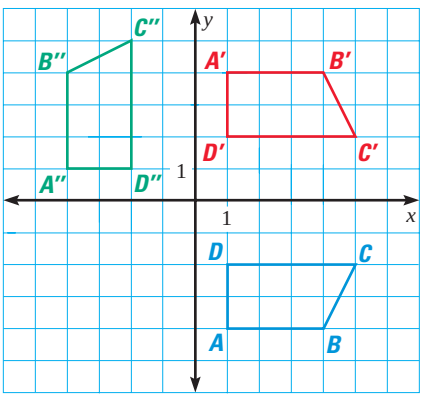


Describe the composition of the transformations.

14.



15.



7.6 FRIEZE PATTERNS

Examples on pp. 437–439

EXAMPLE The corn snake frieze pattern at the right can be classified as TRHVG because the pattern can be mapped onto itself by a translation, 180° rotation, horizontal line reflection, vertical line reflection, and glide reflection.

Classify the snakeskin frieze pattern.

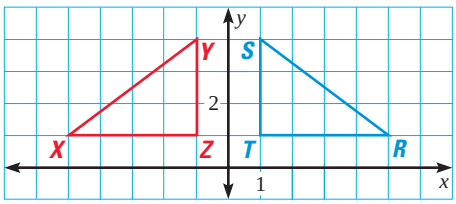
16. Rainbow boa

17. Gray-banded kingsnake

CHAPTER 7

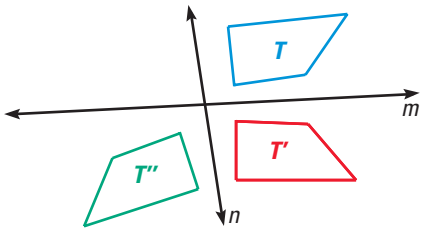
Chapter Test

In Exercises 1–4, use the diagram.



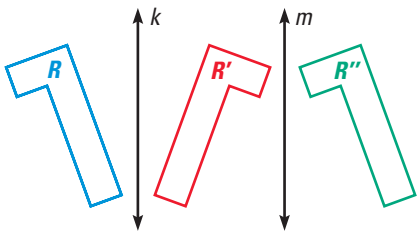
1. Identify the transformation $\triangle RST \rightarrow \triangle XYZ$.
2. Is $\overline{RT}$ congruent to $\overline{XZ}$?
3. What is the image of T?
4. What is the preimage of Y?
5. Sketch a polygon that has line symmetry, but not rotational symmetry.
6. Sketch a polygon that has rotational symmetry, but not line symmetry.

Use the diagram, in which lines *m* and *n* are lines of reflection.



7. Identify the transformation that maps figure T onto figure T'.
8. Identify the transformation that maps figure T onto figure T''.
9. If the measure of the acute angle between m and n is 85°, what is the angle of rotation from figure T to figure T''?

In Exercises 10–12, use the diagram, in which $k \parallel m$.



10. Identify the transformation that maps figure R onto figure R'.
11. Identify the transformation that maps figure R onto figure R''.
12. If the distance between k and m is 5 units, what is the distance between corresponding parts of figure R and figure R''?
13. What type of transformation is a composition of a translation followed by a reflection in a line parallel to the translation vector?

Give an example of the described composition of transformations.

14. The order in which two transformations are performed affects the final image.
15. The order in which two transformations are performed does not affect the final image.

FLAGS Identify any symmetry in the flag.

16. Switzerland

17. Jamaica

18. United Kingdom

Name all of the isometries that map the frieze pattern onto itself.

19.

20.

21.

8.1 Ratio and Proportion

What you should learn

GOAL 1 Find and simplify the ratio of two numbers.

GOAL 2 Use proportions to solve **real-life** problems, such as computing the width of a painting in **Example 6**.

Why you should learn it

▼ To solve **real-life** problems, such as using a scale model to determine the dimensions of a sculpture like the baseball glove below and the baseball bat in **Exs. 51–53**.

REAL LIFE

GOAL 1 COMPUTING RATIOS

If a and b are two quantities that are measured in the *same* units, then the **ratio of a to b** is $\frac{a}{b}$. The ratio of a to b can also be written as $a:b$. Because a ratio is a quotient, its denominator cannot be zero.

Ratios are usually expressed in simplified form. For instance, the ratio of $6:8$ is usually simplified as $3:4$.

EXAMPLE 1 *Simplifying Ratios*

Simplify the ratios.

a. $\frac{12 \text{ cm}}{4 \text{ m}}$ **b.** $\frac{6 \text{ ft}}{18 \text{ in.}}$

SOLUTION

To simplify ratios with unlike units, convert to like units so that the units divide out. Then simplify the fraction, if possible.

a. $\frac{12 \text{ cm}}{4 \text{ m}} = \frac{12 \text{ cm}}{4 \cdot 100 \text{ cm}} = \frac{12}{400} = \frac{3}{100}$

b. $\frac{6 \text{ ft}}{18 \text{ in.}} = \frac{6 \cdot 12 \text{ in.}}{18 \text{ in.}} = \frac{72}{18} = \frac{4}{1}$

ACTIVITY Developing Concepts

Investigating Ratios

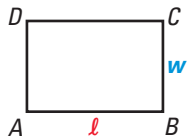
1. Use a tape measure to measure the circumference of the base of your thumb, the circumference of your wrist, and the circumference of your neck. Record the results in a table.
2. Compute the ratio of your wrist measurement to your thumb measurement. Then, compute the ratio of your neck measurement to your wrist measurement.
3. Compare the two ratios.
4. Compare your ratios to those of others in the class.
5. Does it matter whether you record your measurements all in inches or all in centimeters? Explain.

STUDENT HELP

Look Back
For help with perimeter, see p. 51.

EXAMPLE 2 *Using Ratios*

The perimeter of rectangle $ABCD$ is 60 centimeters. The ratio of $AB:BC$ is $3:2$. Find the length and width of the rectangle.



SOLUTION

Because the ratio of $AB:BC$ is $3:2$, you can represent the length AB as $3x$ and the width BC as $2x$.

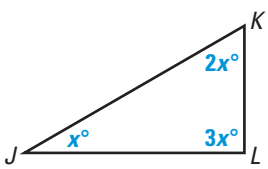
$2\ell + 2w = P$	**Formula for perimeter of rectangle**
$2(3x) + 2(2x) = 60$	**Substitute for ℓ, w, and P.**
$6x + 4x = 60$	**Multiply.**
$10x = 60$	**Combine like terms.**
$x = 6$	**Divide each side by 10.**

▶ So, $ABCD$ has a length of 18 centimeters and a width of 12 centimeters.

Using Algebra

EXAMPLE 3 *Using Extended Ratios*

The measure of the angles in $\triangle JKL$ are in the *extended ratio* of $1:2:3$. Find the measures of the angles.



SOLUTION

Begin by sketching a triangle. Then use the extended ratio of $1:2:3$ to label the measures of the angles as $x°$, $2x°$, and $3x°$.

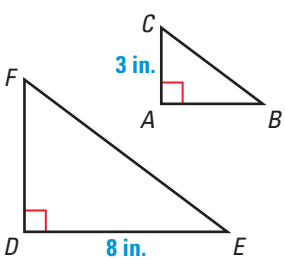
$x° + 2x° + 3x° = 180°$	**Triangle Sum Theorem**
$6x = 180$	**Combine like terms.**
$x = 30$	**Divide each side by 6.**

▶ So, the angle measures are $30°$, $2(30°) = 60°$, and $3(30°) = 90°$.

Logical Reasoning

EXAMPLE 4 *Using Ratios*

The ratios of the side lengths of $\triangle DEF$ to the corresponding side lengths of $\triangle ABC$ are $2:1$. Find the unknown lengths.



SOLUTION

- DE is twice AB and $DE = 8$, so $AB = \frac{1}{2}(8) = 4$.
- Using the Pythagorean Theorem, you can determine that $BC = 5$.
- DF is twice AC and $AC = 3$, so $DF = 2(3) = 6$.
- EF is twice BC and $BC = 5$, so $EF = 2(5) = 10$.

STUDENT HELP

Look Back
For help with the Pythagorean Theorem, see p. 20.

GOAL 2 USING PROPORTIONS

An equation that equates two ratios is a **proportion**. For instance, if the ratio $\frac{a}{b}$ is equal to the ratio $\frac{c}{d}$, then the following proportion can be written:

Means — Extremes

$$\frac{a}{b} = \frac{c}{d}$$

The numbers a and d are the **extremes** of the proportion. The numbers b and c are the **means** of the proportion.

PROPERTIES OF PROPORTIONS

1. **CROSS PRODUCT PROPERTY** The product of the extremes equals the product of the means.

   If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

2. **RECIPROCAL PROPERTY** If two ratios are equal, then their reciprocals are also equal.

   If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

STUDENT HELP

Skills Review
For help with reciprocals, see p. 788.

To *solve the proportion* you find the value of the variable.

EXAMPLE 5 *Solving Proportions*

Using Algebra

Solve the proportions.

a. $\frac{4}{x} = \frac{5}{7}$ **b.** $\frac{3}{y+2} = \frac{2}{y}$

SOLUTION

a. $\frac{4}{x} = \frac{5}{7}$ **Write original proportion.**

$\frac{x}{4} = \frac{7}{5}$ **Reciprocal property**

$x = 4\left(\frac{7}{5}\right)$ **Multiply each side by 4.**

$x = \frac{28}{5}$ **Simplify.**

b. $\frac{3}{y+2} = \frac{2}{y}$ **Write original proportion.**

$3y = 2(y+2)$ **Cross product property**

$3y = 2y + 4$ **Distributive property**

$y = 4$ **Subtract $2y$ from each side.**

▶ The solution is 4. Check this by substituting in the original proportion.

EXAMPLE 6 *Solving a Proportion*

PAINTING The photo shows Bev Dolittle's painting *Music in the Wind*. Her actual painting is 12 inches high. How wide is it?

$1\frac{1}{4}$ in.

$4\frac{3}{8}$ in.

SOLUTION

You can reason that in the photograph all measurements of the artist's painting have been reduced by the same ratio. That is, the ratio of the actual width to the reduced width is equal to the ratio of the actual height to the reduced height. The photograph is $1\frac{1}{4}$ inches by $4\frac{3}{8}$ inches.

PROBLEM SOLVING STRATEGY

VERBAL MODEL

$$\frac{\text{Width of painting}}{\text{Width of photo}} = \frac{\text{Height of painting}}{\text{Height of photo}}$$

LABELS

Width of painting $= x$ Height of painting $= \mathbf{12}$ (inches)

Width of photo $= \mathbf{4.375}$ Height of photo $= \mathbf{1.25}$ (inches)

REASONING

$$\frac{x}{4.375} = \frac{12}{1.25} \quad \text{Substitute.}$$

$$x = 4.375\left(\frac{12}{1.25}\right) \quad \text{Multiply each side by 4.375.}$$

$$x = 42 \quad \text{Use a calculator.}$$

▶ So, the actual painting is 42 inches wide.

EXAMPLE 7 *Solving a Proportion*

Estimate the length of the hidden flute in Bev Doolittle's actual painting.

SOLUTION

In the photo, the flute is about $1\frac{7}{8}$ inches long. Using the reasoning from above you can say that:

$$\frac{\text{Length of flute in painting}}{\text{Length of flute in photo}} = \frac{\text{Height of painting}}{\text{Height of photo}}.$$

$$\frac{f}{1.875} = \frac{12}{1.25} \quad \text{Substitute.}$$

$$f = 18 \quad \text{Multiply each side by 1.875 and simplify.}$$

▶ So, the flute is about 18 inches long in the painting.

GUIDED PRACTICE

Vocabulary Check ✓

1. In the proportion $\frac{r}{s} = \frac{p}{q}$, the variables s and p are the __?__ of the proportion and r and q are the __?__ of the proportion.

Concept Check ✓

ERROR ANALYSIS **In Exercises 2 and 3, find and correct the errors.**

2. ~~A table is 18 inches wide and 3 feet long. The ratio of length to width is 1 : 6.~~

3. $\cancel{\frac{10}{x+6} = \frac{4}{x}}$

$\cancel{10x = 4x + 6}$

$\cancel{6x = 6}$

$\cancel{x = 1}$

Skill Check ✓

Given that the track team won 8 meets and lost 2, find the ratios.

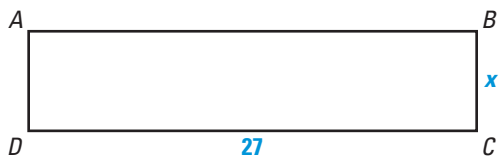
4. What is the ratio of wins to losses? What is the ratio of losses to wins?

5. What is the ratio of wins to the total number of track meets?

In Exercises 6–8, solve the proportion.

6. $\frac{2}{x} = \frac{3}{9}$ **7.** $\frac{5}{8} = \frac{6}{z}$ **8.** $\frac{2}{b+3} = \frac{4}{b}$

9. The ratio $BC:DC$ is $2:9$. Find the value of x.



PRACTICE AND APPLICATIONS

STUDENT HELP

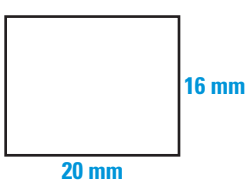
Extra Practice to help you master skills is on p. 817.

SIMPLIFYING RATIOS **Simplify the ratio.**

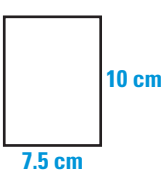
10. $\frac{16 \text{ students}}{24 \text{ students}}$ **11.** $\frac{48 \text{ marbles}}{8 \text{ marbles}}$ **12.** $\frac{22 \text{ feet}}{52 \text{ feet}}$ **13.** $\frac{6 \text{ meters}}{9 \text{ meters}}$

WRITING RATIOS **Find the width to length ratio of each rectangle. Then simplify the ratio.**

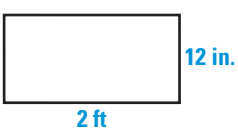
14.



15.



16.



STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 10–24
Example 2: Exs. 29, 30
Example 3: Exs. 31, 32
Example 4: Exs. 57, 58

continued on p. 462

CONVERTING UNITS **Rewrite the fraction so that the numerator and denominator have the same units. Then simplify.**

17. $\frac{3 \text{ ft}}{12 \text{ in.}}$ **18.** $\frac{60 \text{ cm}}{1 \text{ m}}$ **19.** $\frac{350 \text{ g}}{1 \text{ kg}}$ **20.** $\frac{2 \text{ mi}}{3000 \text{ ft}}$

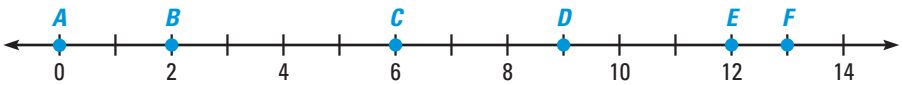
21. $\frac{6 \text{ yd}}{10 \text{ ft}}$ **22.** $\frac{2 \text{ lb}}{20 \text{ oz}}$ **23.** $\frac{400 \text{ m}}{0.5 \text{ km}}$ **24.** $\frac{20 \text{ oz}}{4 \text{ lb}}$

STUDENT HELP

HOMEWORK HELP
continued from p. 461

Example 5: Exs. 33–44
Example 6: Exs. 48–53, 59–61
Example 7: Exs. 48–53, 59–61

FINDING RATIOS **Use the number line to find the ratio of the distances.**



25. $\frac{AB}{CD} = \underline{\ ?\ }$ **26.** $\frac{BD}{CF} = \underline{\ ?\ }$ **27.** $\frac{BF}{AD} = \underline{\ ?\ }$ **28.** $\frac{CF}{AB} = \underline{\ ?\ }$

29. The perimeter of a rectangle is 84 feet. The ratio of the width to the length is 2:5. Find the length and the width.

30. The area of a rectangle is 108 cm^2. The ratio of the width to the length is 3:4. Find the length and the width.

31. The measures of the angles in a triangle are in the extended ratio of 1:4:7. Find the measures of the angles.

32. The measures of the angles in a triangle are in the extended ratio of 2:15:19. Find the measures of the angles.

SOLVING PROPORTIONS **Solve the proportion.**

33. $\frac{x}{4} = \frac{5}{7}$ **34.** $\frac{y}{8} = \frac{9}{10}$ **35.** $\frac{7}{z} = \frac{10}{25}$

36. $\frac{4}{b} = \frac{10}{3}$ **37.** $\frac{30}{5} = \frac{14}{c}$ **38.** $\frac{16}{3} = \frac{d}{6}$

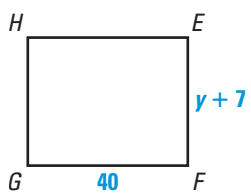
39. $\frac{5}{x+3} = \frac{4}{x}$ **40.** $\frac{4}{y-3} = \frac{8}{y}$ **41.** $\frac{7}{2z+5} = \frac{3}{z}$

42. $\frac{3x-8}{6} = \frac{2x}{10}$ **43.** $\frac{5y-8}{7} = \frac{5y}{6}$ **44.** $\frac{4}{2z+6} = \frac{10}{7z-2}$

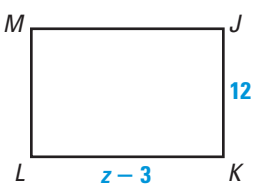
USING PROPORTIONS **In Exercises 45–47, the ratio of the width to the length for each rectangle is given. Solve for the variable.**

45. $AB:BC$ is 3:8.

46. $EF:FG$ is 4:5.



47. $JK:KL$ is 2:3.



SCIENCE CONNECTION **Use the following information.**

The table gives the ratios of the gravity of four different planets to the gravity of Earth. Round your answers to the nearest whole number.

Planet	Venus	Mars	Jupiter	Pluto
Ratio of gravity	$\frac{9}{10}$	$\frac{38}{100}$	$\frac{236}{100}$	$\frac{7}{100}$

48. Which of the planets listed above has a gravity closest to the gravity of Earth?

49. Estimate how much a person who weighs 140 pounds on Earth would weigh on Venus, Mars, Jupiter, and Pluto.

50. If a person weighed 46 pounds on Mars, estimate how much he or she would weigh on Earth.

FOCUS ON APPLICATIONS

REAL LIFE

MOON'S GRAVITY
Neil Armstrong's space suit weighed about 185 pounds on Earth and just over 30 pounds on the moon, due to the weaker force of gravity.

INTERNET APPLICATION LINK
www.mcdougallittell.com

BASEBALL BAT SCULPTURE **A huge, free-standing baseball bat sculpture stands outside a sports museum in Louisville, Kentucky. It was patterned after Babe Ruth's 35 inch bat. The sculpture is 120 feet long. Round your answers to the nearest tenth of an inch.**

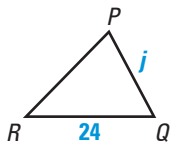
51. How long is the sculpture in inches?

52. The diameter of the sculpture near the base is 9 feet. Estimate the corresponding diameter of Babe Ruth's bat.

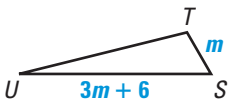
53. The diameter of the handle of the sculpture is 3.5 feet. Estimate the diameter of the handle of Babe Ruth's bat.

USING PROPORTIONS **In Exercises 54–56, the ratio of two side lengths of the triangle is given. Solve for the variable.**

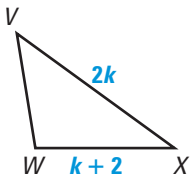
54. $PQ:QR$ is $3:4$.



55. $SU:ST$ is $4:1$.



56. $WX:XV$ is $5:7$.

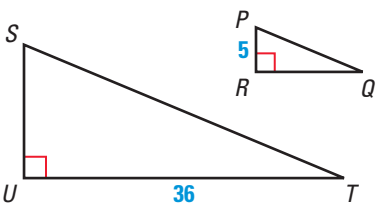


STUDENT HELP

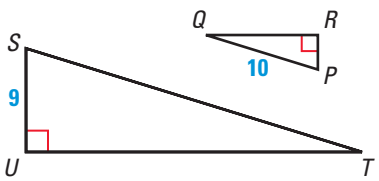
HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 57 and 58.

PYTHAGOREAN THEOREM **The ratios of the side lengths of $\triangle PQR$ to the corresponding side lengths of $\triangle STU$ are 1:3. Find the unknown lengths.**

57.



58.



GULLIVER'S TRAVELS **In Exercises 59–61, use the following information.**

Gulliver's Travels was written by Jonathan Swift in 1726. In the story, Gulliver is shipwrecked and wanders ashore to the island of Lilliput. The average height of the people in Lilliput is 6 inches.

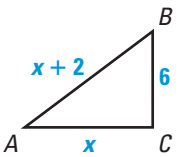
59. Gulliver is 6 feet tall. What is the ratio of his height to the average height of a Lilliputian?

60. After leaving Lilliput, Gulliver visits the island of Brobdingnag. The ratio of the average height of these natives to Gulliver's height is proportional to the ratio of Gulliver's height to the average height of a Lilliputian. What is the average height of a Brobdingnagian?

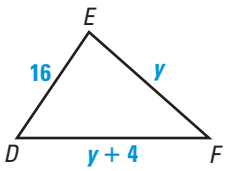
61. What is the ratio of the average height of a Brobdingnagian to the average height of a Lilliputian?

USING ALGEBRA **You are given an extended ratio that compares the lengths of the sides of the triangle. Find the lengths of all unknown sides.**

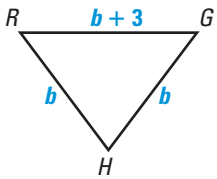
62. $BC:AC:AB$ is $3:4:5$.



63. $DE:EF:DF$ is $4:5:6$.



64. $GH:HR:GR$ is $5:5:6$.



Test Preparation

65. MULTIPLE CHOICE For planting roses, a gardener uses a special mixture of soil that contains sand, peat moss, and compost in the ratio $2:5:3$. How many pounds of compost does she need to add if she uses three 10 pound bags of peat moss?

(A) 12 (B) 14 (C) 15 (D) 18 (E) 20

66. MULTIPLE CHOICE If the measures of the angles of a triangle have the ratio $2:3:7$, the triangle is

(A) acute. (B) right. (C) isosceles.

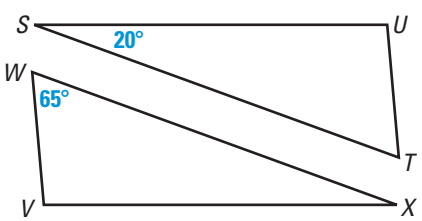
(D) obtuse. (E) equilateral.

★ Challenge

67. FINDING SEGMENT LENGTHS Suppose the points B and C lie on $\overline{AD}$. What is the length of $\overline{AC}$ if $\frac{AB}{BD} = \frac{2}{3}$, $\frac{CD}{AC} = \frac{1}{9}$, and $BD = 24$?

MIXED REVIEW

FINDING UNKNOWN MEASURES **Use the figure shown, in which $\triangle STU \cong \triangle XWV$.** (Review 4.2)



68. What is the measure of $\angle X$?

69. What is the measure of $\angle V$?

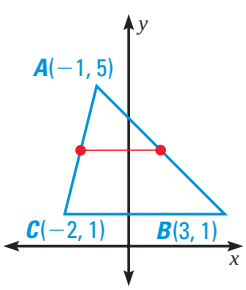
70. What is the measure of $\angle T$?

71. What is the measure of $\angle U$?

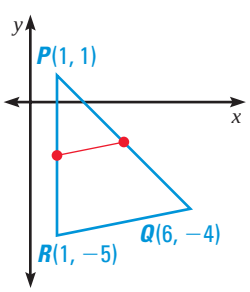
72. Which side is congruent to $\overline{TU}$?

FINDING COORDINATES **Find the coordinates of the endpoints of each midsegment shown in red.** (Review 5.4 for 8.2)

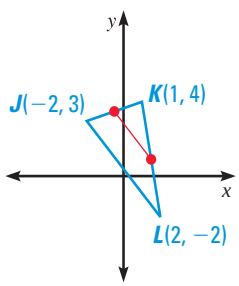
73.



74.



75.



76. A line segment has endpoints $A(1, -3)$ and $B(6, -7)$. Graph $\overline{AB}$ and its image $\overline{A'B'}$ if $\overline{AB}$ is reflected in the line $x = 2$. (Review 7.2)

8.2 Problem Solving in Geometry with Proportions

What you should learn

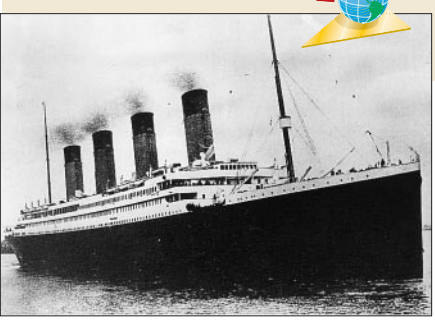
GOAL 1 Use properties of proportions.

GOAL 2 Use proportions to solve **real-life** problems, such as using the scale of a map in **Exs. 41 and 42**.

Why you should learn it

▼ To solve **real-life** problems, such as using a scale model to calculate the dimensions of the Titanic in **Example 4**.

REAL LIFE



GOAL 1 USING PROPERTIES OF PROPORTIONS

In Lesson 8.1, you studied the reciprocal property and the cross product property. Two more properties of proportions, which are especially useful in geometry, are given below.

You can use the cross product property and the reciprocal property to help prove these properties in Exercises 36 and 37.

ADDITIONAL PROPERTIES OF PROPORTIONS

3. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

4. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

EXAMPLE 1 *Using Properties of Proportions*

Tell whether the statement is true.

a. If $\frac{p}{6} = \frac{r}{10}$, then $\frac{p}{r} = \frac{3}{5}$.

b. If $\frac{a}{3} = \frac{c}{4}$, then $\frac{a+3}{3} = \frac{c+3}{4}$.

SOLUTION

a. $\frac{p}{6} = \frac{r}{10}$ **Given**

$\frac{p}{r} = \frac{6}{10}$ **If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.**

$\frac{p}{r} = \frac{3}{5}$ **Simplify.**

▶ The statement is true.

b. $\frac{a}{3} = \frac{c}{4}$ **Given**

$\frac{a+3}{3} = \frac{c+4}{4}$ **If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.**

Because $\frac{c+4}{4} \neq \frac{c+3}{4}$, the conclusions are not equivalent.

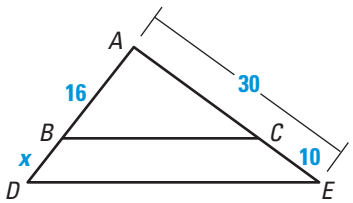
▶ The statement is false.

North Carolina Standards and Assessments
MA.B.3.4.1, MA.C.3.4.1

Using Algebra

EXAMPLE 2 *Using Properties of Proportions*

In the diagram $\frac{AB}{BD} = \frac{AC}{CE}$. Find the length of $\overline{BD}$.



SOLUTION

$\frac{AB}{BD} = \frac{AC}{CE}$	**Given**
$\frac{16}{x} = \frac{30 - 10}{10}$	**Substitute.**
$\frac{16}{x} = \frac{20}{10}$	**Simplify.**
$20x = 160$	**Cross product property**
$x = 8$	**Divide each side by 20.**

▶ So, the length of $\overline{BD}$ is 8.

STUDENT HELP

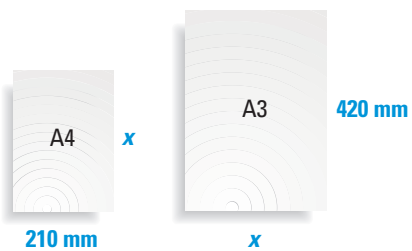
INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

The **geometric mean** of two positive numbers a and b is the positive number x such that $\frac{a}{x} = \frac{x}{b}$. If you solve this proportion for x, you find that $x = \sqrt{a \cdot b}$, which is a positive number.

For example, the geometric mean of 8 and 18 is **12**, because $\frac{8}{12} = \frac{12}{18}$, and also because $\sqrt{8 \cdot 18} = \sqrt{144} = 12$.

EXAMPLE 3 *Using a Geometric Mean*

REAL LIFE **PAPER SIZES** International standard paper sizes are commonly used all over the world. The various sizes all have the same width-to-length ratios. Two sizes of paper are shown, called A4 and A3. The distance labeled x is the geometric mean of 210 mm and 420 mm. Find the value of x.



STUDENT HELP

Skills Review For help with simplifying square roots, see p. 799.

SOLUTION

$\frac{210}{x} = \frac{x}{420}$	**Write proportion.**
$x^2 = 210 \cdot 420$	**Cross product property**
$x = \sqrt{210 \cdot 420}$	**Simplify.**
$x = \sqrt{210 \cdot 210 \cdot 2}$	**Factor.**
$x = 210\sqrt{2}$	**Simplify.**

▶ The geometric mean of 210 and 420 is $210\sqrt{2}$, or about 297. So, the distance labeled x in the diagram is about 297 mm.

GOAL 2 USING PROPORTIONS IN REAL LIFE

In general, when solving word problems that involve proportions, there is more than one correct way to set up the proportion.

EXAMPLE 4 *Solving a Proportion*

REAL LIFE **MODEL BUILDING** A scale model of the Titanic is 107.5 inches long and 11.25 inches wide. The Titanic itself was 882.75 feet long. How wide was it?

SOLUTION

One way to solve this problem is to set up a proportion that compares the measurements of the Titanic to the measurements of the scale model.

PROBLEM SOLVING STRATEGY

VERBAL MODEL

$$\frac{\text{Width of Titanic}}{\text{Width of model ship}} = \frac{\text{Length of Titanic}}{\text{Length of model ship}}$$

LABELS

Width of Titanic = x (feet)

Width of model ship = **11.25** (inches)

Length of Titanic = **882.75** (feet)

Length of model ship = **107.5** (inches)

REASONING

$$\frac{x \text{ ft}}{11.25 \text{ in.}} = \frac{882.75 \text{ ft}}{107.5 \text{ in.}}$$ **Substitute.**

$$x = \frac{11.25 \cdot (882.75)}{107.5}$$ **Multiply each side by 11.25.**

$$x \approx 92.4$$ **Use a calculator.**

▶ So, the Titanic was about 92.4 feet wide.

Notice that the proportion in Example 4 contains measurements that are not in the same units. When writing a proportion with unlike units, the numerators should have the same units and the denominators should have the same units.

GUIDED PRACTICE

Vocabulary Check ✓

1. If x is the *geometric mean* of two positive numbers a and b, write a proportion that relates a, b, and x. $\frac{a}{x} = \frac{x}{b}$

Concept Check ✓

2. If $\frac{x}{4} = \frac{y}{5}$, then $\frac{x+4}{4} = \frac{?}{5}$. $y + 5$

3. If $\frac{b}{6} = \frac{c}{2}$, then $\frac{b}{c} = \frac{?}{?}$. $\frac{6}{2}$; or 3

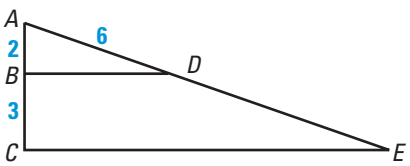
Skill Check ✓

4. Decide whether the statement is *true* or *false*. true

$$\text{If } \frac{r}{s} = \frac{6}{15}, \text{ then } \frac{15}{s} = \frac{6}{r}.$$

5. Find the geometric mean of 3 and 12. 6

6. In the diagram $\frac{AB}{BC} = \frac{AD}{DE}$. Substitute the known values into the proportion and solve for DE. 9



7. **UNITED STATES FLAG** The official height-to-width ratio of the United States flag is 1 : 1.9. If a United States flag is 6 feet high, how wide is it? 11.4 ft

8. **UNITED STATES FLAG** The blue portion of the United States flag is called the union. What is the ratio of the height of the union to the height of the flag? $\frac{7}{13}$

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 817.

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 9–16
Example 2: Exs. 23–28
Example 3: Exs. 17–22, 43
Example 4: Exs. 29–32, 38–42

LOGICAL REASONING **Complete the sentence.**

9. If $\frac{2}{x} = \frac{7}{y}$, then $\frac{2}{7} = \frac{?}{?}$. $\frac{x}{y}$

10. If $\frac{x}{6} = \frac{y}{34}$, then $\frac{x}{y} = \frac{?}{?}$. $\frac{6}{34}$ or $\frac{3}{17}$

11. If $\frac{x}{5} = \frac{y}{12}$, then $\frac{x+5}{5} = \frac{?}{?}$. $\frac{y+12}{12}$

12. If $\frac{13}{7} = \frac{x}{y}$, then $\frac{20}{7} = \frac{?}{?}$. $\frac{x+y}{y}$

LOGICAL REASONING **Decide whether the statement is *true* or *false*.**

13. If $\frac{7}{a} = \frac{b}{2}$, then $\frac{7+a}{a} = \frac{b+2}{2}$. true

14. If $\frac{3}{4} = \frac{p}{r}$, then $\frac{4}{3} = \frac{p}{r}$. false

15. If $\frac{c}{6} = \frac{d+2}{10}$, then $\frac{c}{d+2} = \frac{6}{10}$. true

16. If $\frac{12+m}{12} = \frac{3+n}{n}$, then $\frac{m}{12} = \frac{3}{n}$. true

GEOMETRIC MEAN **Find the geometric mean of the two numbers.**

17. 3 and 27 9

18. 4 and 16 8

19. 7 and 28 14

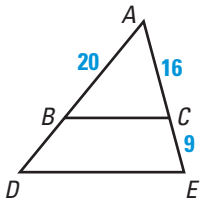
20. 2 and 40 $4\sqrt{5}$

21. 8 and 20 $4\sqrt{10}$

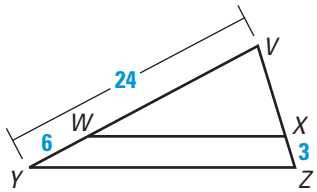
22. 5 and 15 $5\sqrt{3}$

PROPERTIES OF PROPORTIONS **Use the diagram and the given information to find the unknown length.**

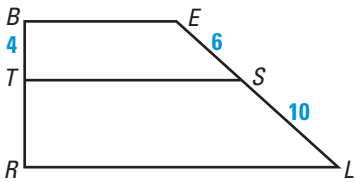
23. GIVEN ▶ $\frac{AB}{BD} = \frac{AC}{CE}$, find BD. **11.25**



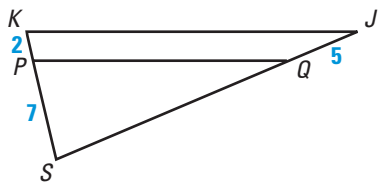
24. GIVEN ▶ $\frac{VW}{WY} = \frac{VX}{XZ}$, find VX. **9**



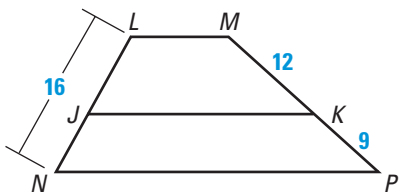
25. GIVEN ▶ $\frac{BT}{TR} = \frac{ES}{SL}$, find TR. **$6\frac{2}{3}$**



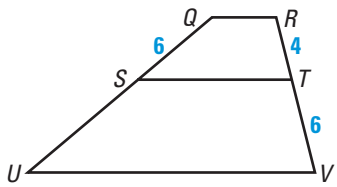
26. GIVEN ▶ $\frac{SP}{SK} = \frac{SQ}{SJ}$, find SQ. **$17\frac{1}{2}$**



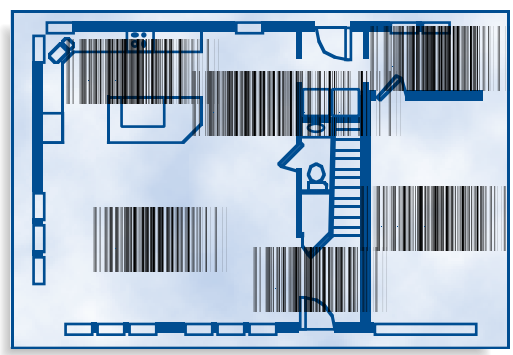
27. GIVEN ▶ $\frac{LJ}{JN} = \frac{MK}{KP}$, find JN. **$6\frac{6}{7}$**



28. GIVEN ▶ $\frac{QU}{QS} = \frac{RV}{RT}$, find SU. **9**



BLUEPRINTS **In Exercises 29 and 30, use the blueprint of the house in which $\frac{1}{16}$ inch = 1 foot. Use a ruler to approximate the dimension.**



29. Find the approximate width of the house to the nearest 5 feet. **about 25 ft**

30. Find the approximate length of the house to the nearest 5 feet. **about 40 ft**

31. **BATTING AVERAGE** The batting average of a baseball player is the ratio of the number of hits to the number of official at-bats. In 1998, Sammy Sosa of the Chicago Cubs had 643 official at-bats and a batting average of .308. Use the following verbal model to find the number of hits Sammy Sosa got. **198 hits**

$$\frac{\textit{Number of hits}}{\textit{Number of at-bats}} = \frac{\textit{Batting average}}{1.000}$$

32. **CURRENCY EXCHANGE** Natalie has relatives in Russia. She decides to take a trip to Russia to visit them. She took 500 U.S. dollars to the bank to exchange for Russian rubles. The exchange rate on that day was 22.76 rubles per U.S. dollar. How many rubles did she get in exchange for the 500 U.S. dollars? ▶ Source: Russia Today **11,380 rubles**

FOCUS ON PEOPLE

SAMMY SOSA was the National League Most Valuable Player in 1998. He hit 66 home runs to finish second to Mark McGwire who hit 70, a new record.

APPLICATION LINK
www.mcdougallittell.com

35. Each side of the equation represents the slope of the line through two of the points; if the points are collinear, the slopes are the same.

40. *Sample answer:* Construct a ramp consisting of two ramps in opposite directions, each 18 ft long. The first should be 3 ft high at its beginning and $1\frac{1}{2}$ ft high at its end, for a rise:run ratio of $\frac{1}{12}$. The second would be $1\frac{1}{2}$ ft high at its beginning and ground level at its end. The second ramp also has a rise:run ratio of $\frac{1}{12}$.

43. If the two sizes share a dimension, the shorter dimension of A5 paper must be the longer dimension of A6 paper. That is, the length of A6 paper must be 148 mm. Let x be the width of A6 paper; 148 is the geometric mean of x and 210. Then $\frac{x}{148} = \frac{148}{210}$ and $x \approx 104$ mm.

33. COORDINATE GEOMETRY The points $(-4, -1)$, $(1, 1)$, and $(x, 5)$ are collinear. Find the value of x by solving the proportion below. **11**

$$\frac{1-(-1)}{1-(-4)} = \frac{5-1}{x-1}$$

34. COORDINATE GEOMETRY The points $(2, 8)$, $(6, 18)$, and $(8, y)$ are collinear. Find the value of y by solving the proportion below. **23**

$$\frac{18-8}{6-2} = \frac{y-18}{8-6}$$

35. CRITICAL THINKING Explain why the method used in Exercises 33 and 34 is a correct way to express that three given points are collinear. **See margin.**

36. PROOF Prove property 3 of proportions (see page 465). **See margin.**

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

37. PROOF Prove property 4 of proportions (see page 465). **See margin.**

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

RAMP DESIGN Assume that a wheelchair ramp has a slope of $\frac{1}{12}$, which is the maximum slope recommended for a wheelchair ramp.

38. A wheelchair ramp has a 15 foot run. What is its rise? **$1\frac{1}{4}$ ft**

39. A wheelchair ramp rises 2 feet. What is its run? **24 ft**

40. You are constructing a wheelchair ramp that must rise 3 feet. Because of space limitations, you cannot build a continuous ramp with a length greater than 21 feet. Design a ramp that solves this problem. **See margin.**

FOCUS ON PEOPLE

SACAGAWEA Representing liberty on the new dollar coin is Sacagawea, who played a crucial role in the Lewis and Clark expedition. She acted as an interpreter and guide, and is now given credit for much of the mission's success.

HISTORY CONNECTION Part of the Lewis and Clark Trail on which Sacagawea acted as guide is now known as the Lolo Trail. The map, which shows a portion of the trail, has a scale of 1 inch = 6.7 miles.

41. Use a ruler to estimate the distance (measured in a straight line) between Lewis and Clark Grove and Pheasant Camp. Then calculate the actual distance in miles. **about $\frac{3}{8}$ in.; about $2\frac{1}{2}$ mi**

42. Estimate the distance along the trail between Portable Soup Camp and Full Stomach Camp. Then calculate the actual distance in miles. **about $1\frac{1}{4}$ in.; about $8\frac{3}{8}$ mi**

43. *Writing* Size A5 paper has a width of 148 mm and a length of 210 mm. Size A6, which is the next smaller size, shares a dimension with size A5. Use the proportional relationship stated in Example 3 and geometric mean to explain how to determine the length and width of size A6 paper. **See margin.**

Test Preparation

44. MULTIPLE CHOICE There are 24 fish in an aquarium. If $\frac{1}{8}$ of the fish are tetras, and $\frac{2}{3}$ of the remaining fish are guppies, how many guppies are in the aquarium? **D**

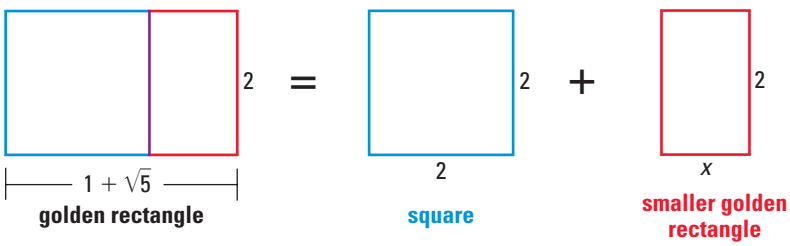
Ⓐ 2 Ⓑ 3 Ⓒ 10 Ⓓ 14 Ⓔ 16

45. MULTIPLE CHOICE A basketball team had a ratio of wins to losses of 3 : 1. After winning 6 games in a row, the team's ratio of wins to losses was 5 : 1. How many games had the team won before it won the 6 games in a row? **C**

Ⓐ 3 Ⓑ 6 Ⓒ 9 Ⓓ 15 Ⓔ 24

★ Challenge

46. GOLDEN RECTANGLE A golden rectangle has its length and width in the golden ratio $\frac{1+\sqrt{5}}{2}$. If you cut a square away from a golden rectangle, the shape that remains is also a golden rectangle.



a. The diagram indicates that $1 + \sqrt{5} = 2 + x$. Find x. $-1 + \sqrt{5}$

b. To prove that the large and small rectangles are both golden rectangles, show that $\frac{1+\sqrt{5}}{2} = \frac{2}{x}$.

c. Give a decimal approximation for the golden ratio to six decimal places. **1.618034**

46. b. $\frac{1+\sqrt{5}}{2} = \frac{2}{x}$ **if 2 is the geometric mean of $1 + \sqrt{5}$ and x. The geometric mean of $1 + \sqrt{5}$ and $-1 + \sqrt{5}$ is $\sqrt{(1+\sqrt{5})(-1+\sqrt{5})} = \sqrt{4} = 2$.**

EXTRA CHALLENGE
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MIXED REVIEW

FINDING AREA **Find the area of the figure described.** (Review 1.7)

47. Rectangle: width = 3 m, length = 4 m **12 m²**

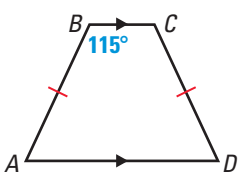
48. Square: side = 3 cm **9 cm²**

49. Triangle: base = 13 cm, height = 4 cm **26 cm²**

50. Circle: diameter = 11 ft **about 95 ft²**

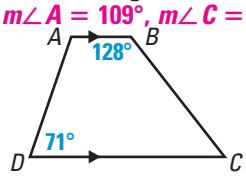
FINDING ANGLE MEASURES **Find the angle measures.** (Review 6.5 for 8.3)

51.

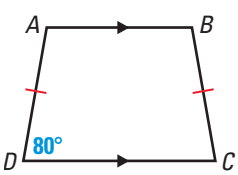


$m\angle C = 115°$, $m\angle A = m\angle D = 65°$

52. **$m\angle A = 109°$, $m\angle C = 52°$**

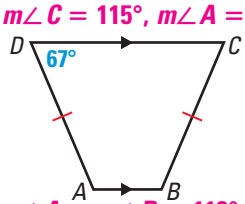


53.



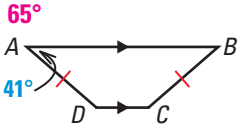
$m\angle A = m\angle B = 100°$, $m\angle C = 80°$

54.



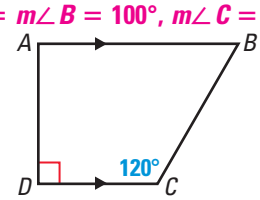
$m\angle A = m\angle B = 113°$, $m\angle C = 67°$

55.



$m\angle B = 41°$, $m\angle C = m\angle D = 139°$

56.



$m\angle A = 90°$, $m\angle B = 60°$

57. PENTAGON Describe any symmetry in a regular pentagon $ABCDE$. (Review 7.2, 7.3)

57. A regular pentagon has 5 lines of symmetry (one from each vertex to the midpoint of the opposite side) and rotational symmetries of 72° and 144°, clockwise and counterclockwise, about the center of the pentagon.

8.3 Similar Polygons

What you should learn

GOAL 1 Identify similar polygons.

GOAL 2 Use similar polygons to solve **real-life** problems, such as making an enlargement similar to an original photo in **Example 3**.

Why you should learn it

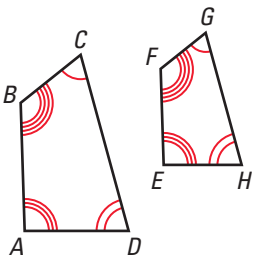
▼ To solve **real-life** problems, such as comparing television screen sizes in **Exs. 43 and 44**.

REAL LIFE

GOAL 1 IDENTIFYING SIMILAR POLYGONS

When there is a correspondence between two polygons such that their corresponding angles are congruent and the lengths of corresponding sides are proportional the two polygons are called **similar polygons**.

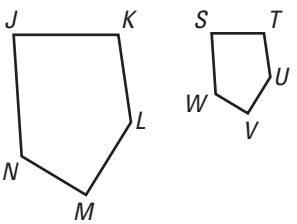
In the diagram, *ABCD* is similar to *EFGH*. The symbol ~ is used to indicate similarity. So, *ABCD* ~ *EFGH*.



$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH} = \frac{DA}{HE}$$

EXAMPLE 1 *Writing Similarity Statements*

Pentagons *JKLMN* and *STUVW* are similar. List all the pairs of congruent angles. Write the ratios of the corresponding sides in a statement of proportionality.



SOLUTION

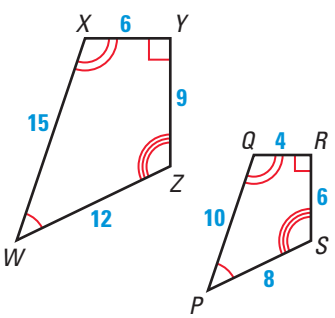
Because *JKLMN* ~ *STUVW*, you can write $\angle J \cong \angle S$, $\angle K \cong \angle T$, $\angle L \cong \angle U$, $\angle M \cong \angle V$, and $\angle N \cong \angle W$.

You can write the statement of proportionality as follows:

$$\frac{JK}{ST} = \frac{KL}{TU} = \frac{LM}{UV} = \frac{MN}{VW} = \frac{NJ}{WS}.$$

EXAMPLE 2 *Comparing Similar Polygons*

Decide whether the figures are similar. If they are similar, write a similarity statement.



SOLUTION

As shown, the corresponding angles of *WXYZ* and *PQRS* are congruent. Also, the corresponding side lengths are proportional.

$$\frac{WX}{PQ} = \frac{15}{10} = \frac{3}{2} \qquad \frac{XY}{QR} = \frac{6}{4} = \frac{3}{2}$$

$$\frac{YZ}{RS} = \frac{9}{6} = \frac{3}{2} \qquad \frac{ZW}{SP} = \frac{12}{8} = \frac{3}{2}$$

▶ So, the two figures are similar and you can write *WXYZ* ~ *PQRS*.

STUDENT HELP

Study Tip

When you refer to similar polygons, their corresponding vertices must be listed in the same order.

GOAL 2 USING SIMILAR POLYGONS IN REAL LIFE

EXAMPLE 3 *Comparing Photographic Enlargements*

REAL LIFE **POSTER DESIGN** You have been asked to create a poster to advertise a field trip to see the Liberty Bell. You have a 3.5 inch by 5 inch photo that you want to enlarge. You want the enlargement to be 16 inches wide. How long will it be?

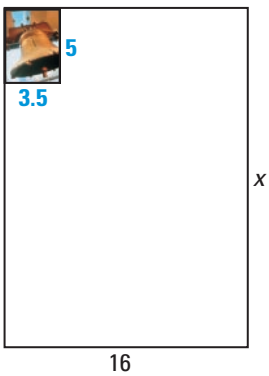
SOLUTION

To find the length of the enlargement, you can compare the enlargement to the original measurements of the photo.

$$\frac{16 \text{ in.}}{3.5 \text{ in.}} = \frac{x \text{ in.}}{5 \text{ in.}}$$

$$x = \frac{16}{3.5} \cdot 5$$

$$x \approx 22.9 \text{ inches}$$

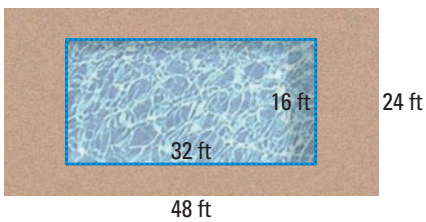


▶ The length of the enlargement will be about 23 inches.

If two polygons are similar, then the ratio of the lengths of two corresponding sides is called the **scale factor**. In Example 2 on the previous page, the common ratio of $\frac{3}{2}$ is the scale factor of *WXYZ* to *PQRS*.

EXAMPLE 4 *Using Similar Polygons*

The rectangular patio around a pool is similar to the pool as shown. Calculate the scale factor of the patio to the pool, and find the ratio of their perimeters.



SOLUTION

Because the rectangles are similar, the scale factor of the patio to the pool is 48 ft : 32 ft, which is 3 : 2 in simplified form.

The perimeter of the patio is $2(24) + 2(48) = 144$ feet and the perimeter of the pool is $2(16) + 2(32) = 96$ feet. The ratio of the perimeters is $\frac{144}{96}$, or $\frac{3}{2}$.

Notice in Example 4 that the ratio of the perimeters is the same as the scale factor of the rectangles. This observation is generalized in the following theorem. You are asked to prove Theorem 8.1 for two similar rectangles in Exercise 45.

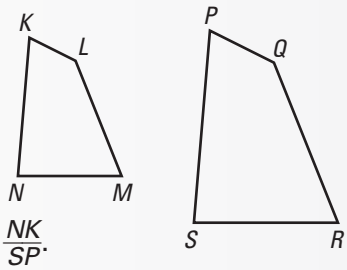
THEOREM

THEOREM 8.1

If two polygons are similar, then the ratio of their perimeters is equal to the ratios of their corresponding side lengths.

If $KLMN \sim PQRS$, then

$$\frac{KL + LM + MN + NK}{PQ + QR + RS + SP} = \frac{KL}{PQ} = \frac{LM}{QR} = \frac{MN}{RS} = \frac{NK}{SP}.$$

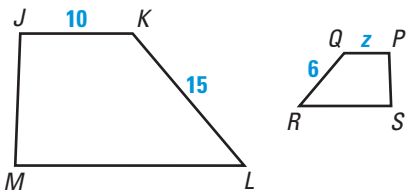


Using Algebra

EXAMPLE 5 *Using Similar Polygons*

Quadrilateral $JKLM$ is similar to quadrilateral $PQRS$.

Find the value of z.



SOLUTION

Set up a proportion that contains PQ.

$\frac{KL}{QR} = \frac{JK}{PQ}$ **Write proportion.**

$\frac{15}{6} = \frac{10}{z}$ **Substitute.**

$z = 4$ **Cross multiply and divide by 15.**

GUIDED PRACTICE

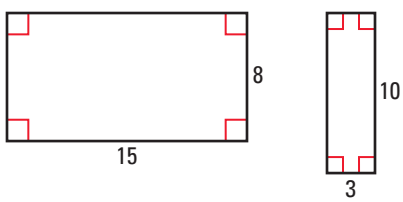
Vocabulary Check ✓

1. If two polygons are similar, must they also be congruent? Explain.

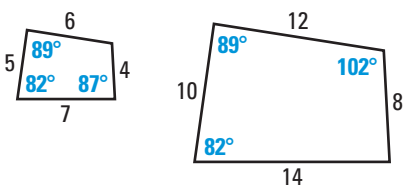
Concept Check ✓

Decide whether the figures are similar. Explain your reasoning.

2.

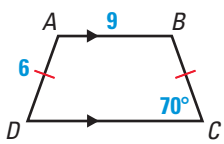


3.



Skill Check ✓

In the diagram, $TUVW \sim ABCD$.

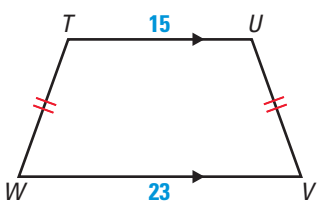


4. List all pairs of congruent angles and write the statement of proportionality for the polygons.

5. Find the scale factor of $TUVW$ to $ABCD$.

6. Find the length of $\overline{TW}$.

7. Find the measure of $\angle TUV$.



PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 817.

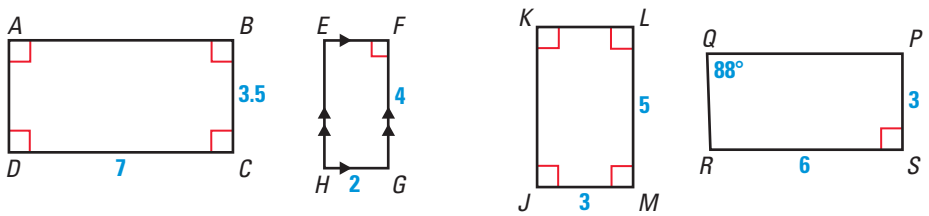
WRITING SIMILARITY STATEMENTS **Use the information given to list all pairs of congruent angles and write the statement of proportionality for the figures.**

8. $\triangle DEF \sim \triangle PQR$

9. $\square JKLM \sim \square WXYZ$

10. $QRSTU \sim ABCDE$

DETERMINING SIMILARITY **Decide whether the quadrilaterals are similar. Explain your reasoning.**



11. *ABCD* and *FGHE*

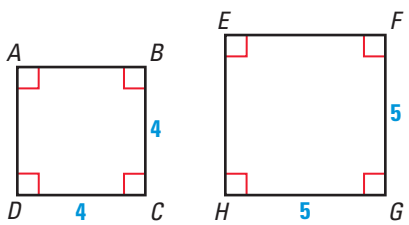
12. *ABCD* and *JKLM*

13. *ABCD* and *PQRS*

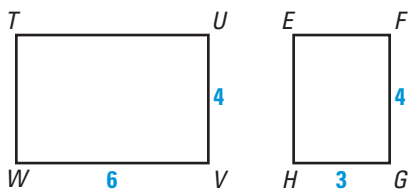
14. *JKLM* and *PQRS*

DETERMINING SIMILARITY **Decide whether the polygons are similar. If so, write a similarity statement.**

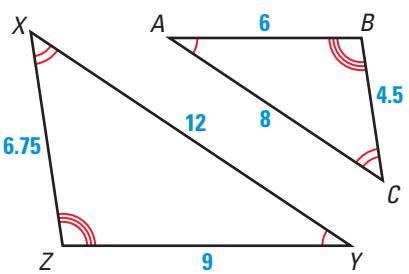
15.



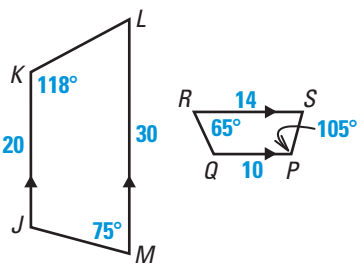
16.



17.



18.



USING SIMILAR POLYGONS ***PQRS* ~ *JKLM*.**

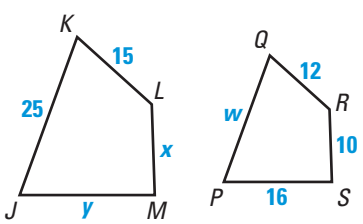
19. Find the scale factor of *PQRS* to *JKLM*.

20. Find the scale factor of *JKLM* to *PQRS*.

21. Find the values of w, x, and y.

22. Find the perimeter of each polygon.

23. Find the ratio of the perimeter of *PQRS* to the perimeter of *JKLM*.

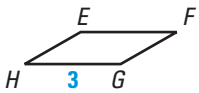


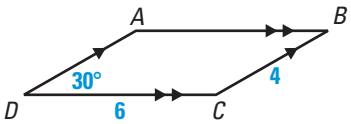
STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 8–10
Example 2: Exs. 11–18
Example 3: Exs. 19–30, 43, 44
Example 4: Exs. 19–30, 46–48
Example 5: Exs. 39–42

USING SIMILAR POLYGONS ▱*ABCD* ~ ▱*EFGH*.





24. Find the scale factor of ▱*ABCD* to ▱*EFGH*.

25. Find the length of $\overline{EH}$.

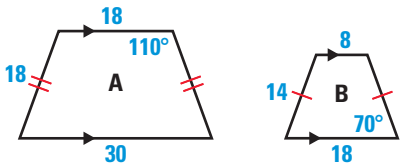
26. Find the measure of $\angle G$.

27. Find the perimeter of ▱*EFGH*.

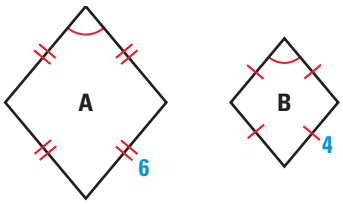
28. Find the ratio of the perimeter of ▱*EFGH* to the perimeter of ▱*ABCD*.

DETERMINING SIMILARITY **Decide whether the polygons are similar. If so, find the scale factor of Figure A to Figure B.**

29.



30.



LOGICAL REASONING **Tell whether the polygons are *always*, *sometimes*, or *never* similar.**

31. Two isosceles triangles

32. Two regular polygons

33. Two isosceles trapezoids

34. Two rhombuses

35. Two squares

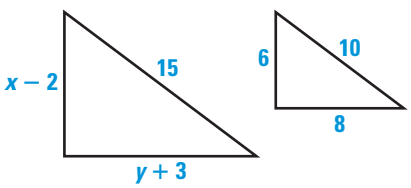
36. An isosceles and a scalene triangle

37. Two equilateral triangles

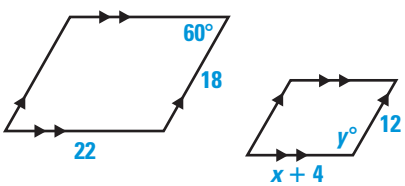
38. A right and an isosceles triangle

USING ALGEBRA **The two polygons are similar. Find the values of *x* and *y*.**

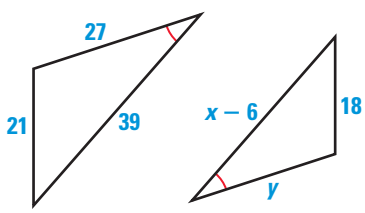
39.



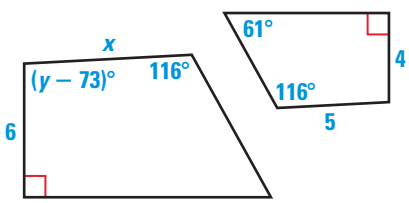
40.



41.



42.



TV SCREENS **In Exercises 43 and 44, use the following information.**
Television screen sizes are based on the length of the diagonal of the screen. The *aspect ratio* refers to the length to width ratio of the screen. A standard 27 inch analog television screen has an aspect ratio of 4:3. A 27 inch digital television screen has an aspect ratio of 16:9.

43. Make a scale drawing of each television screen. Use proportions and the Pythagorean Theorem to calculate the lengths and widths of the screens in inches.

44. Are the television screens similar? Explain.

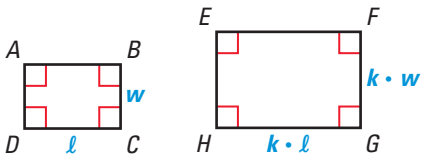
STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 31–38.

FOCUS ON APPLICATIONS

REAL LIFE **DIGITAL TELEVISION** screens contain over 6 times as many pixels (the tiny dots that make up the picture) as standard analog screens.

45. **PROOF** Prove Theorem 8.1 for two similar rectangles.



GIVEN ▶ $ABCD \sim EFGH$

PROVE ▶ $\dfrac{\text{perimeter of } ABCD}{\text{perimeter of } EFGH} = \dfrac{AB}{EF}$

46. **SCALE** The ratio of the perimeter of $WXYZ$ to the perimeter of $QRST$ is $7.5:2$. Find the scale factor of $QRST$ to $WXYZ$.

47. **SCALE** The ratio of one side of $\triangle CDE$ to the corresponding side of similar $\triangle FGH$ is $2:5$. The perimeter of $\triangle FGH$ is 28 inches. Find the perimeter of $\triangle CDE$.

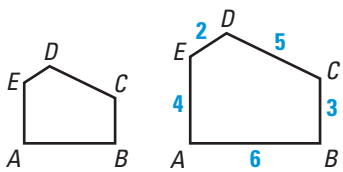
48. **SCALE** The perimeter of $\square PQRS$ is 94 centimeters. The perimeter of $\square JKLM$ is 18.8 centimeters, and $\square JKLM \sim \square PQRS$. The lengths of the sides of $\square PQRS$ are 15 centimeters and 32 centimeters. Find the scale factor of $\square PQRS$ to $\square JKLM$, and the lengths of the sides of $\square JKLM$.

Test Preparation

49. **MULTI-STEP PROBLEM** Use the similar figures shown. The scale factor of Figure 1 to Figure 2 is $7:10$.

a. Copy and complete the table.

	AB	BC	CD	DE	EA
Figure 1	?	?	?	?	?
Figure 2	6.0	3.0	5.0	2.0	4.0



b. Graph the data in the table. Let x represent the length of a side in Figure 1 and let y represent the length of the corresponding side in Figure 2. Determine an equation that relates x and y.

c. **ANALYZING DATA** The equation you obtained in part (b) should be linear. What is its slope? How does its slope compare to the scale factor?

★ Challenge

TOTAL ECLIPSE **Use the following information in Exercises 50–52.** From your perspective on Earth during a total eclipse of the sun, the moon is directly in line with the sun and blocks the sun's rays. The ratio of the radius of the moon to its distance to Earth is about the same as the ratio of the radius of the sun to its distance to Earth.

Distance between Earth and the moon: 240,000 miles

Distance between Earth and the sun: 93,000,000 miles

Radius of the sun: 432,500 miles

50. Make a sketch of Earth, the moon, and the sun during a total eclipse of the sun. Include the given distances in your sketch.

51. Your sketch should contain some similar triangles. Use the similar triangles in your sketch to explain a total eclipse of the sun.

52. Write a statement of proportionality for the similar triangles. Then use the given distances to estimate the radius of the moon.

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

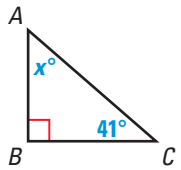
FINDING SLOPE **Find the slope of the line that passes through the given points.** (Review 3.6 for 8.4)

53. $A(-1, 4)$, $B(3, 8)$ **54.** $P(0, -7)$, $Q(-6, -3)$ **55.** $J(9, 4)$, $K(2, 5)$

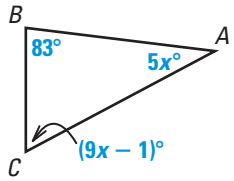
56. $L(-2, -3)$, $M(1, 10)$ **57.** $S(-4, 5)$, $T(2, -2)$ **58.** $Y(-1, 6)$, $Z(5, -5)$

FINDING ANGLE MEASURES **Find the value of *x*.** (Review 4.1 for 8.4)

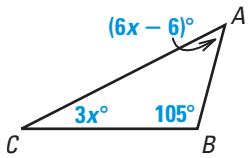
59.



60.



61.



SOLVING PROPORTIONS **Solve the proportion.** (Review 8.1)

62. $\frac{x}{9} = \frac{6}{27}$ **63.** $\frac{4}{y} = \frac{2}{19}$ **64.** $\frac{5}{24} = \frac{25}{z}$

65. $\frac{4}{13} = \frac{b}{8}$ **66.** $\frac{11}{x+2} = \frac{9}{x}$ **67.** $\frac{3x+7}{5} = \frac{4x}{6}$

QUIZ 1

Self-Test for Lessons 8.1–8.3

Solve the proportions. (Lesson 8.1)

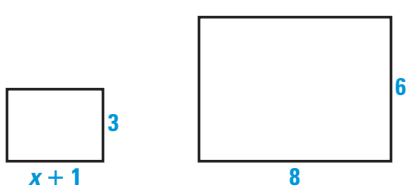
1. $\frac{p}{15} = \frac{2}{3}$ **2.** $\frac{5}{7} = \frac{20}{d}$ **3.** $\frac{4}{2x-6} = \frac{16}{x}$

Find the geometric mean of the two numbers. (Lesson 8.2)

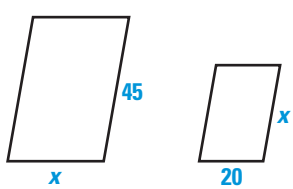
4. 7 and 63 **5.** 5 and 11 **6.** 10 and 7

In Exercises 7 and 8, the two polygons are similar. Find the value of *x*. Then find the scale factor and the ratio of the perimeters. (Lesson 8.3)

7.



8.



COMPARING PHOTO SIZES **Use the following information.** (Lesson 8.3)
You are ordering your school pictures. You decide to order one 8×10 (8 inches by 10 inches), two 5×7's (5 inches by 7 inches), and 24 wallets $\left(2\frac{1}{4}\text{ inches by } 3\frac{1}{4}\text{ inches}\right)$.

9. Are any of these sizes similar to each other?

10. Suppose you want the wallet photos to be similar to the 8×10 photo. If the wallet photo were $2\frac{1}{2}$ inches wide, how tall would it be?

8.4 Similar Triangles

What you should learn

GOAL 1 Identify similar triangles.

GOAL 2 Use similar triangles in **real-life** problems, such as using shadows to determine the height of the Great Pyramid in **Ex. 55**.

Why you should learn it

▼ To solve **real-life** problems, such as using similar triangles to understand aerial photography in **Example 4**.

REAL LIFE

GOAL 1 IDENTIFYING SIMILAR TRIANGLES

In this lesson, you will continue the study of similar polygons by looking at properties of similar triangles. The activity that follows Example 1 allows you to explore one of these properties.

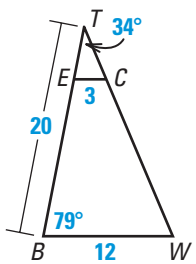
EXAMPLE 1 *Writing Proportionality Statements*

In the diagram, $\triangle BTW \sim \triangle ETC$.

a. Write the statement of proportionality.

b. Find $m\angle TEC$.

c. Find ET and BE.



SOLUTION

a. $\dfrac{ET}{BT} = \dfrac{TC}{TW} = \dfrac{CE}{WB}$

b. $\angle B \cong \angle TEC$, so $m\angle TEC = 79°$.

c.

$$\frac{CE}{WB} = \frac{ET}{BT} \quad \textbf{Write proportion.}$$

$$\frac{3}{12} = \frac{ET}{20} \quad \textbf{Substitute.}$$

$$\frac{3(20)}{12} = ET \quad \textbf{Multiply each side by 20.}$$

$$5 = ET \quad \textbf{Simplify.}$$

Because $BE = BT - ET$, $BE = 20 - 5 = 15$.

▶ So, ET is 5 units and BE is 15 units.

ACTIVITY Developing Concepts

Investigating Similar Triangles

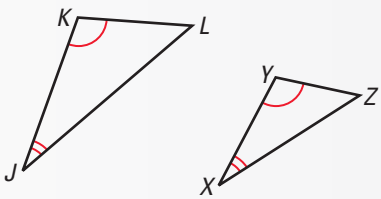
Use a protractor and a ruler to draw two noncongruent triangles so that each triangle has a 40° angle and a 60° angle. Check your drawing by measuring the third angle of each triangle—it should be 80°. Why? Measure the lengths of the sides of the triangles and compute the ratios of the lengths of corresponding sides. Are the triangles similar?

POSTULATE

POSTULATE 25 ***Angle-Angle (AA) Similarity Postulate***

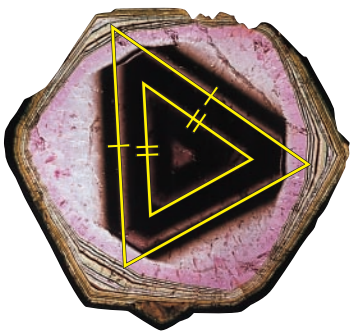
If two angles of one triangle are congruent to two angles of another triangle, then the two triangles are similar.

If $\angle JKL \cong \angle XYZ$ and $\angle KJL \cong \angle YXZ$, then $\triangle JKL \sim \triangle XYZ$.



EXAMPLE 2 *Proving that Two Triangles are Similar*

Color variations in the tourmaline crystal shown lie along the sides of isosceles triangles. In the triangles each vertex angle measures 52°. Explain why the triangles are similar.



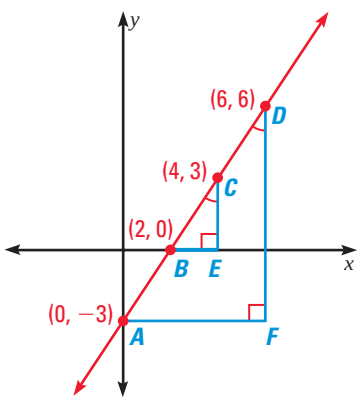
SOLUTION

Because the triangles are isosceles, you can determine that each base angle is 64°. Using the AA Similarity Postulate, you can conclude that the triangles are similar.

EXAMPLE 3 *Why a Line Has Only One Slope*

Using Algebra

Use properties of similar triangles to explain why any two points on a line can be used to calculate the slope. Find the slope of the line using both pairs of points shown.



SOLUTION

By the AA Similarity Postulate $\triangle BEC \sim \triangle AFD$, so the ratios of corresponding sides are the same. In particular, $\frac{CE}{DF} = \frac{BE}{AF}$.

By a property of proportions, $\frac{CE}{BE} = \frac{DF}{AF}$.

STUDENT HELP

Look Back
For help with finding slope, see p. 165.

The slope of a line is the ratio of the change in y to the corresponding change in x. The ratios $\frac{CE}{BE}$ and $\frac{DF}{AF}$ represent the slopes of $\overline{BC}$ and $\overline{AD}$, respectively. Because the two slopes are equal, any two points on a line can be used to calculate its slope. You can verify this with specific values from the diagram.

$$\text{slope of } \overline{BC} = \frac{3-0}{4-2} = \frac{3}{2}$$

$$\text{slope of } \overline{AD} = \frac{6-(-3)}{6-0} = \frac{9}{6} = \frac{3}{2}$$

FOCUS ON CAREERS

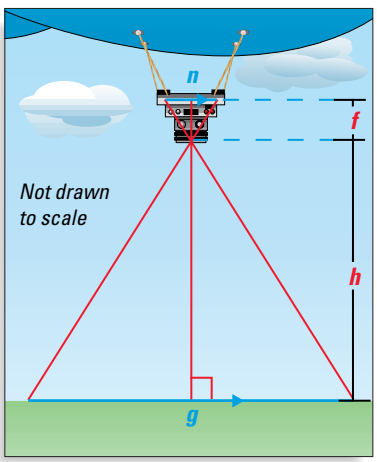
AERIAL PHOTOGRAPHER
An aerial photographer can take photos from a plane or using a remote-controlled blimp as discussed in Example 4.

CAREER LINK
www.mcdougallittell.com

GOAL 2 USING SIMILAR TRIANGLES IN REAL LIFE

EXAMPLE 4 *Using Similar Triangles*

AERIAL PHOTOGRAPHY Low-level aerial photos can be taken using a remote-controlled camera suspended from a blimp. You want to take an aerial photo that covers a ground distance g of 50 meters. Use the proportion $\frac{f}{h} = \frac{n}{g}$ to estimate the altitude h that the blimp should fly at to take the photo. In the proportion, use $f = 8$ cm and $n = 3$ cm. These two variables are determined by the type of camera used.



SOLUTION

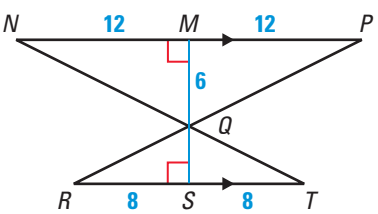
$\frac{f}{h} = \frac{n}{g}$	**Write proportion.**
$\frac{8 \text{ cm}}{h} = \frac{3 \text{ cm}}{50 \text{ m}}$	**Substitute.**
$3h = 400$	**Cross product property**
$h \approx 133$	**Divide each side by 3.**

▶ The blimp should fly at an altitude of about 133 meters to take a photo that covers a ground distance of 50 meters.

In Lesson 8.3, you learned that the perimeters of similar polygons are in the same ratio as the lengths of the corresponding sides. This concept can be generalized as follows. If two polygons are similar, then the ratio of *any two corresponding lengths* (such as altitudes, medians, angle bisector segments, and diagonals) is equal to the scale factor of the similar polygons.

EXAMPLE 5 *Using Scale Factors*

Find the length of the altitude $\overline{QS}$.



SOLUTION

Find the scale factor of $\triangle NQP$ to $\triangle TQR$.

$$\frac{NP}{TR} = \frac{12 + 12}{8 + 8} = \frac{24}{16} = \frac{3}{2}$$

Now, because the ratio of the lengths of the altitudes is equal to the scale factor, you can write the following equation.

$$\frac{QM}{QS} = \frac{3}{2}$$

▶ Substitute 6 for QM and solve for QS to show that $QS = 4$.

STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

GUIDED PRACTICE

Vocabulary Check ✓

1. If $\triangle ABC \sim \triangle XYZ$, $AB = 6$, and $XY = 4$, what is the *scale factor* of the triangles?

Concept Check ✓

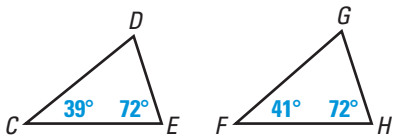
2. The points $A(2, 3)$, $B(-1, 6)$, $C(4, 1)$, and $D(0, 5)$ lie on a line. Which two points could be used to calculate the slope of the line? Explain.

3. Can you assume that corresponding sides and corresponding angles of any two similar triangles are congruent?

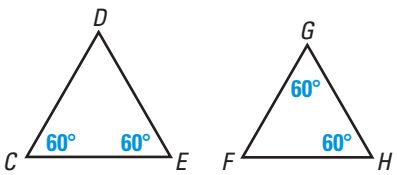
Skill Check ✓

Determine whether $\triangle CDE \sim \triangle FGH$.

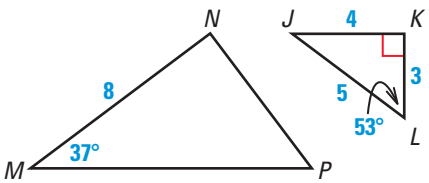
4.



5.



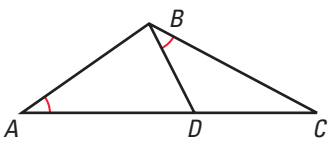
In the diagram shown $\triangle JKL \sim \triangle MNP$.



6. Find $m\angle J$, $m\angle N$, and $m\angle P$.

7. Find MP and PN.

8. Given that $\angle CAB \cong \angle CBD$, how do you know that $\triangle ABC \sim \triangle BDC$? Explain your answer.



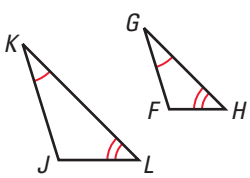
PRACTICE AND APPLICATIONS

STUDENT HELP

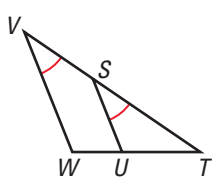
Extra Practice to help you master skills is on p. 818.

USING SIMILARITY STATEMENTS **The triangles shown are similar. List all the pairs of congruent angles and write the statement of proportionality.**

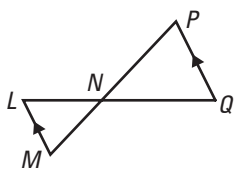
9.



10.



11.



LOGICAL REASONING **Use the diagram to complete the following.**

12. $\triangle PQR \sim$ ___?___

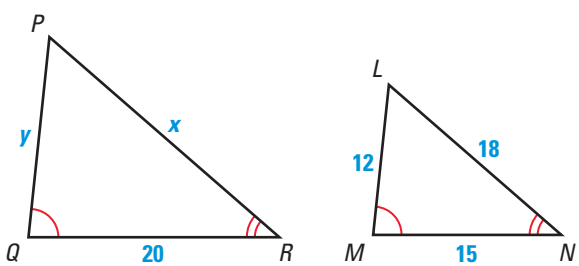
13. $\dfrac{PQ}{?} = \dfrac{QR}{?} = \dfrac{RP}{?}$

14. $\dfrac{20}{?} = \dfrac{?}{12}$

15. $\dfrac{?}{20} = \dfrac{18}{?}$

16. $y =$ ___?___

17. $x =$ ___?___



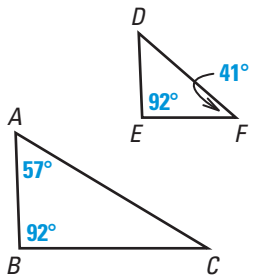
STUDENT HELP

HOMEWORK HELP

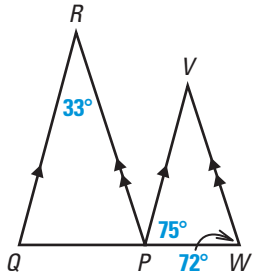
Example 1: Exs. 9–17, 33–38
Example 2: Exs. 18–26
Example 3: Exs. 27–32
Example 4: Exs. 39–44, 53, 55, 56
Example 5: Exs. 45–47

DETERMINING SIMILARITY **Determine whether the triangles can be proved similar. If they are similar, write a similarity statement. If they are not similar, explain why.**

18.



19.



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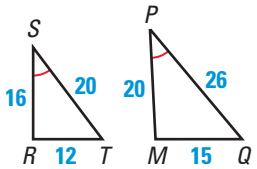
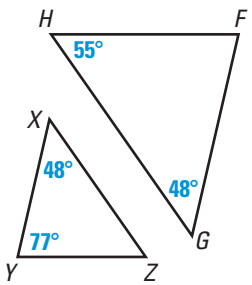


21.



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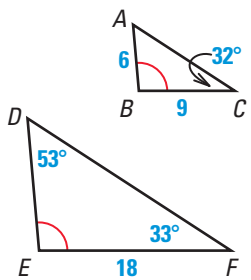
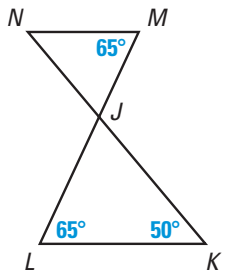


23.



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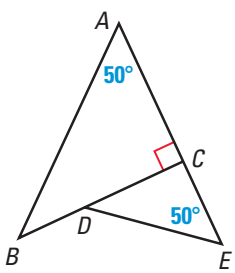

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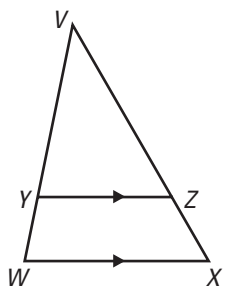
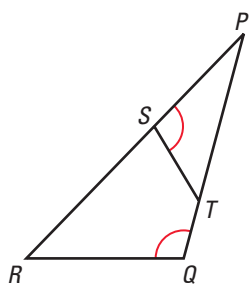
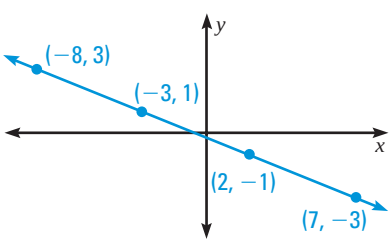


26.

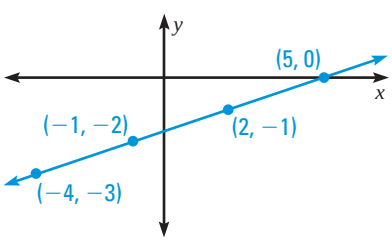


USING ALGEBRA **Using the labeled points, find the slope of the line. To verify your answer, choose another pair of points and find the slope using the new points. Compare the results.**

27.



28.



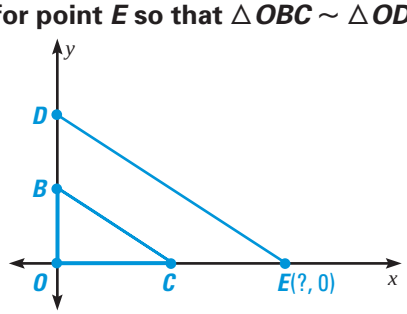
USING ALGEBRA **Find coordinates for point *E* so that $\triangle OBC \sim \triangle ODE$.**

29. $O(0, 0)$, $B(0, 3)$, $C(6, 0)$, $D(0, 5)$

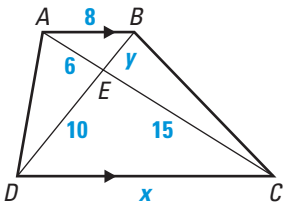
30. $O(0, 0)$, $B(0, 4)$, $C(3, 0)$, $D(0, 7)$

31. $O(0, 0)$, $B(0, 1)$, $C(5, 0)$, $D(0, 6)$

32. $O(0, 0)$, $B(0, 8)$, $C(4, 0)$, $D(0, 9)$



USING ALGEBRA You are given that *ABCD* is a trapezoid, $AB = 8$, $AE = 6$, $EC = 15$, and $DE = 10$.



33. $\triangle ABE \sim \triangle$ ___?___

34. $\frac{AB}{?} = \frac{AE}{?} = \frac{BE}{?}$

35. $\frac{6}{?} = \frac{8}{?}$

36. $\frac{15}{?} = \frac{10}{?}$

37. $x =$ ___?___

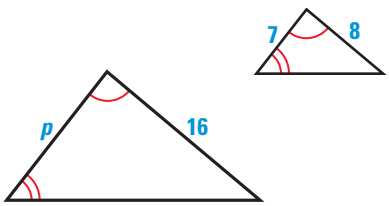
38. $y =$ ___?___

STUDENT HELP

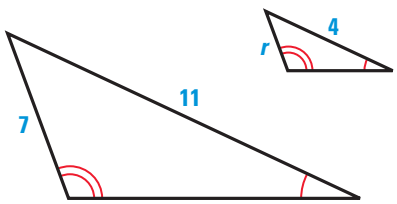
HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 39–44.

SIMILAR TRIANGLES The triangles are similar. Find the value of the variable.

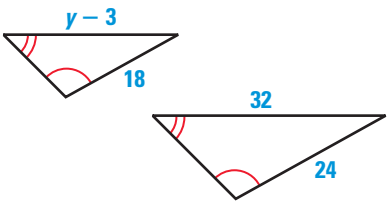
39.



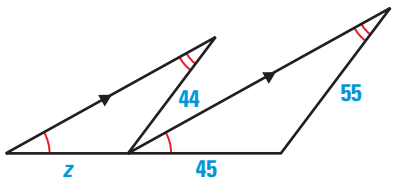
40.



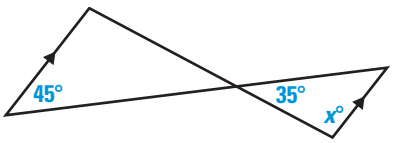
41.



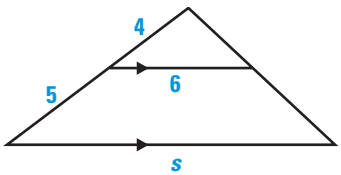
42.



43.

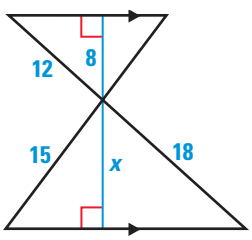


44.

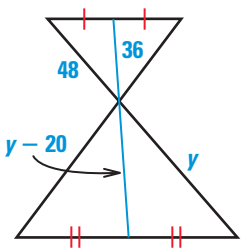


SIMILAR TRIANGLES The segments in blue are special segments in the similar triangles. Find the value of the variable.

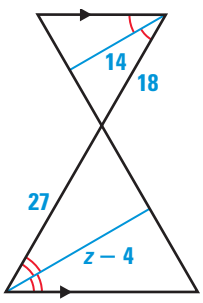
45.



46.



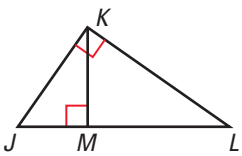
47.



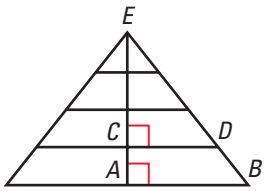
48. **PROOF** Write a paragraph or two-column proof.

GIVEN ▶ $\overline{KM} \perp \overline{JL}$, $\overline{JK} \perp \overline{KL}$

PROVE ▶ $\triangle JKL \sim \triangle JMK$



49. **PROOF** Write a paragraph proof or a two-column proof. The National Humanities Center is located in Research Triangle Park in North Carolina. Some of its windows consist of nested right triangles, as shown in the diagram. Prove that $\triangle ABE \sim \triangle CDE$.



GIVEN ▶ $\angle ECD$ is a right angle, $\angle EAB$ is a right angle.

PROVE ▶ $\triangle ABE \sim \triangle CDE$

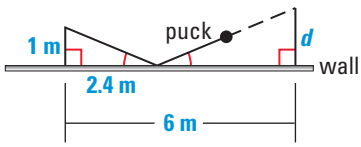
LOGICAL REASONING **In Exercises 50–52, decide whether the statement is *true* or *false*. Explain your reasoning.**

50. If an acute angle of a right triangle is congruent to an acute angle of another right triangle, then the triangles are similar.

51. Some equilateral triangles are not similar.

52. All isosceles triangles with a 40° vertex angle are similar.

53. **ICE HOCKEY** A hockey player passes the puck to a teammate by bouncing the puck off the wall of the rink as shown. From physics, the angles that the path of the puck makes with the wall are congruent. How far from the wall will the pass be picked up by his teammate?

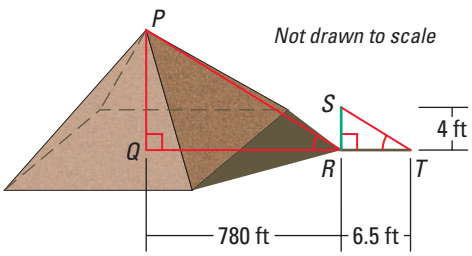


STUDENT HELP

SOFTWARE HELP Visit our Web site www.mcdougallittell.com to see instructions for several software applications.

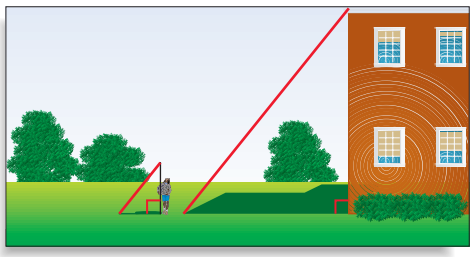
54. **TECHNOLOGY** Use geometry software to verify that any two points on a line can be used to calculate the slope of the line. Draw a line k with a negative slope in a coordinate plane. Draw two right triangles of different size whose hypotenuses lie along line k and whose other sides are parallel to the x- and y-axes. Calculate the slope of each triangle by finding the ratio of the vertical side length to the horizontal side length. Are the slopes equal?

55. **THE GREAT PYRAMID** The Greek mathematician Thales (640–546 B.C.) calculated the height of the Great Pyramid in Egypt by placing a rod at the tip of the pyramid's shadow and using similar triangles.



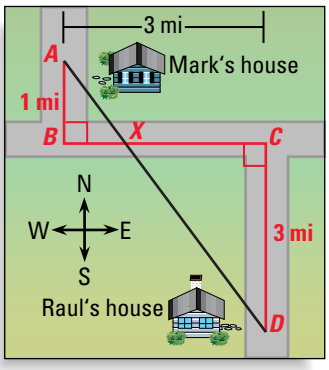
In the figure, $\overline{PQ} \perp \overline{QT}$, $\overline{SR} \perp \overline{QT}$, and $\overline{PR} \parallel \overline{ST}$. Write a paragraph proof to show that the height of the pyramid is 480 feet.

56. **ESTIMATING HEIGHT** On a sunny day, use a rod or pole to estimate the height of your school building. Use the method that Thales used to estimate the height of the Great Pyramid in Exercise 55.



Test Preparation

57. MULTI-STEP PROBLEM Use the following information.
Going from his own house to Raul's house, Mark drives due south one mile, due east three miles, and due south again three miles. What is the distance between the two houses as the crow flies?



a. Explain how to prove that $\triangle ABX \sim \triangle DCX$.

b. Use corresponding side lengths of the triangles to calculate BX.

c. Use the Pythagorean Theorem to calculate AX, and then DX. Then find AD.

d. *Writing* Using the properties of rectangles, explain a way that a point E could be added to the diagram so that $\overline{AD}$ would be the hypotenuse of $\triangle AED$, and $\overline{AE}$ and $\overline{ED}$ would be its legs of known length.

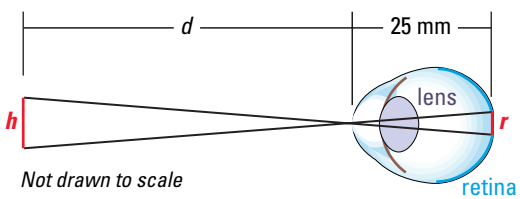
★ Challenge

HUMAN VISION In Exercises 58–60, use the following information.
The diagram shows how similar triangles relate to human vision. An image similar to a viewed object appears on the retina. The actual height of the object h is proportional to the size of the image as it appears on the retina r. In the same manner, the distances from the object to the lens of the eye d and from the lens to the retina, 25 mm in the diagram, are also proportional.

58. Write a proportion that relates r, d, h, and 25 mm.

59. An object that is 10 meters away appears on the retina as 1 mm tall. Find the height of the object.

60. An object that is 1 meter tall appears on the retina as 1 mm tall. How far away is the object?



EXTRA CHALLENGE
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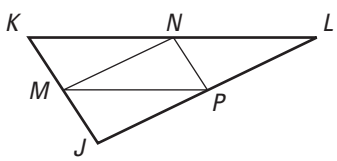
MIXED REVIEW

61. USING THE DISTANCE FORMULA Find the distance between the points $A(-17, 12)$ and $B(14, -21)$. **(Review 1.3)**

TRIANGLE MIDSEGMENTS *M, N,* and *P* are the midpoints of the sides of $\triangle JKL$. Complete the statement. **(Review 5.4 for 8.5)**



62. $\overline{NP} \parallel$ ___?___

63. If $NP = 23$, then $KJ =$ ___?___.

64. If $KN = 16$, then $MP =$ ___?___.

65. If $JL = 24$, then $MN =$ ___?___.

PROPORTIONS Solve the proportion. **(Review 8.1)**

66. $\frac{x}{12} = \frac{3}{8}$

67. $\frac{3}{y} = \frac{12}{32}$

68. $\frac{17}{x} = \frac{11}{33}$

69. $\frac{34}{11} = \frac{x+6}{3}$

70. $\frac{23}{24} = \frac{x}{72}$

71. $\frac{8}{x} = \frac{x}{32}$

8.5 Proving Triangles are Similar

What you should learn

GOAL 1 Use similarity theorems to prove that two triangles are similar.

GOAL 2 Use similar triangles to solve **real-life** problems, such as finding the height of a climbing wall in **Example 5**.

Why you should learn it

▼ To solve **real-life** problems, such as estimating the height of the Unisphere in **Ex. 29**.

REAL LIFE

GOAL 1 USING SIMILARITY THEOREMS

In this lesson, you will study two additional ways to prove that two triangles are similar: the Side-Side-Side (SSS) Similarity Theorem and the Side-Angle-Side (SAS) Similarity Theorem. The first theorem is proved in Example 1 and you are asked to prove the second theorem in Exercise 31.

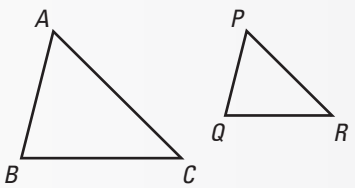
THEOREMS

THEOREM 8.2 ***Side-Side-Side (SSS) Similarity Theorem***

If the lengths of the corresponding sides of two triangles are proportional, then the triangles are similar.

If $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$,

then $\triangle ABC \sim \triangle PQR$.

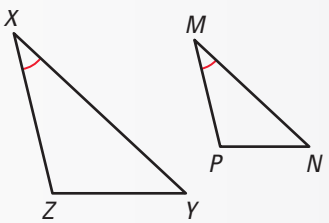


THEOREM 8.3 ***Side-Angle-Side (SAS) Similarity Theorem***

If an angle of one triangle is congruent to an angle of a second triangle and the lengths of the sides including these angles are proportional, then the triangles are similar.

If $\angle X \cong \angle M$ and $\frac{ZX}{PM} = \frac{XY}{MN}$,

then $\triangle XYZ \sim \triangle MNP$.

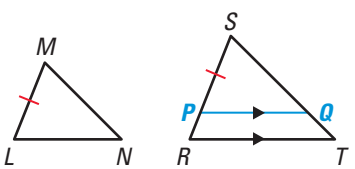


EXAMPLE 1 ***Proof of Theorem 8.2***

Proof

GIVEN ▶ $\frac{RS}{LM} = \frac{ST}{MN} = \frac{TR}{NL}$

PROVE ▶ $\triangle RST \sim \triangle LMN$



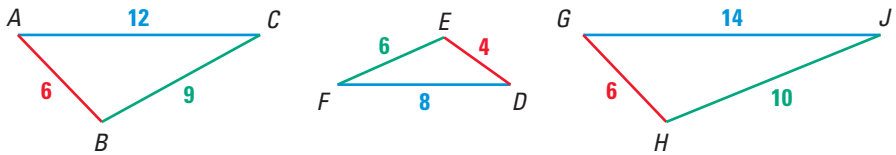
SOLUTION

Paragraph Proof Locate P on $\overline{RS}$ so that $PS = LM$. Draw $\overline{PQ}$ so that $\overline{PQ} \parallel \overline{RT}$. Then $\triangle RST \sim \triangle PSQ$, by the AA Similarity Postulate, and $\frac{RS}{PS} = \frac{ST}{SQ} = \frac{TR}{QP}$. Because $PS = LM$, you can substitute in the given proportion and find that $SQ = MN$ and $QP = NL$. By the SSS Congruence Theorem, it follows that $\triangle PSQ \cong \triangle LMN$. Finally, use the definition of congruent triangles and the AA Similarity Postulate to conclude that $\triangle RST \sim \triangle LMN$.

EXAMPLE 2 *Using the SSS Similarity Theorem*

Logical Reasoning

Which of the following three triangles are similar?



STUDENT HELP

Study Tip
Note that when using the SSS Similarity Theorem it is useful to compare the shortest sides, the longest sides, and then the remaining sides.

SOLUTION

To decide which, if any, of the triangles are similar, you need to consider the ratios of the lengths of corresponding sides.

Ratios of Side Lengths of $\triangle ABC$ *and* $\triangle DEF$

$\frac{AB}{DE} = \frac{6}{4} = \frac{3}{2}$, $\frac{CA}{FD} = \frac{12}{8} = \frac{3}{2}$, $\frac{BC}{EF} = \frac{9}{6} = \frac{3}{2}$

Shortest sides **Longest sides** **Remaining sides**

▶ Because all of the ratios are equal, $\triangle ABC \sim \triangle DEF$.

Ratios of Side Lengths of $\triangle ABC$ *and* $\triangle GHJ$

$\frac{AB}{GH} = \frac{6}{6} = 1$, $\frac{CA}{JG} = \frac{12}{14} = \frac{6}{7}$, $\frac{BC}{HJ} = \frac{9}{10}$

Shortest sides **Longest sides** **Remaining sides**

▶ Because the ratios are not equal, $\triangle ABC$ and $\triangle GHJ$ are not similar.

Since $\triangle ABC$ is similar to $\triangle DEF$ and $\triangle ABC$ is not similar to $\triangle GHJ$, $\triangle DEF$ is not similar to $\triangle GHJ$.

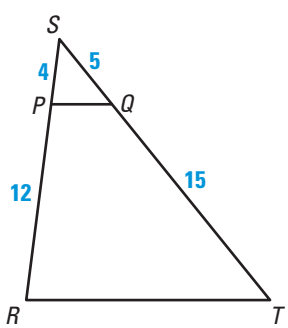
EXAMPLE 3 *Using the SAS Similarity Theorem*

Use the given lengths to prove that $\triangle RST \sim \triangle PSQ$.

SOLUTION

GIVEN ▶ $SP = 4$, $PR = 12$, $SQ = 5$, $QT = 15$

PROVE ▶ $\triangle RST \sim \triangle PSQ$



Paragraph Proof Use the SAS Similarity Theorem. Begin by finding the ratios of the lengths of the corresponding sides.

$$\frac{SR}{SP} = \frac{SP + PR}{SP} = \frac{4 + 12}{4} = \frac{16}{4} = 4$$

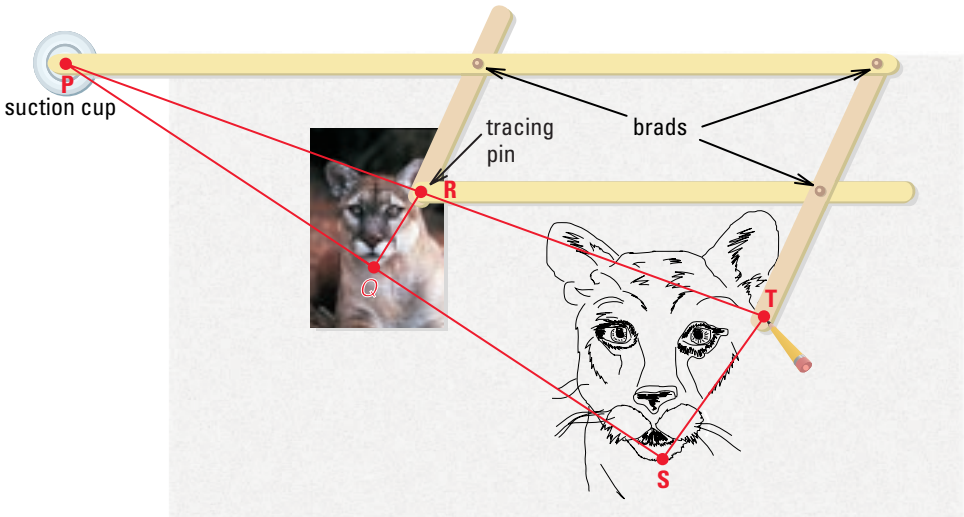
$$\frac{ST}{SQ} = \frac{SQ + QT}{SQ} = \frac{5 + 15}{5} = \frac{20}{5} = 4$$

So, the lengths of sides $\overline{SR}$ and $\overline{ST}$ are proportional to the lengths of the corresponding sides of $\triangle PSQ$. Because $\angle S$ is the included angle in both triangles, use the SAS Similarity Theorem to conclude that $\triangle RST \sim \triangle PSQ$.

GOAL 2 USING SIMILAR TRIANGLES IN REAL LIFE

EXAMPLE 4 *Using a Pantograph*

SCALE DRAWING As you move the tracing pin of a *pantograph* along a figure, the pencil attached to the far end draws an enlargement. As the pantograph expands and contracts, the three brads and the tracing pin always form the vertices of a parallelogram. The ratio of PR to PT is always equal to the ratio of PQ to PS. Also, the suction cup, the tracing pin, and the pencil remain collinear.



a. How can you show that $\triangle PRQ \sim \triangle PTS$?

b. In the diagram, PR is 10 inches and RT is 10 inches. The length of the cat, RQ, in the original print is 2.4 inches. Find the length TS in the enlargement.

SOLUTION

a. You know that $\frac{PR}{PT} = \frac{PQ}{PS}$. Because $\angle P \cong \angle P$, you can apply the SAS Similarity Theorem to conclude that $\triangle PRQ \sim \triangle PTS$.

b. Because the triangles are similar, you can set up a proportion to find the length of the cat in the enlarged drawing.

$\frac{PR}{PT} = \frac{RQ}{TS}$ **Write proportion.**

$\frac{10}{20} = \frac{2.4}{TS}$ **Substitute.**

$TS = 4.8$ **Solve for *TS*.**

▶ So, the length of the cat in the enlarged drawing is 4.8 inches.

FOCUS ON APPLICATIONS

PANTOGRAPH Before photocopiers, people used pantographs to make enlargements. As the tracing pin is guided over the figure, the pencil draws an enlargement.

Similar triangles can be used to find distances that are difficult to measure directly. One technique is called *Thales' shadow method* (page 486), named after the Greek geometer Thales who used it to calculate the height of the Great Pyramid.

FOCUS ON APPLICATIONS

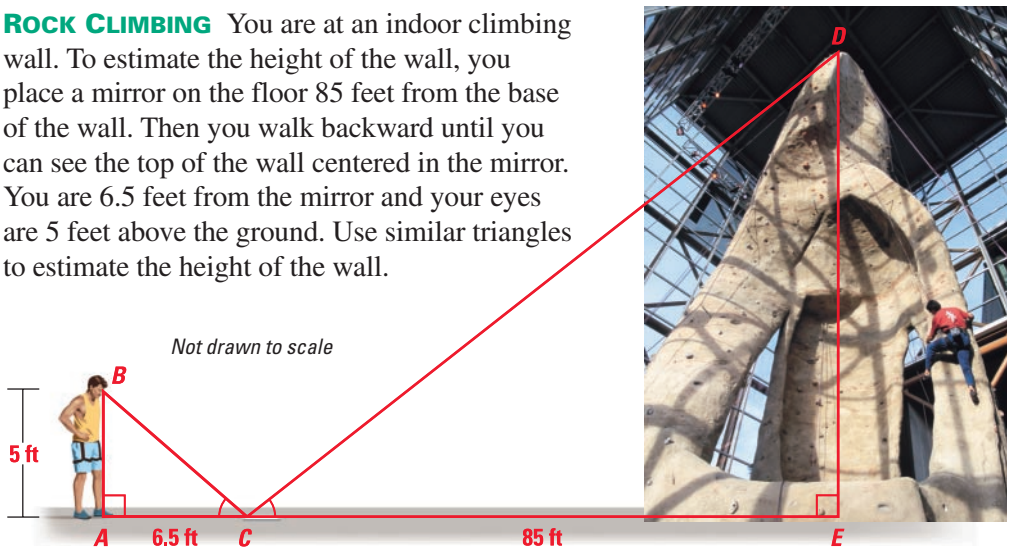
ROCK CLIMBING Interest in rock climbing appears to be growing. From 1988 to 1998, over 700 indoor rock climbing gyms opened in the United States.

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

EXAMPLE 5 *Finding Distance Indirectly*

ROCK CLIMBING You are at an indoor climbing wall. To estimate the height of the wall, you place a mirror on the floor 85 feet from the base of the wall. Then you walk backward until you can see the top of the wall centered in the mirror. You are 6.5 feet from the mirror and your eyes are 5 feet above the ground. Use similar triangles to estimate the height of the wall.



SOLUTION

Due to the reflective property of mirrors, you can reason that $\angle ACB \cong \angle ECD$. Using the fact that $\triangle ABC$ and $\triangle EDC$ are right triangles, you can apply the AA Similarity Postulate to conclude that these two triangles are similar.

$\frac{DE}{BA} = \frac{EC}{AC}$ **Ratios of lengths of corresponding sides are equal.**

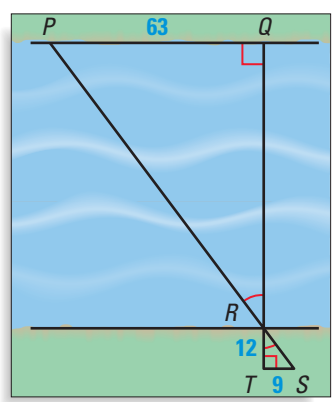
$\frac{DE}{5} = \frac{85}{6.5}$ **Substitute.**

$65.38 \approx DE$ **Multiply each side by 5 and simplify.**

▶ So, the height of the wall is about 65 feet.

EXAMPLE 6 *Finding Distance Indirectly*

INDIRECT MEASUREMENT To measure the width of a river, you use a surveying technique, as shown in the diagram. Use the given lengths (measured in feet) to find RQ.



SOLUTION

By the AA Similarity Postulate, $\triangle PQR \sim \triangle STR$.

$\frac{RQ}{RT} = \frac{PQ}{ST}$ **Write proportion.**

$\frac{RQ}{12} = \frac{63}{9}$ **Substitute.**

$RQ = 12 \cdot 7$ **Multiply each side by 12.**

$RQ = 84$ **Simplify.**

▶ So, the river is 84 feet wide.

GUIDED PRACTICE

Vocabulary Check ✓

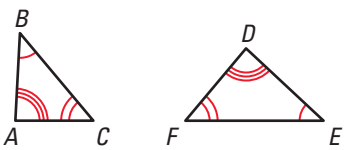
1. You want to prove that $\triangle FHG$ is similar to $\triangle RXS$ by the SSS Similarity Theorem. Complete the proportion that is needed to use this theorem.

$$\frac{FH}{?} = \frac{?}{XS} = \frac{FG}{?}$$

Concept Check ✓

Name a postulate or theorem that can be used to prove that the two triangles are similar. Then, write a similarity statement.

2.



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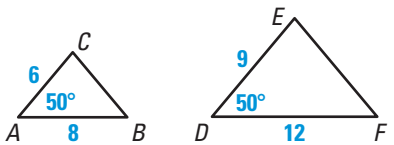
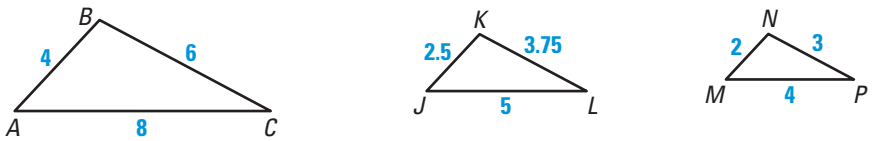


Skill Check ✓

4. Which triangles are similar to $\triangle ABC$? Explain.



5. The side lengths of $\triangle ABC$ are 2, 5, and 6, and $\triangle DEF$ has side lengths of 12, 30, and 36. Find the ratios of the lengths of the corresponding sides of $\triangle ABC$ to $\triangle DEF$. Are the two triangles similar? Explain.

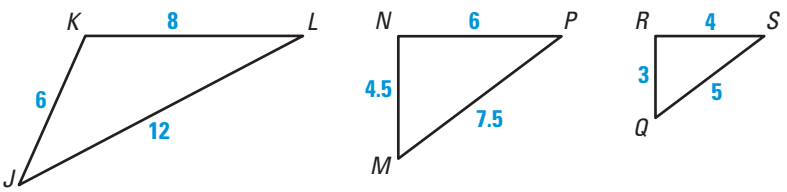
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 818.

DETERMINING SIMILARITY **In Exercises 6–8, determine which two of the three given triangles are similar. Find the scale factor for the pair.**

6.



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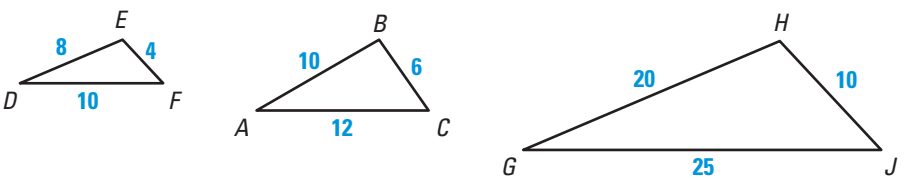
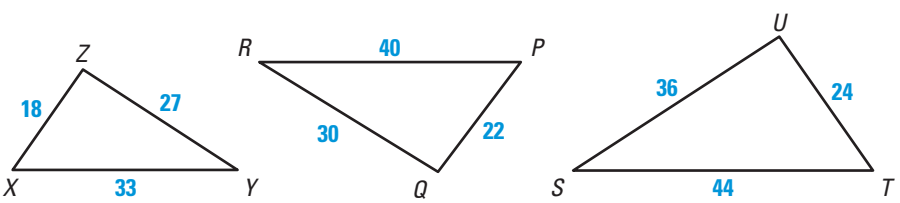


8.



STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 30, 31
Example 2: Exs. 6–18

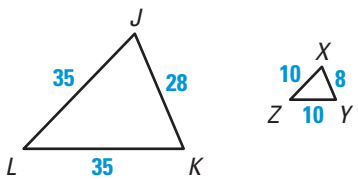
STUDENT HELP

HOMEWORK HELP

Example 3: Exs. 6–18, 30, 31
Example 4: Exs. 19–26, 29, 32–35
Example 5: Exs. 29, 32–35
Example 6: Exs. 29, 32–35

DETERMINING SIMILARITY **Are the triangles similar? If so, state the similarity and the postulate or theorem that justifies your answer.**

9.



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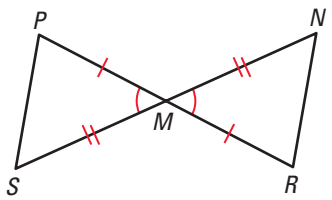

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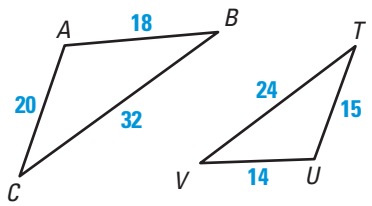

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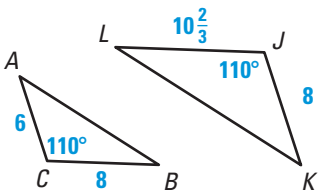

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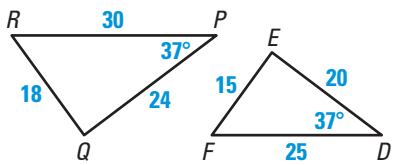

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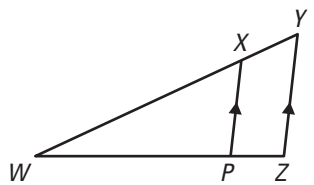


LOGICAL REASONING **Draw the given triangles roughly to scale. Then, name a postulate or theorem that can be used to prove that the triangles are similar.**

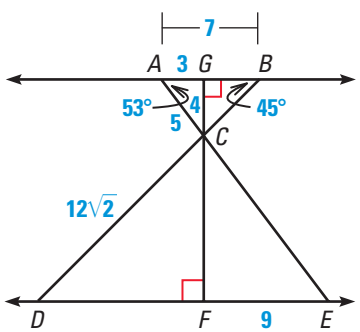
15. The side lengths of $\triangle PQR$ are 16, 8, and 18, and the side lengths of $\triangle XYZ$ are 9, 8, and 4.

16. In $\triangle ABC$, $m\angle A = 28°$ and $m\angle B = 62°$. In $\triangle DEF$, $m\angle D = 28°$ and $m\angle F = 90°$.

17. In $\triangle STU$, the length of $\overline{ST}$ is 18, the length of $\overline{SU}$ is 24, and $m\angle S = 65°$. The length of $\overline{JK}$ is 6, $m\angle J = 65°$, and the length of $\overline{JL}$ is 8 in $\triangle JKL$.

18. The ratio of VW to MN is 6 to 1. In $\triangle VWX$, $m\angle W = 30°$, and in $\triangle MNP$, $m\angle N = 30°$. The ratio of WX to NP is 6 to 1.

FINDING MEASURES AND LENGTHS **Use the diagram shown to complete the statements.**



19. $m\angle CED =$ __?__.

20. $m\angle EDC =$ __?__.

21. $m\angle DCE =$ __?__.

22. $FC =$ __?__.

23. $EC =$ __?__.

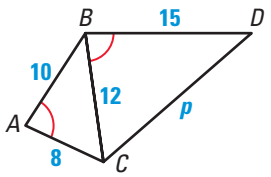
24. $DE =$ __?__.

25. $CB =$ __?__.

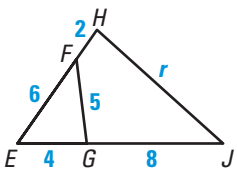
26. Name the three pairs of triangles that are similar in the figure.

DETERMINING SIMILARITY **Determine whether the triangles are similar. If they are, write a similarity statement and solve for the variable.**

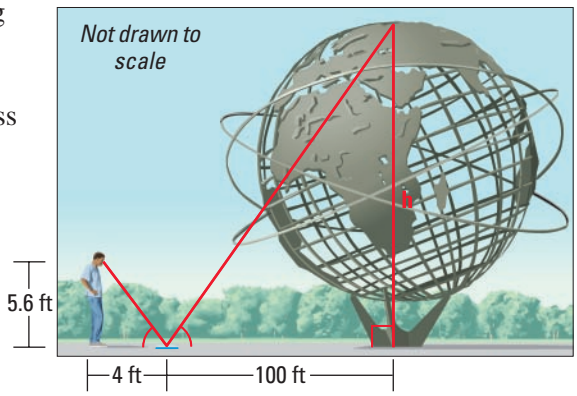
27.



28.



29. **UNISPHERE** You are visiting the Unisphere at Flushing Meadow Park in New York. To estimate the height of the stainless steel model of Earth, you place a mirror on the ground and stand where you can see the top of the model in the mirror. Use the diagram shown to estimate the height of the model.

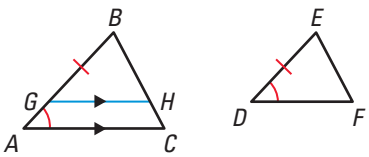


30. **PARAGRAPH PROOF** Two isosceles triangles are similar if the vertex angle of one triangle is congruent to the vertex angle of the other triangle. Write a paragraph proof of this statement and include a labeled figure.

31. **PARAGRAPH PROOF** Write a paragraph proof of Theorem 8.3.

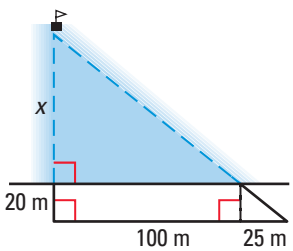
GIVEN ▶ $\angle A \cong \angle D$, $\frac{AB}{DE} = \frac{AC}{DF}$

PROVE ▶ $\triangle ABC \sim \triangle DEF$

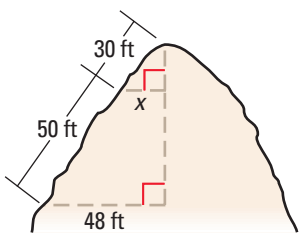


FINDING DISTANCES INDIRECTLY **Find the distance labeled *x*.**

32.

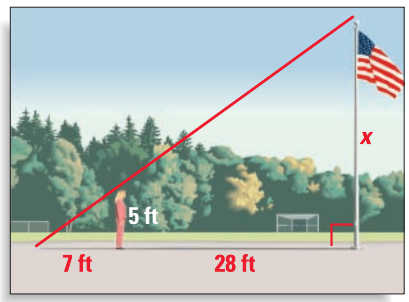


33.



FLAGPOLE HEIGHT **In Exercises 34 and 35, use the following information.**

Julia uses the shadow of the flagpole to estimate its height. She stands so that the tip of her shadow coincides with the tip of the flagpole's shadow as shown. Julia is 5 feet tall. The distance from the flagpole to Julia is 28 feet and the distance between the tip of the shadows and Julia is 7 feet.

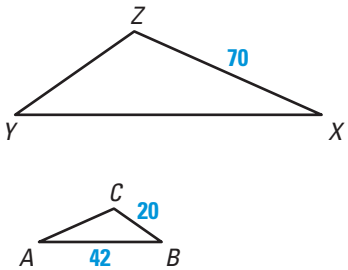


34. Calculate the height of the flagpole.

35. Explain why Julia's shadow method works.

Test Preparation

QUANTITATIVE COMPARISON In Exercises 36 and 37, use the diagram, in which $\triangle ABC \sim \triangle XYZ$, and the ratio $AB:XY$ is 2:5. Choose the statement that is true about the given quantities.



Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
36.	The perimeter of $\triangle ABC$	The length XY
37.	The distance $XY + BC$	The distance $XZ + YZ$

★ Challenge

EXTRA CHALLENGE
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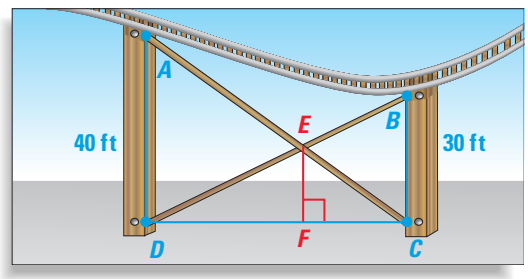
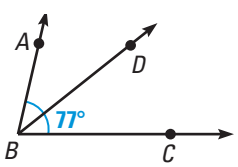
38. DESIGNING THE LOOP A portion of an amusement park ride called the Loop is shown. Find the length of $\overline{EF}$. (*Hint:* Use similar triangles.)



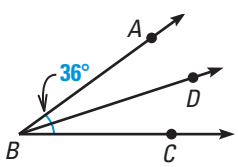
MIXED REVIEW

ANALYZING ANGLE BISECTORS $\overrightarrow{BD}$ is the angle bisector of $\angle ABC$. Find any angle measures not given in the diagram. (Review 1.5 for 8.6)

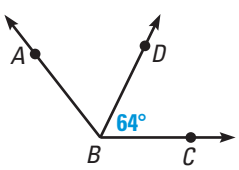
39.



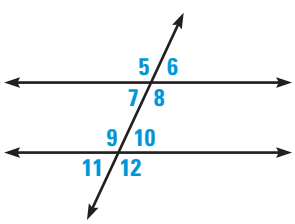
40.



41.



RECOGNIZING ANGLES Use the diagram shown to complete the statement. (Review 3.1 for 8.6)



42. $\angle 5$ and __?__ are alternate exterior angles.

43. $\angle 8$ and __?__ are consecutive interior angles.

44. $\angle 10$ and __?__ are alternate interior angles.

45. $\angle 9$ and __?__ are corresponding angles.

FINDING COORDINATES Find the coordinates of the image after the reflection without using a coordinate plane. (Review 7.2)

46. $T(0, 5)$ reflected in the x-axis

47. $P(-2, 7)$ reflected in the y-axis

48. $B(-3, -10)$ reflected in the y-axis

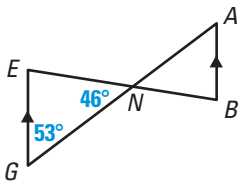
49. $C(-5, -1)$ reflected in the x-axis

QUIZ 2

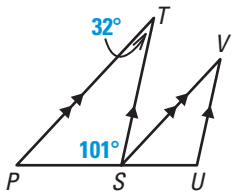
Self-Test for Lessons 8.4 and 8.5

Determine whether you can show that the triangles are similar. State any angle measures that are not given. (Lesson 8.4)

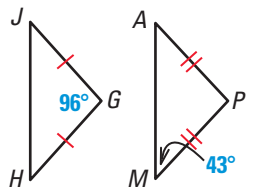
1.



2.



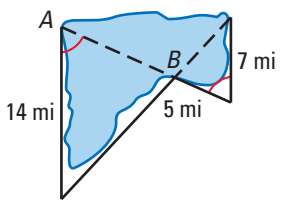
3.



In Exercises 4–6, you are given the ratios of the lengths of the sides of $\triangle DEF$. If $\triangle ABC$ has sides of lengths 3, 6, and 7 units, are the triangles similar? (Lesson 8.5)

4. $4:7:8$ **5.** $6:12:14$ **6.** $1:2:\frac{7}{3}$

7. **DISTANCE ACROSS WATER** Use the known distances in the diagram to find the distance across the lake from A to B. (Lesson 8.5)



MATH & History — The Golden Rectangle

APPLICATION LINK
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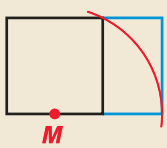
THEN

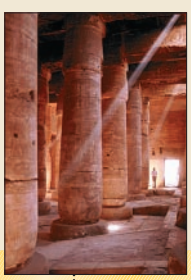
THOUSANDS OF YEARS AGO, Greek mathematicians became interested in the *golden ratio*, a ratio of about 1 : 1.618. A rectangle whose side lengths are in the golden ratio is called a *golden rectangle*. Such rectangles are believed to be especially pleasing to look at.

NOW

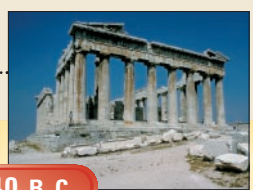
THE GOLDEN RATIO has been found in the proportions of many works of art and architecture, including the works shown in the timeline below.



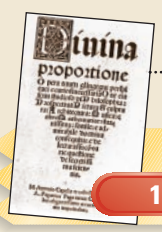
1. Follow the steps below to construct a golden rectangle. When you are done, check to see whether the ratio of the width to the length is 1 : 1.618.
   - Construct a square. Mark the midpoint M of the bottom side.
   - Place the compass point at M and draw an arc through the upper right corner of the square.
   - Extend the bottom side of the square to intersect with the arc. The intersection point is the corner of a golden rectangle. Complete the rectangle.



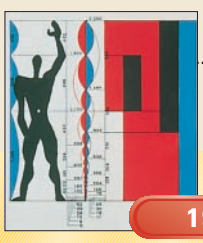
c. 1300 B.C. The Osirion (underground Egyptian temple)



c. 440 B.C. The Parthenon, Athens, Greece



1509 Leonardo da Vinci illustrates Luca Pacioli's book on the golden ratio.



1956 Le Corbusier uses golden ratios based on this human figure in his architecture.

8.6 Proportions and Similar Triangles

What you should learn

GOAL 1 Use proportionality theorems to calculate segment lengths.

GOAL 2 To solve **real-life** problems, such as determining the dimensions of a piece of land in **Exs. 29 and 30**.

Why you should learn it

▼ Model **real-life** situations using proportionality theorems, as in the construction problem in **Example 5**.

REAL LIFE

GOAL 1 USING PROPORTIONALITY THEOREMS

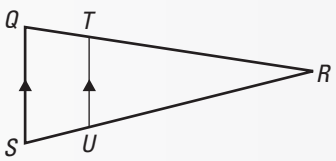
In this lesson, you will study four proportionality theorems. Similar triangles are used to prove each theorem. You are asked to prove the theorems in Exercises 31–33 and 38.

THEOREMS

THEOREM 8.4 ***Triangle Proportionality Theorem***

If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally.

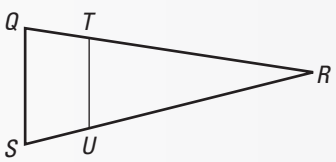
If $\overline{TU} \parallel \overline{QS}$, then $\frac{RT}{TQ} = \frac{RU}{US}$.



THEOREM 8.5 ***Converse of the Triangle Proportionality Theorem***

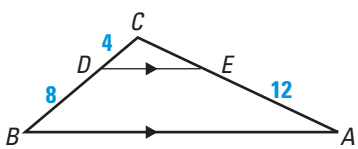
If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

If $\frac{RT}{TQ} = \frac{RU}{US}$, then $\overline{TU} \parallel \overline{QS}$.



EXAMPLE 1 *Finding the Length of a Segment*

In the diagram $\overline{AB} \parallel \overline{ED}$, $BD = 8$, $DC = 4$, and $AE = 12$. What is the length of $\overline{EC}$?



SOLUTION

$\frac{DC}{BD} = \frac{EC}{AE}$ **Triangle Proportionality Theorem**

$\frac{4}{8} = \frac{EC}{12}$ **Substitute.**

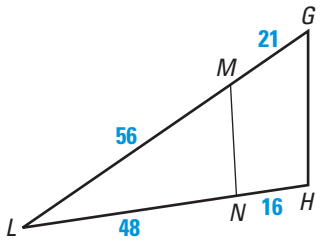
$\frac{4(12)}{8} = EC$ **Multiply each side by 12.**

$6 = EC$ **Simplify.**

▶ So, the length of $\overline{EC}$ is 6.

EXAMPLE 2 *Determining Parallels*

Given the diagram, determine whether $\overline{MN} \parallel \overline{GH}$.



SOLUTION

Begin by finding and simplifying the ratios of the two sides divided by $\overline{MN}$.

$$\frac{LM}{MG} = \frac{56}{21} = \frac{8}{3} \qquad \frac{LN}{NH} = \frac{48}{16} = \frac{3}{1}$$

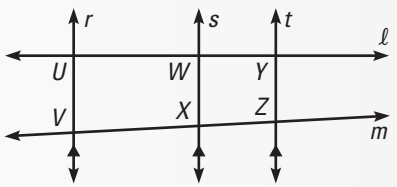
Because $\frac{8}{3} \neq \frac{3}{1}$, $\overline{MN}$ is not parallel to $\overline{GH}$.

THEOREMS

THEOREM 8.6

If three parallel lines intersect two transversals, then they divide the transversals proportionally.

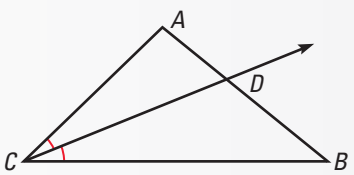
If $r \parallel s$ and $s \parallel t$, and ℓ and m intersect r, s, and t, then $\frac{UW}{WY} = \frac{VX}{XZ}$.



THEOREM 8.7

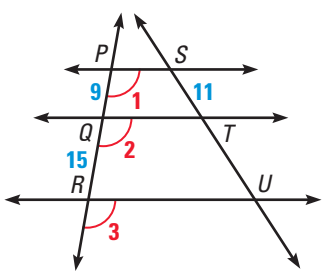
If a ray bisects an angle of a triangle, then it divides the opposite side into segments whose lengths are proportional to the lengths of the other two sides.

If $\overrightarrow{CD}$ bisects $\angle ACB$, then $\frac{AD}{DB} = \frac{CA}{CB}$.



EXAMPLE 3 *Using Proportionality Theorems*

In the diagram, $\angle 1 \cong \angle 2 \cong \angle 3$, and $PQ = 9$, $QR = 15$, and $ST = 11$. What is the length of $\overline{TU}$?



SOLUTION

Because corresponding angles are congruent the lines are parallel and you can use Theorem 8.6.

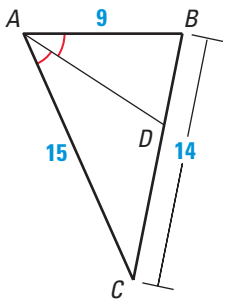
$\frac{PQ}{QR} = \frac{ST}{TU}$	**Parallel lines divide transversals proportionally.**
$\frac{9}{15} = \frac{11}{TU}$	**Substitute.**
$9 \cdot TU = 15 \cdot 11$	**Cross product property**
$TU = \frac{15(11)}{9} = \frac{55}{3}$	**Divide each side by 9 and simplify.**

▶ So, the length of $\overline{TU}$ is $\frac{55}{3}$, or $18\frac{1}{3}$.

Using Algebra

EXAMPLE 4 *Using Proportionality Theorems*

In the diagram, $\angle CAD \cong \angle DAB$. Use the given side lengths to find the length of $\overline{DC}$.



SOLUTION

Since $\overline{AD}$ is an angle bisector of $\angle CAB$, you can apply Theorem 8.7.

Let $x = DC$. Then, $BD = 14 - x$.

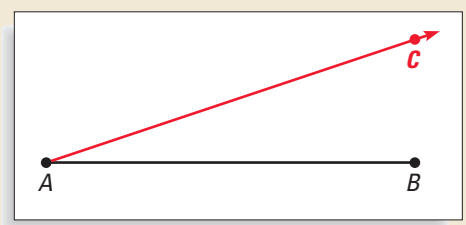
$\frac{AB}{AC} = \frac{BD}{DC}$	**Apply Theorem 8.7.**
$\frac{9}{15} = \frac{14 - x}{x}$	**Substitute.**
$9 \cdot x = 15(14 - x)$	**Cross product property**
$9x = 210 - 15x$	**Distributive property**
$24x = 210$	**Add $15x$ to each side.**
$x = 8.75$	**Divide each side by 24.**

▶ So, the length of $\overline{DC}$ is 8.75 units.

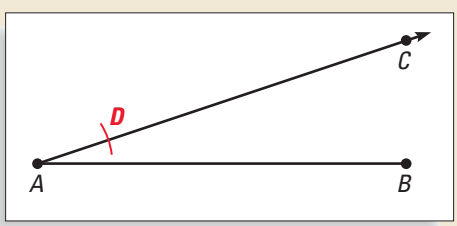
ACTIVITY
Construction

Dividing a Segment into Equal Parts (4 shown)

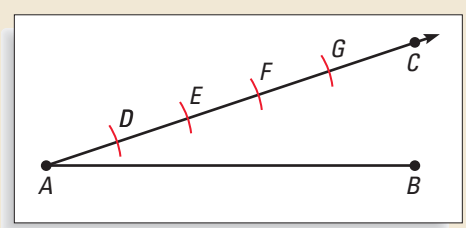
1. Draw a line segment that is about 3 inches long. Label the endpoints A and B. Choose any point C not on $\overleftrightarrow{AB}$. Draw $\overrightarrow{AC}$.



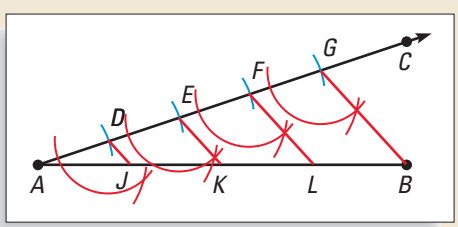
2. Using any length, place the compass point at A and make an arc intersecting $\overrightarrow{AC}$ at D.



3. Using the same compass setting, make additional arcs on $\overrightarrow{AC}$. Label the points E, F, and G so that $AD = DE = EF = FG$.



4. Draw $\overline{GB}$. Construct a line parallel to $\overline{GB}$ through D. Continue constructing parallel lines and label the points as shown. Explain why $AJ = JK = KL = LB$.



GOAL 2 USING PROPORTIONALITY THEOREMS IN REAL LIFE

EXAMPLE 5 *Finding the Length of a Segment*

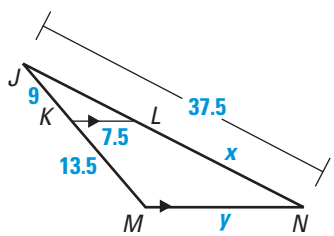
REAL LIFE **BUILDING CONSTRUCTION** You are insulating your attic, as shown. The vertical 2×4 studs are evenly spaced. Explain why the diagonal cuts at the tops of the strips of insulation should have the same lengths.

SOLUTION

Because the studs $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are each vertical, you know that they are parallel to each other. Using Theorem 8.6, you can conclude that $\frac{DE}{EF} = \frac{AB}{BC}$. Because the studs are evenly spaced, you know that $DE = EF$. So, you can conclude that $AB = BC$, which means that the diagonal cuts at the tops of the strips have the same lengths.

EXAMPLE 6 *Finding Segment Lengths*

In the diagram $\overline{KL} \parallel \overline{MN}$. Find the values of the variables.



SOLUTION

To find the value of x, you can set up a proportion.

$\frac{9}{13.5} = \frac{37.5 - x}{x}$	**Write proportion.**
$13.5(37.5 - x) = 9x$	**Cross product property**
$506.25 - 13.5x = 9x$	**Distributive property**
$506.25 = 22.5x$	**Add $13.5x$ to each side.**
$22.5 = x$	**Divide each side by 22.5.**

Since $\overline{KL} \parallel \overline{MN}$, $\triangle JKL \sim \triangle JMN$ and $\frac{JK}{JM} = \frac{KL}{MN}$.

$\frac{9}{13.5 + 9} = \frac{7.5}{y}$	**Write proportion.**
$9y = 7.5(22.5)$	**Cross product property**
$y = 18.75$	**Divide each side by 9.**

GUIDED PRACTICE

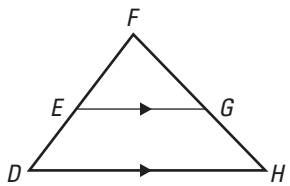
Vocabulary Check ✓

1. Complete the following: If a line divides two sides of a triangle proportionally, then it is __?__ to the third side. This theorem is known as the __?__.

Concept Check ✓

2. In $\triangle ABC$, $\overrightarrow{AR}$ bisects $\angle CAB$. Write the proportionality statement for the triangle that is based on Theorem 8.7.

Determine whether the statement is *true* or *false*. Explain your reasoning.



3. $\frac{FE}{ED} = \frac{FG}{GH}$

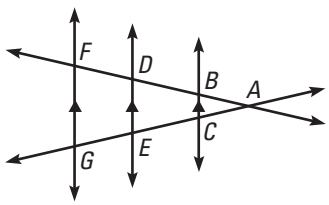
4. $\frac{FE}{FD} = \frac{FG}{FH}$

5. $\frac{EG}{DH} = \frac{EF}{DF}$

6. $\frac{ED}{FE} = \frac{EG}{DH}$

Skill Check ✓

Use the figure to complete the proportion.



7. $\frac{BD}{BF} = \frac{?}{CG}$

8. $\frac{AE}{CE} = \frac{?}{BD}$

9. $\frac{?}{GA} = \frac{FD}{FA}$

10. $\frac{GA}{?} = \frac{FA}{DA}$

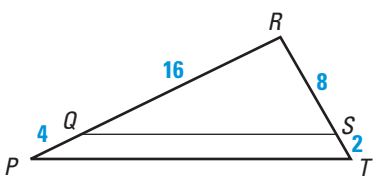
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 818.

LOGICAL REASONING **Determine whether the given information implies that $\overline{QS} \parallel \overline{PT}$. Explain.**

11.



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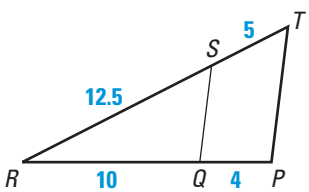

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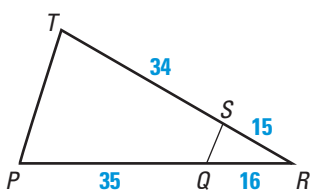
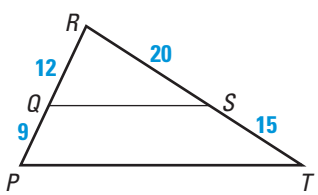


14.

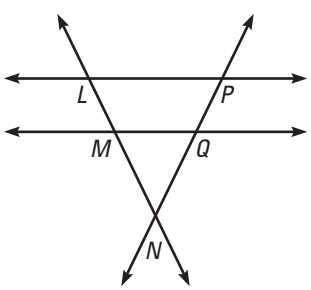


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 21–28
Example 2: Exs. 11–20
Example 3: Exs. 21–28
Example 4: Exs. 21–28
Example 5: Exs. 29, 30, 36, 37
Example 6: Exs. 34–37

LOGICAL REASONING **Use the diagram shown to decide if you are given enough information to conclude that $\overline{LP} \parallel \overline{MQ}$. If so, state the reason.**



15. $\frac{NM}{ML} = \frac{NQ}{QP}$

16. $\angle MNQ \cong \angle LNP$

17. $\angle NLP \cong \angle NMQ$

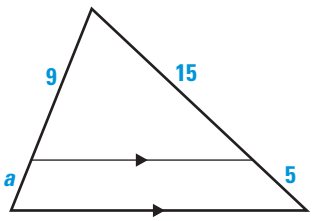
18. $\angle MQN \cong \angle LPN$

19. $\frac{LM}{MN} = \frac{LP}{MQ}$

20. $\triangle LPN \sim \triangle MQN$

USING PROPORTIONALITY THEOREMS **Find the value of the variable.**

21.



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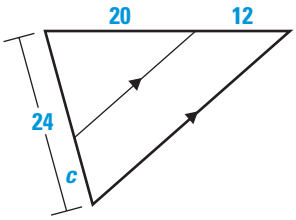

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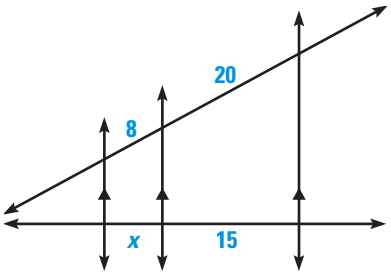
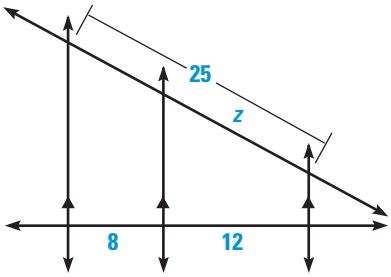
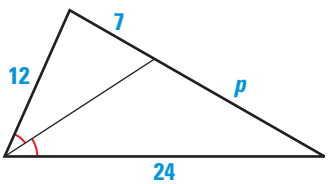


24.

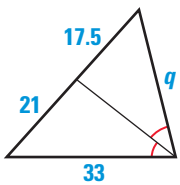


USING ALGEBRA **Find the value of the variable.**

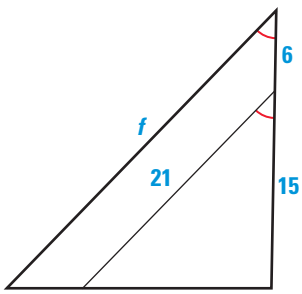
25.



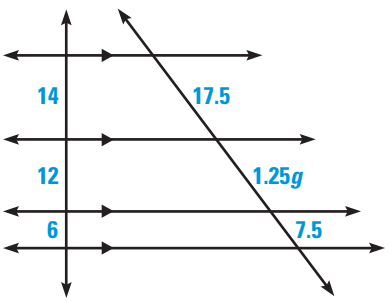
26.



27.



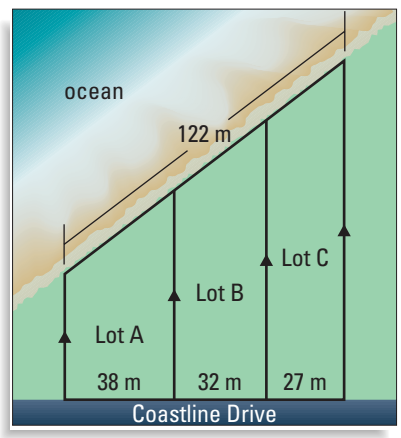
28.



LOT PRICES **The real estate term for the distance along the edge of a piece of property that touches the ocean is "ocean frontage."**

29. Find the ocean frontage (to the nearest tenth of a meter) for each lot shown.

30. **CRITICAL THINKING** In general, the more ocean frontage a lot has, the higher its selling price. Which of the lots should be listed for the highest price?



FOCUS ON CAREERS

REAL ESTATE SALESPERSON

A real estate salesperson can help a seller establish a price for their property as discussed in Exercise 30.

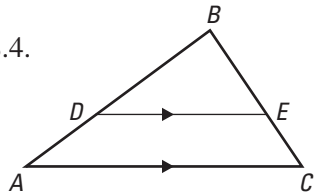
CAREER LINK

www.mcdougallittell.com

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with proofs in Exs. 31–33.

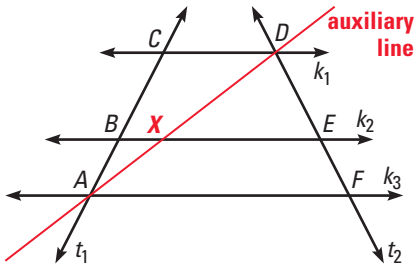
31. TWO-COLUMN PROOF Use the diagram shown to write a two-column proof of Theorem 8.4.



GIVEN ▶ $\overline{DE} \parallel \overline{AC}$

PROVE ▶ $\frac{DA}{BD} = \frac{EC}{BE}$

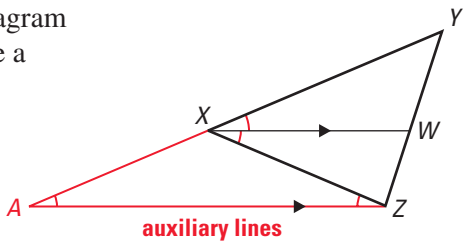
32. PARAGRAPH PROOF Use the diagram with the auxiliary line drawn to write a paragraph proof of Theorem 8.6.



GIVEN ▶ $k_1 \parallel k_2$, $k_2 \parallel k_3$

PROVE ▶ $\frac{CB}{BA} = \frac{DE}{EF}$

33. PARAGRAPH PROOF Use the diagram with the auxiliary lines drawn to write a paragraph proof of Theorem 8.7.

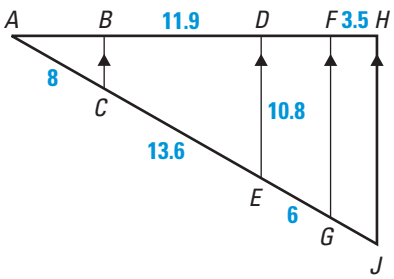


GIVEN ▶ $\angle YXW \cong \angle WXZ$

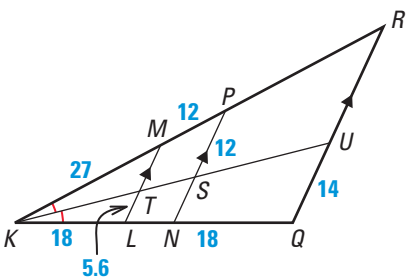
PROVE ▶ $\frac{YW}{WZ} = \frac{XY}{XZ}$

FINDING SEGMENT LENGTHS **Use the diagram to determine the lengths of the missing segments.**

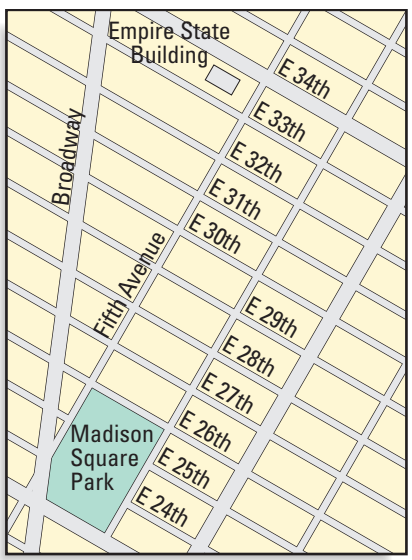
34.



35.



NEW YORK CITY **Use the following information and the map of New York City.** On Fifth Avenue, the distance between E 33rd Street and E 24th Street is about 2600 feet. The distance between those same streets on Broadway is about 2800 feet. All numbered streets are parallel.



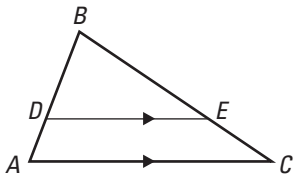
36. On Fifth Avenue, the distance between E 24th Street and E 29th Street is about 1300 feet. What is the distance between these two streets on Broadway?

37. On Broadway, the distance between E 33rd Street and E 30th Street is about 1120 feet. What is the distance between these two streets on Fifth Avenue?

38. *Writing* Use the diagram given for the proof of Theorem 8.4 from Exercise 31 to explain how you can prove the Triangle Proportionality Converse, Theorem 8.5.

Test Preparation

39. MULTI-STEP PROBLEM Use the diagram shown.



a. If $DB = 6$, $AD = 2$, and $CB = 20$, find EB.

b. Use the diagram to state three correct proportions.

c. If $DB = 4$, $AB = 10$, and $CB = 20$, find CE.

d. *Writing* Explain how you know that $\triangle ABC$ is similar to $\triangle DBE$.

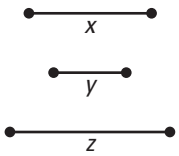
★ Challenge

40. CONSTRUCTION Perform the following construction.

GIVEN ▶ Segments with lengths x, y, and z

CONSTRUCT ▶ A segment of length p, such that $\frac{x}{y} = \frac{z}{p}$.

(*Hint:* This construction is like the construction on page 500.)



MIXED REVIEW

USING THE DISTANCE FORMULA Find the distance between the two points. (Review 1.3)

41. $A(10, 5)$
$B(-6, -4)$

42. $A(7, -3)$
$B(-9, 4)$

43. $A(-1, -9)$
$B(6, -2)$

44. $A(0, 11)$
$B(-5, 2)$

45. $A(0, -10)$
$B(4, 7)$

46. $A(8, -5)$
$B(0, 4)$

USING THE DISTANCE FORMULA Place the figure in a coordinate plane and find the requested information. (Review 4.7)

47. Draw a right triangle with legs of 12 units and 9 units. Find the length of the hypotenuse.

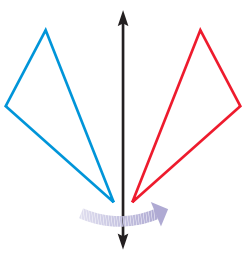
48. Draw a rectangle with length 16 units and width 12 units. Find the length of a diagonal.

49. Draw an isosceles right triangle with legs of 6 units. Find the length of the hypotenuse.

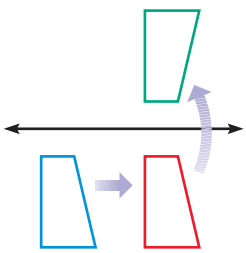
50. Draw an isosceles triangle with base of 16 units and height of 6 units. Find the length of the legs.

TRANSFORMATIONS Name the type of transformation. (Review 7.1– 7.3, 7.5 for 8.7)

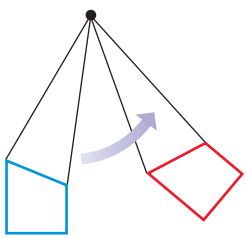
51.



52.



53.



8.7 Dilations

What you should learn

GOAL 1 Identify dilations.

GOAL 2 Use properties of dilations to create a **real-life** perspective drawing in **Ex. 34**.

Why you should learn it

▼ To solve **real-life** problems, such as estimating the height of the shadow of a shadow puppet in **Example 3**.

REAL LIFE

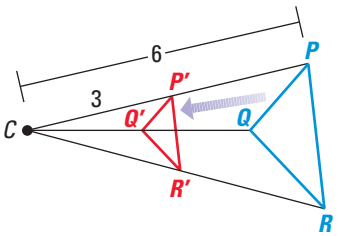
GOAL 1 IDENTIFYING DILATIONS

In Chapter 7, you studied rigid transformations, in which the image and preimage of a figure are *congruent*. In this lesson, you will study a type of nonrigid transformation called a *dilation*, in which the image and preimage of a figure are *similar*.

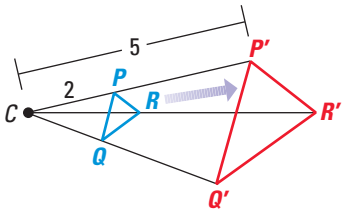
A **dilation** with center C and scale factor k is a transformation that maps every point P in the plane to a point P' so that the following properties are true.

1. If P is not the center point C, then the image point P' lies on $\overrightarrow{CP}$. The scale factor k is a positive number such that $k = \frac{CP'}{CP}$, and $k \neq 1$.
2. If P is the center point C, then $P = P'$.

The dilation is a **reduction** if $0 < k < 1$ and it is an **enlargement** if $k > 1$.



Reduction: $k = \frac{CP'}{CP} = \frac{3}{6} = \frac{1}{2}$



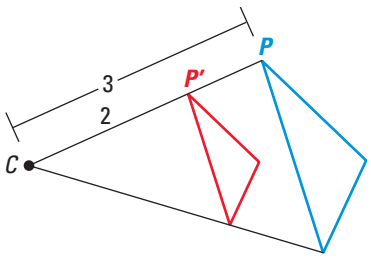
Enlargement: $k = \frac{CP'}{CP} = \frac{5}{2}$

Because $\triangle PQR \sim \triangle P'Q'R'$, $\frac{P'Q'}{PQ}$ is equal to the scale factor of the dilation.

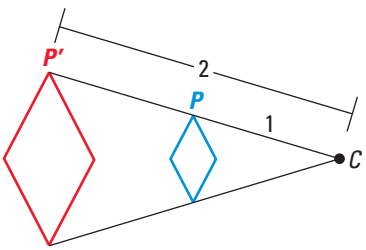
EXAMPLE 1 *Identifying Dilations*

Identify the dilation and find its scale factor.

a.



b.



SOLUTION

a. Because $\frac{CP'}{CP} = \frac{2}{3}$, the scale factor is $k = \frac{2}{3}$. This is a reduction.

b. Because $\frac{CP'}{CP} = \frac{2}{1}$, the scale factor is $k = 2$. This is an enlargement.

STUDENT HELP

Look Back
For help with the blue to red color scheme used in transformations, see p. 396.

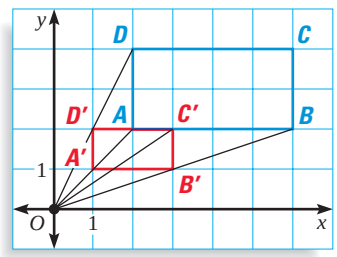
In a coordinate plane, dilations whose centers are the origin have the property that the image of $P(x, y)$ is $P'(kx, ky)$.

EXAMPLE 2 *Dilation in a Coordinate Plane*

Draw a dilation of rectangle $ABCD$ with $A(2, 2)$, $B(6, 2)$, $C(6, 4)$, and $D(2, 4)$. Use the origin as the center and use a scale factor of $\frac{1}{2}$. How does the perimeter of the preimage compare to the perimeter of the image?

SOLUTION

Because the center of the dilation is the origin, you can find the image of each vertex by multiplying its coordinates by the scale factor.



$A(2, 2) \rightarrow A'(1, 1)$

$B(6, 2) \rightarrow B'(3, 1)$

$C(6, 4) \rightarrow C'(3, 2)$

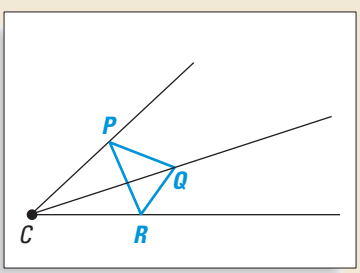
$D(2, 4) \rightarrow D'(1, 2)$

From the graph, you can see that the preimage has a perimeter of 12 and the image has a perimeter of 6. A preimage and its image after a dilation are similar figures. Therefore, the ratio of the perimeters of a preimage and its image is equal to the scale factor of the dilation.

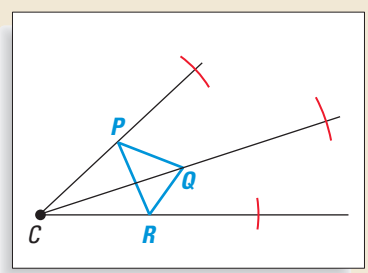
ACTIVITY
Construction

Drawing a Dilation

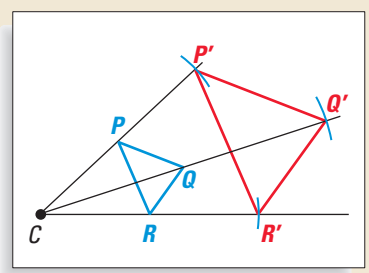
Use the following steps to construct a dilation ($k = 2$) of a triangle using a straightedge and a compass.



1 Draw $\triangle PQR$ and choose the center of the dilation C. Use a straightedge to draw lines from C through the vertices of the triangle.



2 Use the compass to locate P' on $\overrightarrow{CP}$ so that $CP' = 2(CP)$. Locate Q' and R' in the same way.



3 Connect the points P', Q', and R'.

In the construction above, notice that $\triangle PQR \sim \triangle P'Q'R'$. You can prove this by using the SAS and SSS Similarity Theorems.

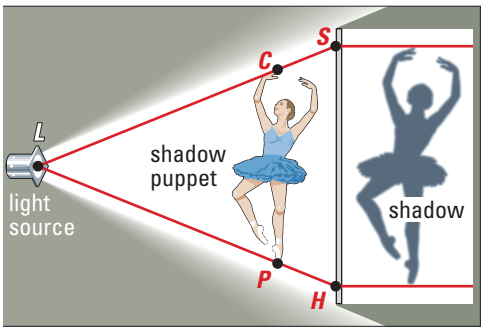
GOAL 2 USING DILATIONS IN REAL LIFE

EXAMPLE 3 *Finding the Scale Factor*

FOCUS ON APPLICATIONS

REAL LIFE SHADOW PUPPETS Some experienced shadowmaster puppeteers can manipulate over 20 carved leather puppets at the same time.

SHADOW PUPPETS Shadow puppets have been used in many countries for hundreds of years. A flat figure is held between a light and a screen. The audience on the other side of the screen sees the puppet's shadow. The shadow is a dilation, or enlargement, of the shadow puppet. When looking at a cross sectional view, $\triangle LCP \sim \triangle LSH$.



The shadow puppet shown is 12 inches tall (CP in the diagram). Find the height of the shadow, SH, for each distance from the screen. In each case, by what percent is the shadow larger than the puppet?

a. $LC = LP = 59$ in.; $LS = LH = 74$ in.

b. $LC = LP = 66$ in.; $LS = LH = 74$ in.

SOLUTION

a. $\frac{59}{74} = \frac{12}{SH}$ $\qquad \frac{LC}{LS} = \frac{CP}{SH}$

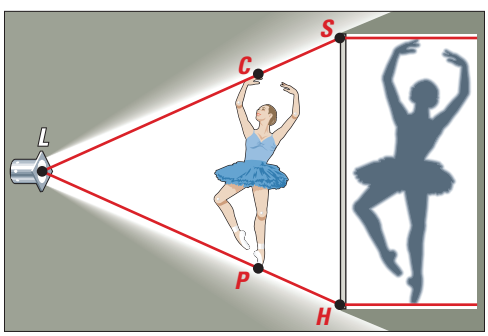
$59(SH) = 888$

$SH \approx 15$ inches

To find the percent of size increase, use the scale factor of the dilation.

$\text{scale factor} = \frac{SH}{CP}$

$\frac{15}{12} = 1.25$



▶ So, the shadow is 25% larger than the puppet.

b. $\frac{66}{74} = \frac{12}{SH}$

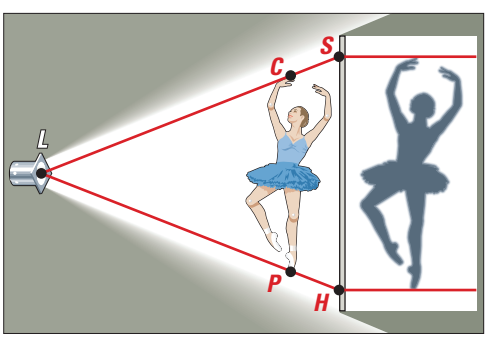
$66(SH) = 888$

$SH \approx 13.45$ inches

Use the scale factor again to find the percent of size increase.

$\text{scale factor} = \frac{SH}{CP}$

$\frac{13.45}{12} \approx 1.12$



▶ So, the shadow is about 12% larger than the puppet.

Notice that as the puppet moves closer to the screen, the shadow height decreases.

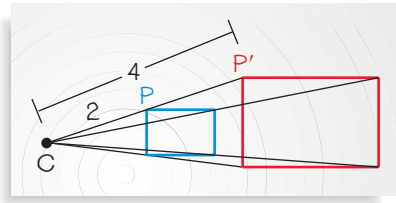
GUIDED PRACTICE

Vocabulary Check ✓

1. In a *dilation* every image is _?_ to its preimage.

Concept Check ✓

2. **ERROR ANALYSIS** Katie found the scale factor of the dilation shown to be $\frac{1}{2}$. What did Katie do wrong?



3. Is the dilation shown a reduction or an enlargement? How do you know?

Skill Check ✓

$\triangle PQR$ is mapped onto $\triangle P'Q'R'$ by a dilation with center C. Complete the statement.

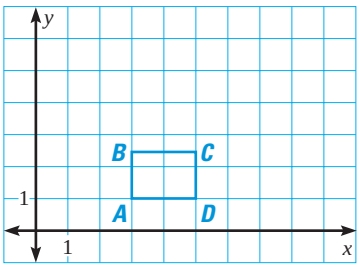
4. $\triangle PQR$ is (similar, congruent) to $\triangle P'Q'R'$.

5. If $\frac{CP'}{CP} = \frac{4}{3}$, then $\triangle P'Q'R'$ is (larger, smaller) than $\triangle PQR$, and the dilation is (a reduction, an enlargement).

Use the following information to draw a dilation of rectangle *ABCD*.

6. Draw a dilation of rectangle $ABCD$ on a coordinate plane, with $A(3, 1)$, $B(3, 2.5)$, $C(5, 2.5)$, and $D(5, 1)$. Use the origin as the center and use a scale factor of 2.

7. Is $ABCD \sim A'B'C'D'$? Explain your answer.



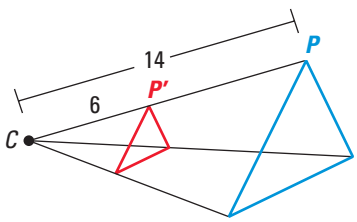
PRACTICE AND APPLICATIONS

STUDENT HELP

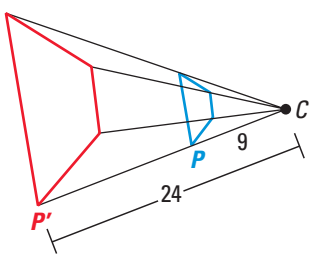
Extra Practice to help you master skills is on p. 818.

IDENTIFYING DILATIONS **Identify the dilation and find its scale factor.**

8.

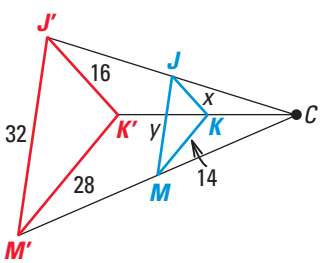


9.

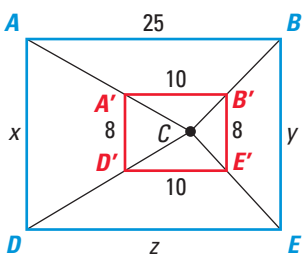


FINDING SCALE FACTORS **Identify the dilation, and find its scale factor. Then, find the values of the variables.**

10.



11.



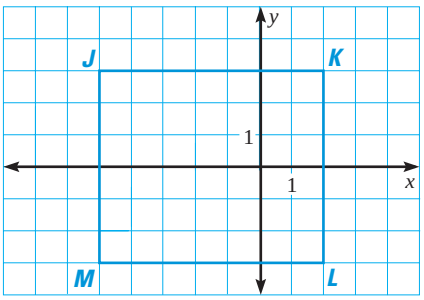
STUDENT HELP

HOMEWORK HELP

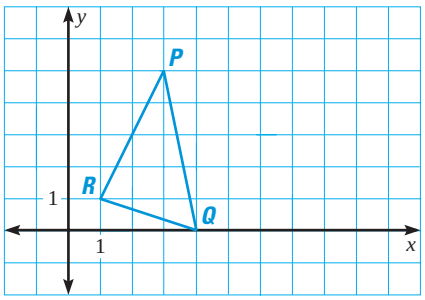
Example 1: Exs. 8–11, 20–23
Example 2: Exs. 12–15
Example 3: Exs. 24–26, 33

DILATIONS IN A COORDINATE PLANE **Use the origin as the center of the dilation and the given scale factor to find the coordinates of the vertices of the image of the polygon.**

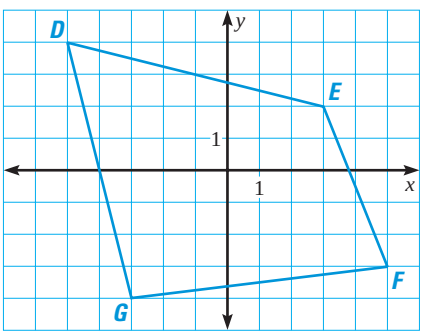
12. $k = \frac{1}{2}$



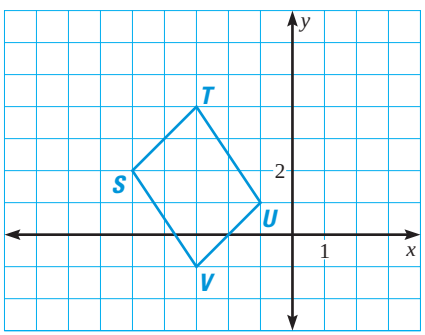
13. $k = 2$



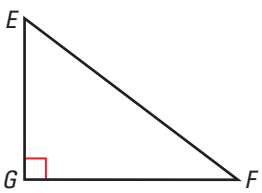
14. $k = \frac{1}{3}$



15. $k = 4$



16. **COMPARING RATIOS** Use the triangle shown. Let P and Q be the midpoints of the sides $\overline{EG}$ and $\overline{FG}$, respectively. Find the scale factor and the center of the dilation that enlarges $\triangle PQG$ to $\triangle EFG$. Find the ratio of EF to PQ. How does this ratio compare to the scale factor?

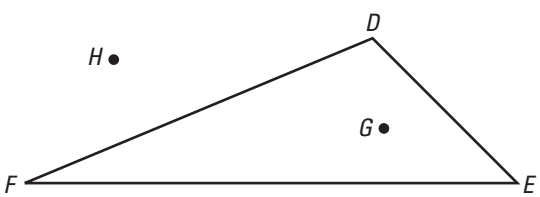


CONSTRUCTION **Copy $\triangle DEF$ and points G and H as shown. Then, use a straightedge and a compass to construct the dilation.**

17. $k = 3$; Center: G

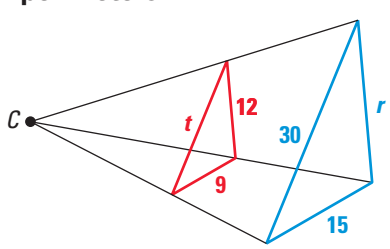
18. $k = \frac{1}{2}$; Center: H

19. $k = 2$; Center: E

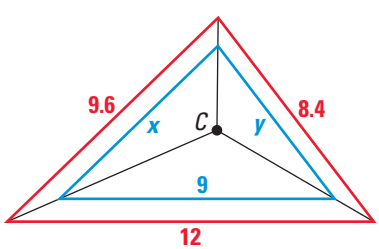


SIMILAR TRIANGLES **The red triangle is the image of the blue triangle after a dilation. Find the values of the variables. Then find the ratio of their perimeters.**

20.



21.



STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 22 and 23.

IDENTIFYING DILATIONS **$\triangle ABC$ is mapped onto $\triangle A'B'C'$ by a dilation. Use the given information to sketch the dilation, identify it as a reduction or an enlargement, and find the scale factor. Then find the missing lengths.**

22. In $\triangle ABC$, $AB = 6$, $BC = 9$, and $AC = 12$. In $\triangle A'B'C'$, $A'B' = 2$. Find the lengths of $\overline{B'C'}$ and $\overline{A'C'}$.

23. In $\triangle ABC$, $AB = 5$ and $BC = 7$. In $\triangle A'B'C'$, $A'B' = 20$ and $A'C' = 36$. Find the lengths of $\overline{AC}$ and $\overline{B'C'}$.

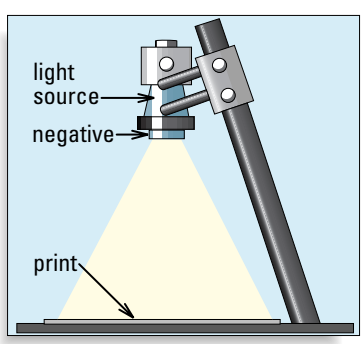
FLASHLIGHT IMAGE **In Exercises 24–26, use the following information.** You are projecting images onto a wall with a flashlight. The lamp of the flashlight is 8.3 centimeters away from the wall. The preimage is imprinted onto a clear cap that fits over the end of the flashlight. This cap has a diameter of 3 centimeters. The preimage has a height of 2 centimeters, and the lamp of the flashlight is located 2.7 centimeters from the preimage.

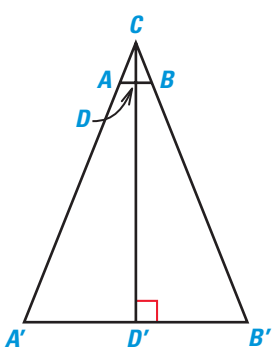
24. Sketch a diagram of the dilation.

25. Find the diameter of the circle of light projected onto the wall from the flashlight.

26. Find the height of the image projected onto the wall.

ENLARGEMENTS **In Exercises 27 and 28, use the following information.** By adjusting the distance between the negative and the enlarged print in the photographic enlarger shown, you can make prints of different sizes. In the diagram shown, you want the enlarged print to be 7 inches wide ($A'B'$). The negative is 1 inch wide (AB), and the distance between the light source and the negative is 1.25 inches (CD).





27. What is the scale factor of the enlargement?

28. What is the distance between the light source and the enlarged print?

DIMENSIONS OF PHOTOS **Use the diagram from Exercise 27 to determine the missing information.**

	CD	CD'	AB	$A'B'$
29.	1.2 in.	7.2 in.	0.8 in.	?
30.	?	14 cm	2 cm	12 cm
31.	2 in.	10 in.	?	8.5 in.

FOCUS ON CAREERS

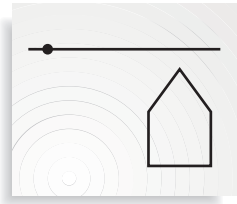
ARCHITECTURAL RENDERING An architectural renderer uses techniques like the one shown in Exercise 34 to create three dimensional drawings of buildings and other structures.

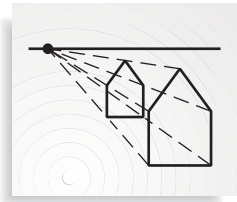
CAREER LINK
www.mcdougallittell.com

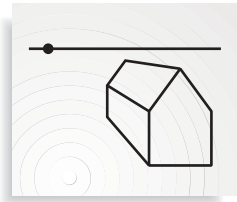
32. LOGICAL REASONING Draw any triangle, and label it $\triangle PQR$. Using a scale factor of 2, draw the image of $\triangle PQR$ after a dilation with a center outside the triangle, with a center inside the triangle, and with a center on the triangle. Explain the relationship between the three images created.

33. *Writing* Use the information about shadow puppet theaters from Example 3, page 508. Explain how you could use a shadow puppet theater to help another student understand the terms *image*, *preimage*, *center of dilation*, and *dilation*. Draw a diagram and label the terms on the diagram.

34. PERSPECTIVE DRAWING Create a perspective drawing by following the given steps.







1. Draw a horizontal line across the paper, and choose a point on this line to be the center of the dilation, also called the *vanishing point*. Next, draw a polygon.
2. Draw rays from the vanishing point to all vertices of the polygon. Draw a reduction of the polygon by locating image points on the rays.
3. Connect the preimage to the image by darkening the segments between them. Erase all hidden lines.

Test Preparation

35. MULTIPLE CHOICE Identify the dilation shown as an enlargement or reduction and find its scale factor.

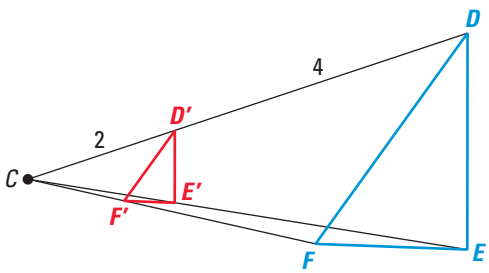
(A) enlargement; $k = 2$

(B) enlargement; $k = \frac{1}{3}$

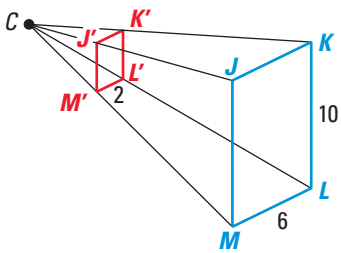
(C) reduction; $k = \frac{1}{3}$

(D) reduction; $k = \frac{1}{2}$

(E) reduction; $k = 3$



36. MULTIPLE CHOICE In the diagram shown, the center of the dilation of $\square JKLM$ is point C. The length of a side of $\square J'K'L'M'$ is what percent of the length of the corresponding side of $\square JKLM$?



(A) 3% (B) 12% (C) 20%

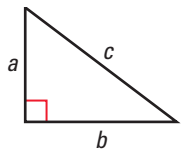
(D) $33\frac{1}{3}\%$ (E) 300%

★ Challenge

37. CREATING NEW IMAGES A polygon is reduced by a dilation with center C and scale factor $\frac{1}{k}$. The image is then enlarged by a dilation with center C and scale factor k. Describe the size and shape of this new image.

MIXED REVIEW

USING THE PYTHAGOREAN THEOREM **Refer to the triangle shown to find the length of the missing side by using the Pythagorean Theorem.** **(Review 1.3 for 9.1)**



38. $a = 5, b = 12$

39. $a = 8, c = 2\sqrt{65}$

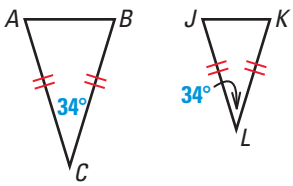
40. $b = 2, c = 5\sqrt{5}$

41. $b = 1, c = \sqrt{50}$

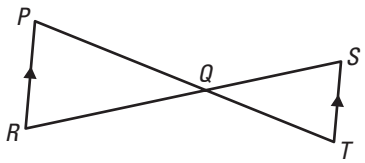
42. Find the geometric mean of 11 and 44. **(Review 8.2 for 9.1)**

DETERMINING SIMILARITY **Determine whether the triangles can be proved similar or not. Explain your reasoning.** **(Review 8.4 and 8.5)**

43.



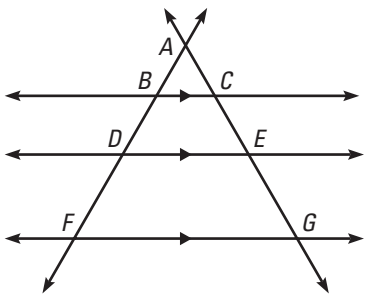
44.



QUIZ 3

Self-Test for Lessons 8.6 and 8.7

Use the figure to complete the proportion. **(Lesson 8.6)**



1. $\frac{AC}{CE} = \frac{AB}{?}$

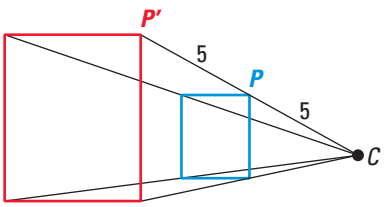
2. $\frac{BD}{BF} = \frac{?}{CG}$

3. $\frac{EG}{AG} = \frac{DF}{?}$

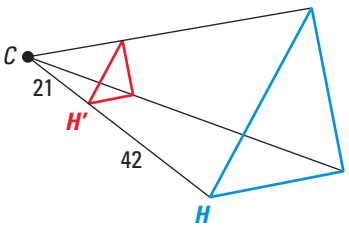
4. $\frac{GA}{EA} = \frac{?}{DA}$

In Exercises 5 and 6, identify the dilation and find its scale factor. **(Lesson 8.7)**

5.



6.



7. $\triangle JKL$ is mapped onto $\triangle J'K'L'$ by a dilation, with center C. If $\frac{CJ'}{CJ} = \frac{5}{6}$, then the dilation is (a reduction, an enlargement) and $\triangle JKL$ is (larger, smaller) than $\triangle J'K'L'$. **(Lesson 8.7)**

8. **ENLARGING PHOTOS** An 8 inch by 10 inch photo is enlarged to produce an 18 inch by $22\frac{1}{2}$ inch photo. What is the scale factor? **(Lesson 8.7)**

CHAPTER 8

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Write and simplify the ratio of two numbers. **(8.1)**	Find the ratio of the track team's wins to losses. **(p. 461)**
Use proportions to solve problems. **(8.1)**	Use measurements of a baseball bat sculpture to find the dimensions of Babe Ruth's bat. **(p. 463)**
Understand properties of proportions. **(8.2)**	Determine the width of the actual Titanic ship from the dimensions of a scale model. **(p. 467)**
Identify similar polygons and use properties of similar polygons. **(8.3)**	Determine whether two television screens are similar. **(p. 477)**
Prove that two triangles are similar using the definition of similar triangles and the AA Similarity Postulate. **(8.4)**	Use similar triangles to determine the altitude of an aerial photography blimp. **(p. 482)**
Prove that two triangles are similar using the SSS Similarity Theorem and the SAS Similarity Theorem. **(8.5)**	Use similar triangles to estimate the height of the Unisphere. **(p. 494)**
Use proportionality theorems to solve problems. **(8.6)**	Explain why the diagonal cuts on insulation strips have the same length. **(p. 501)**
Identify and draw dilations and use properties of dilations. **(8.7)**	Understand how the shadows in a shadow puppet show change size. **(p. 508)**

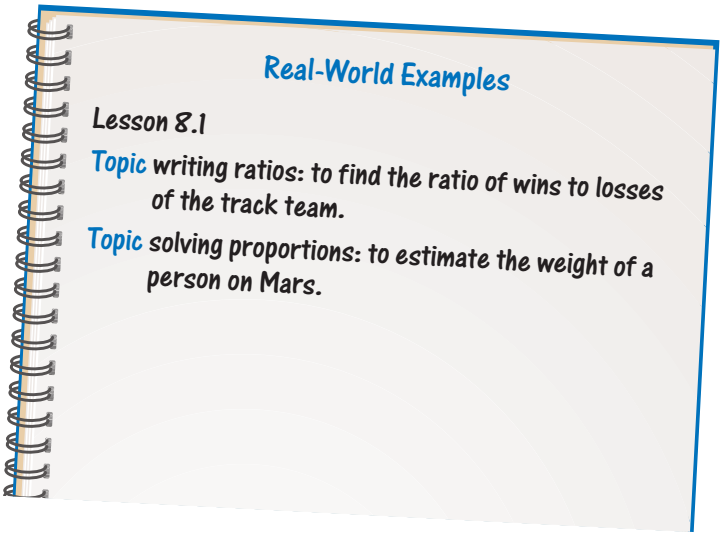
How does Chapter 8 fit into the BIGGER PICTURE of geometry?

In this chapter, you learned that if two polygons are similar, then the lengths of their corresponding sides are proportional. You also studied several connections among real-life situations, geometry, and algebra. For instance, solving a problem that involves similar polygons (geometry) often requires the use of a proportion (algebra). In later chapters, remember that the measures of corresponding angles of similar polygons are equal, but the lengths of corresponding sides of similar polygons are proportional.

STUDY STRATEGY

How did you use your list of real-world examples?

The list of the main topics of the chapter with corresponding real-world examples that you made following the **Study Strategy** on page 456, may resemble this one.



CHAPTER 8

Chapter Review

VOCABULARY

- ratio, p. 457
- proportion, p. 459
- extremes, p. 459
- means, p. 459
- geometric mean, p. 466
- similar polygons, p. 473
- scale factor, p. 474
- dilation, p. 506
- reduction, p. 506
- enlargement, p. 506

8.1 RATIO AND PROPORTION

Examples on pp. 457–460

EXAMPLE You can solve a proportion by finding the value of the variable.

$\frac{x}{12} = \frac{x+6}{30}$	**Write original proportion.**
$30x = 12(x+6)$	**Cross product property**
$30x = 12x + 72$	**Distributive property**
$18x = 72$	**Subtract 12*x* from each side.**
$x = 4$	**Divide each side by 18.**

Solve the proportion.

1. $\frac{3}{x} = \frac{2}{7}$

2. $\frac{a+1}{5} = \frac{2a}{9}$

3. $\frac{2}{x+1} = \frac{4}{x+6}$

4. $\frac{d-4}{d} = \frac{3}{7}$

8.2 PROBLEM SOLVING IN GEOMETRY WITH PROPORTIONS

Examples on pp. 465–467

EXAMPLE In 1997, the ratio of the population of South Carolina to the population of Wyoming was 47 : 6. The population of South Carolina was about 3,760,000. You can find the population of Wyoming by solving a proportion.

$$\frac{47}{6} = \frac{3{,}760{,}000}{x}$$

$$47x = 22{,}560{,}000$$

$x = 480{,}000$ The population of Wyoming was about 480,000.

5. You buy a 13 inch scale model of the sculpture *The Dancer* by Edgar Degas. The ratio of the height of the scale model to the height of the sculpture is 1 : 3. Find the height of the sculpture.

6. The ratio of the birth weight to the adult weight of a male black bear is 3 : 1000. The average birth weight is 12 ounces. Find the average adult weight in pounds.

8.3 SIMILAR POLYGONS

Examples on pp. 473–475

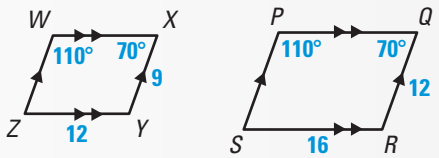
EXAMPLE The two parallelograms shown are similar because their corresponding angles are congruent and the lengths of their corresponding sides are proportional.

$$\frac{WX}{PQ} = \frac{ZY}{SR} = \frac{XY}{QR} = \frac{WZ}{PS} = \frac{3}{4}$$

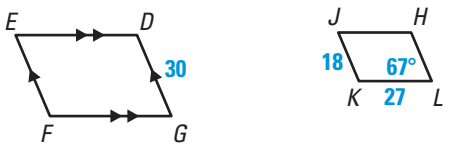
$m\angle P = m\angle R = m\angle W = m\angle Y = 110°$

$m\angle Q = m\angle S = m\angle X = m\angle Z = 70°$

The scale factor of ▱$WXYZ$ to ▱$PQRS$ is $\frac{3}{4}$.



In Exercises 7–9, ▱*DEFG* ~ ▱*HJKL*.



7. Find the scale factor of ▱$DEFG$ to ▱$HJKL$.

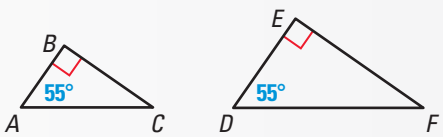
8. Find the length of $\overline{DE}$ and the measure of $\angle F$.

9. Find the ratio of the perimeter of ▱$HJKL$ to the perimeter of ▱$DEFG$.

8.4 SIMILAR TRIANGLES

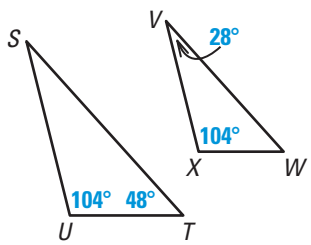
Examples on pp. 480–482

EXAMPLE Because two angles of $\triangle ABC$ are congruent to two angles of $\triangle DEF$, $\triangle ABC \sim \triangle DEF$ by the Angle-Angle (AA) Similarity Postulate.

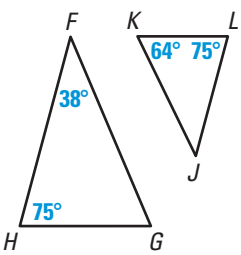


Determine whether the triangles can be proved similar or not. Explain why or why not. If they are similar, write a similarity statement.

10.



11.



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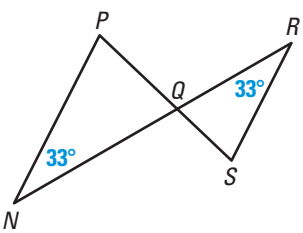
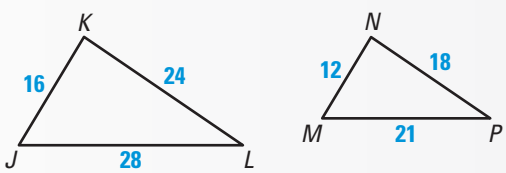


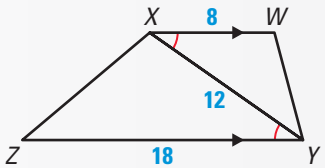
8.5 PROVING TRIANGLES ARE SIMILAR

Examples on pp. 488–491

EXAMPLES Three sides of $\triangle JKL$ are proportional to three sides of $\triangle MNP$, so $\triangle JKL \sim \triangle MNP$ by the Side-Side-Side (SSS) Similarity Theorem.

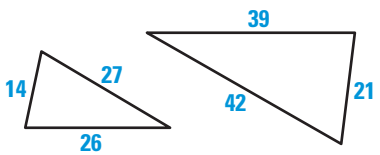


Two sides of $\triangle XYZ$ are proportional to two sides of $\triangle WXY$, and the included angles are congruent. By the Side-Angle-Side (SAS) Similarity Theorem, $\triangle XYZ \sim \triangle WXY$.

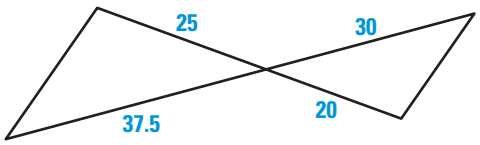


Are the triangles similar? If so, state the similarity and a postulate or theorem that can be used to prove that the triangles are similar.

13.



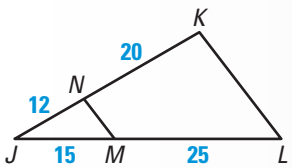
14.

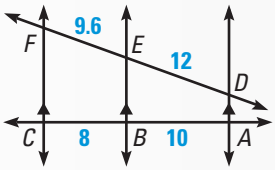


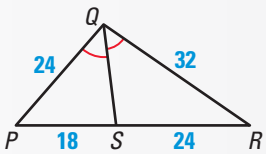
8.6 PROPORTIONS AND SIMILAR TRIANGLES

Examples on pp. 498–501

EXAMPLES You can use proportionality theorems to compare proportional lengths.



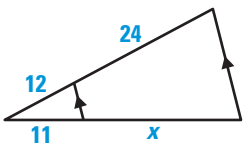




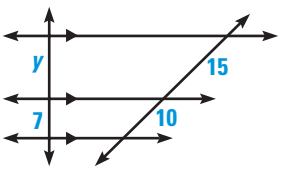
$\frac{JN}{NK} = \frac{12}{20} = \frac{3}{5}$ $\frac{JM}{ML} = \frac{15}{25} = \frac{3}{5}$ $\frac{AB}{BC} = \frac{10}{8} = \frac{5}{4}$ $\frac{DE}{EF} = \frac{12}{9.6} = \frac{5}{4}$ $\frac{QP}{QR} = \frac{24}{32} = \frac{3}{4}$ $\frac{SP}{SR} = \frac{18}{24} = \frac{3}{4}$

Find the value of the variable.

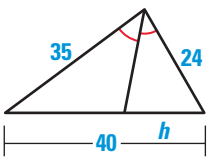
15.



16.



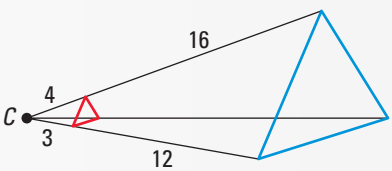
17.



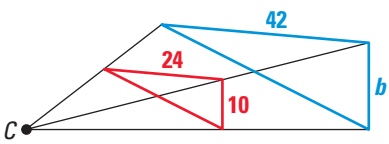
8.7 DILATIONS

Examples on pp. 506–508

EXAMPLE The blue triangle is mapped onto the red triangle by a dilation with center C. The scale factor is $\frac{1}{5}$, so the dilation is a reduction.



18. Identify the dilation, find its scale factor, and find the value of the variable.



CHAPTER 8

Chapter Test

In Exercises 1–3, solve the proportion.

1. $\frac{x}{3} = \frac{12}{9}$

2. $\frac{18}{y} = \frac{15}{20}$

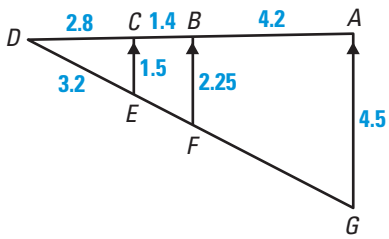
3. $\frac{11}{110} = \frac{z}{10}$

Complete the sentence.

4. If $\frac{5}{2} = \frac{a}{b}$, then $\frac{5}{a} = \frac{?}{b}$.

5. If $\frac{8}{x} = \frac{3}{y}$, then $\frac{8 + x}{x} = \frac{?}{y}$.

In Exercises 6–8, use the figure shown.

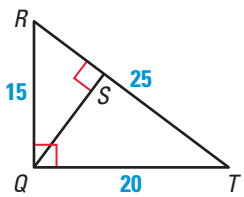


6. Find the length of $\overline{EF}$.

7. Find the length of $\overline{FG}$.

8. Is quadrilateral $FECB$ similar to quadrilateral $GFBA$? If so, what is the scale factor?

In Exercises 9–12, use the figure shown.



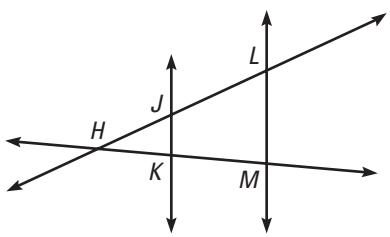
9. Prove that $\triangle RSQ \sim \triangle RQT$.

10. What is the scale factor of $\triangle RSQ$ to $\triangle RQT$?

11. Is $\triangle RSQ$ similar to $\triangle QST$? Explain.

12. Find the length of $\overline{QS}$.

In Exercises 13–15, use the figure shown to decide if you are given enough information to conclude that $\overline{JK} \parallel \overline{LM}$. If so, state the reason.



13. $\frac{LJ}{JH} = \frac{MK}{KH}$

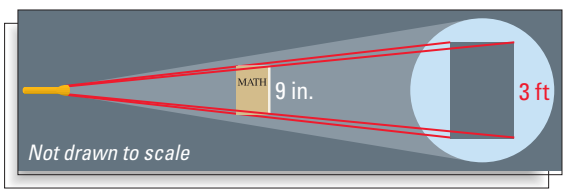
14. $\angle HJK \cong \angle HLM$

15. $\frac{LH}{JH} = \frac{MH}{KH}$

16. The triangle $\triangle RST$ is mapped onto $\triangle R'S'T'$ by a dilation with $RS = 24$, $ST = 12$, $RT = 20$, and $R'S' = 6$. Find the scale factor k, and side lengths $S'T'$ and $R'T'$.

17. Two sides of a triangle have lengths of 14 inches and 18 inches. The measure of the angle included by the sides is 45°. Two sides of a second triangle have lengths of 7 inches and 8 inches. The measure of the angle included by the sides is 45°. Are the two triangles similar? Explain.

18. You shine a flashlight on a book that is 9 inches tall and 6 inches wide. It makes a shadow on the wall that is 3 feet tall and 2 feet wide. What is the scale factor of the book to its shadow?



9.1 Similar Right Triangles

What you should learn

GOAL 1 Solve problems involving similar right triangles formed by the altitude drawn to the hypotenuse of a right triangle.

GOAL 2 Use a geometric mean to solve problems, such as estimating a climbing distance in **Ex. 32**.

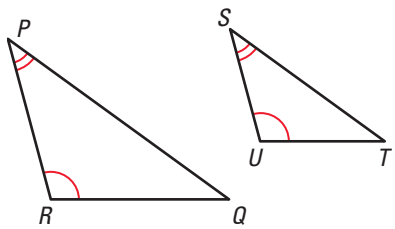
Why you should learn it

▼ You can use right triangles and a geometric mean to help you estimate distances, such as finding the approximate height of a monorail track in **Example 3**.

REAL LIFE

GOAL 1 PROPORTIONS IN RIGHT TRIANGLES

In Lesson 8.4, you learned that two triangles are similar if two of their corresponding angles are congruent. For example, $\triangle PQR \sim \triangle STU$. Recall that the corresponding side lengths of similar triangles are in proportion.

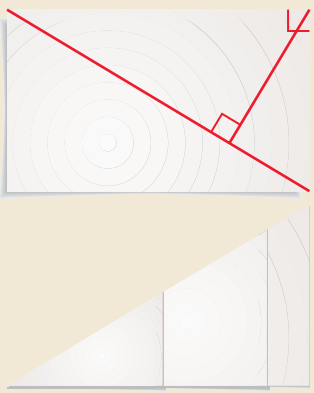


In the activity, you will see how a right triangle can be divided into two similar right triangles.

ACTIVITY Developing Concepts

Investigating Similar Right Triangles

1. Cut an index card along one of its diagonals.
2. On one of the right triangles, draw an altitude from the right angle to the hypotenuse. Cut along the altitude to form two right triangles.
3. You should now have three right triangles. Compare the triangles. What special property do they share? Explain.

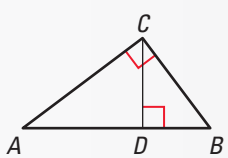


In the activity, you may have discovered the following theorem. A plan for proving the theorem appears on page 528, and you are asked to prove it in Exercise 34 on page 533.

THEOREM

THEOREM 9.1

If the altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other.

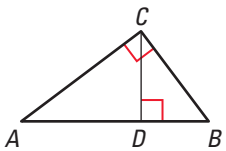


$\triangle CBD \sim \triangle ABC$, $\triangle ACD \sim \triangle ABC$, and $\triangle CBD \sim \triangle ACD$

A plan for proving Theorem 9.1 is shown below.

GIVEN ▶ $\triangle ABC$ is a right triangle; altitude $\overline{CD}$ is drawn to hypotenuse $\overline{AB}$.

PROVE ▶ $\triangle CBD \sim \triangle ABC$, $\triangle ACD \sim \triangle ABC$, and $\triangle CBD \sim \triangle ACD$.



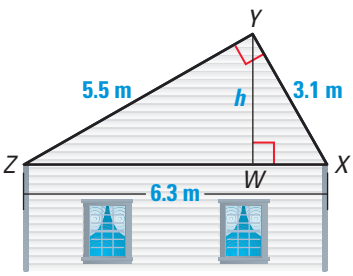
Plan for Proof First prove that $\triangle CBD \sim \triangle ABC$. Each triangle has a right angle, and each includes $\angle B$. The triangles are similar by the AA Similarity Postulate. You can use similar reasoning to show that $\triangle ACD \sim \triangle ABC$. To show that $\triangle CBD \sim \triangle ACD$, begin by showing that $\angle ACD \cong \angle B$ because they are both complementary to $\angle DCB$. Then you can use the AA Similarity Postulate.

EXAMPLE 1 *Finding the Height of a Roof*

ROOF HEIGHT A roof has a cross section that is a right triangle. The diagram shows the approximate dimensions of this cross section.

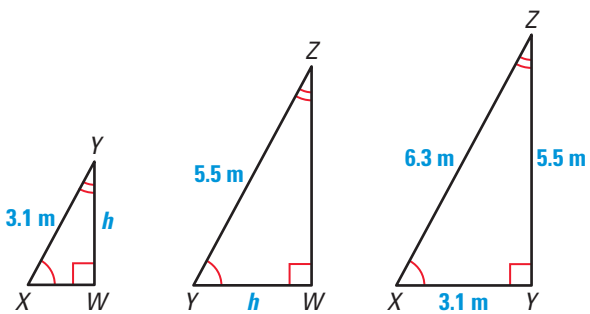
a. Identify the similar triangles.

b. Find the height h of the roof.



SOLUTION

a. You may find it helpful to sketch the three similar right triangles so that the corresponding angles and sides have the same orientation. Mark the congruent angles. Notice that some sides appear in more than one triangle. For instance, $\overline{XY}$ is the hypotenuse in $\triangle XYW$ and the shorter leg in $\triangle XZY$.



▶ $\triangle XYW \sim \triangle YZW \sim \triangle XZY$

b. Use the fact that $\triangle XYW \sim \triangle XZY$ to write a proportion.

$\frac{YW}{ZY} = \frac{XY}{XZ}$	**Corresponding side lengths are in proportion.**
$\frac{h}{5.5} = \frac{3.1}{6.3}$	**Substitute.**
$6.3h = 5.5(3.1)$	**Cross product property**
$h \approx 2.7$	**Solve for *h*.**

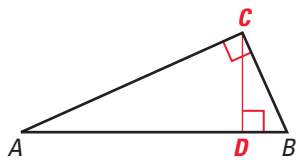
▶ The height of the roof is about 2.7 meters.

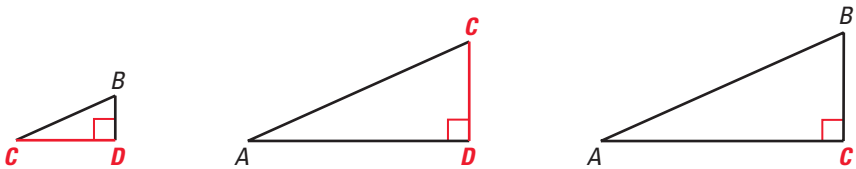
STUDENT HELP

Study Tip
In Example 1, all the side lengths of $\triangle XZY$ are given. This makes it a good choice for setting up a proportion to find an unknown side length of $\triangle XYW$.

GOAL 2 USING A GEOMETRIC MEAN TO SOLVE PROBLEMS

In right $\triangle ABC$, altitude $\overline{CD}$ is drawn to the hypotenuse, forming two smaller right triangles that are similar to $\triangle ABC$. From Theorem 9.1, you know that $\triangle CBD \sim \triangle ACD \sim \triangle ABC$.





STUDENT HELP

Look Back
The geometric mean of two numbers a and b is the positive number x such that $\frac{a}{x} = \frac{x}{b}$. For more help with finding a geometric mean, see p. 466.

Notice that $\overline{CD}$ is the longer leg of $\triangle CBD$ and the shorter leg of $\triangle ACD$. When you write a proportion comparing the leg lengths of $\triangle CBD$ and $\triangle ACD$, you can see that CD is the *geometric mean* of BD and AD.

$$\frac{\text{shorter leg of } \triangle CBD}{\text{shorter leg of } \triangle ACD} \quad \frac{BD}{CD} = \frac{CD}{AD} \quad \frac{\text{longer leg of } \triangle CBD}{\text{longer leg of } \triangle ACD}$$

Sides $\overline{CB}$ and $\overline{AC}$ also appear in more than one triangle. Their side lengths are also geometric means, as shown by the proportions below:

$$\frac{\text{hypotenuse of } \triangle ABC}{\text{hypotenuse of } \triangle CBD} \quad \frac{AB}{CB} = \frac{CB}{DB} \quad \frac{\text{shorter leg of } \triangle ABC}{\text{shorter leg of } \triangle CBD}$$

$$\frac{\text{hypotenuse of } \triangle ABC}{\text{hypotenuse of } \triangle ACD} \quad \frac{AB}{AC} = \frac{AC}{AD} \quad \frac{\text{longer leg of } \triangle ABC}{\text{longer leg of } \triangle ACD}$$

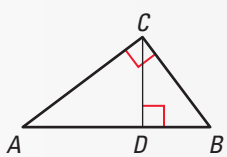
These results are expressed in the theorems below. You are asked to prove the theorems in Exercises 35 and 36.

GEOMETRIC MEAN THEOREMS

THEOREM 9.2

In a right triangle, the altitude from the right angle to the hypotenuse divides the hypotenuse into two segments.

The length of the altitude is the geometric mean of the lengths of the two segments.



$$\frac{BD}{CD} = \frac{CD}{AD}$$

THEOREM 9.3

In a right triangle, the altitude from the right angle to the hypotenuse divides the hypotenuse into two segments.

$$\frac{AB}{CB} = \frac{CB}{DB}$$

The length of each leg of the right triangle is the geometric mean of the lengths of the hypotenuse and the segment of the hypotenuse that is adjacent to the leg.

$$\frac{AB}{AC} = \frac{AC}{AD}$$

EXAMPLE 2 *Using a Geometric Mean*

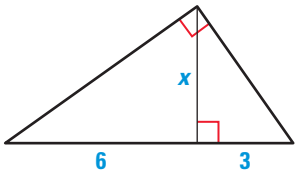
STUDENT HELP

Skills Review
For help with simplifying radicals, see p. 799.

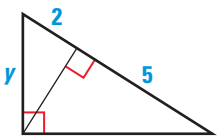
Study Tip
In part (a) of Example 2, the equation $18 = x^2$ has two solutions, $+\sqrt{18}$ and $-\sqrt{18}$. Because you are finding a length, you use the *positive* square root.

Find the value of each variable.

a.



b.



SOLUTION

a. Apply Theorem 9.2.

$$\frac{6}{x} = \frac{x}{3}$$
$$18 = x^2$$
$$\sqrt{18} = x$$
$$\sqrt{9} \cdot \sqrt{2} = x$$
$$3\sqrt{2} = x$$

b. Apply Theorem 9.3.

$$\frac{5+2}{y} = \frac{y}{2}$$
$$\frac{7}{y} = \frac{y}{2}$$
$$14 = y^2$$
$$\sqrt{14} = y$$

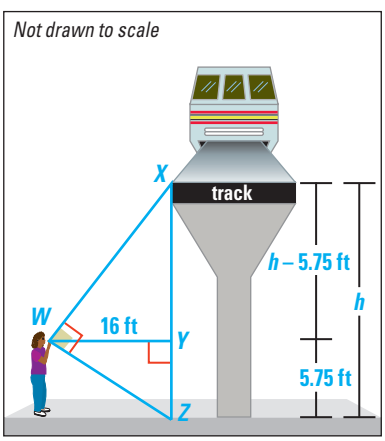
EXAMPLE 3 *Using Indirect Measurement*

FOCUS ON APPLICATIONS

REAL LIFE

MONORAILS are railways that have only one rail. The Jacksonville monorail, shown in the photo above, can travel up to 35 miles per hour and can carry about 56 passengers per car.

MONORAIL TRACK To estimate the height of a monorail track, your friend holds a cardboard square at eye level. Your friend lines up the top edge of the square with the track and the bottom edge with the ground. You measure the distance from the ground to your friend's eye and the distance from your friend to the track.



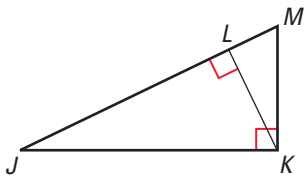
In the diagram, $XY = h - 5.75$ is the difference between the track height h and your friend's eye level. Use Theorem 9.2 to write a proportion involving XY. Then you can solve for h.

$\frac{XY}{WY} = \frac{WY}{ZY}$	**Geometric Mean Theorem 9.2**
$\frac{h - 5.75}{16} = \frac{16}{5.75}$	**Substitute.**
$5.75(h - 5.75) = 16^2$	**Cross product property**
$5.75h - 33.0625 = 256$	**Distributive property**
$5.75h = 289.0625$	**Add 33.0625 to each side.**
$h \approx 50$	**Divide each side by 5.75.**

▶ The height of the track is about 50 feet.

GUIDED PRACTICE

Vocabulary Check ✓ **In Exercises 1–3, use the diagram at the right.**



1. In the diagram, KL is the __?__ of ML and JL.

Concept Check ✓

2. Complete the following statement: $\triangle JKL \sim \triangle$ __?__ $\sim \triangle$ __?__ .

3. Which segment's length is the geometric mean of ML and MJ?

Skill Check ✓ **In Exercises 4–9, use the diagram above. Complete the proportion.**

4. $\frac{KM}{KL} = \frac{?}{JK}$

5. $\frac{JM}{?} = \frac{JK}{JL}$

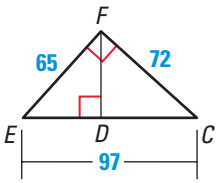
6. $\frac{?}{LK} = \frac{LK}{LM}$

7. $\frac{JM}{?} = \frac{KM}{LM}$

8. $\frac{LK}{LM} = \frac{JK}{?}$

9. $\frac{?}{JK} = \frac{MK}{MJ}$

10. Use the diagram at the right. Find DC. Then find DF. Round decimals to the nearest tenth.

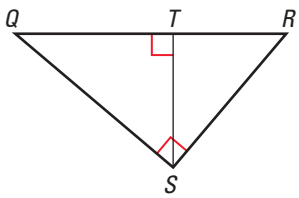


PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 819.

SIMILAR TRIANGLES **Use the diagram.**

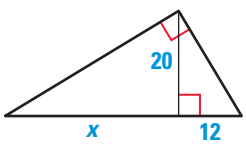


11. Sketch the three similar triangles in the diagram. Label the vertices.

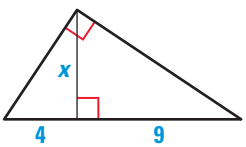
12. Write similarity statements for the three triangles.

USING PROPORTIONS **Complete and solve the proportion.**

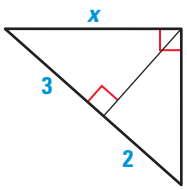
13. $\frac{x}{20} = \frac{?}{12}$



14. $\frac{4}{x} = \frac{x}{?}$

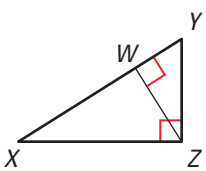


15. $\frac{5}{x} = \frac{x}{?}$

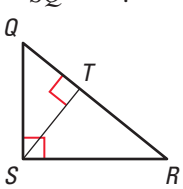


COMPLETING PROPORTIONS **Write similarity statements for the three similar triangles in the diagram. Then complete the proportion.**

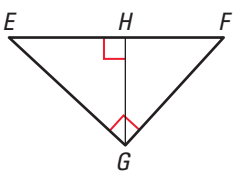
16. $\frac{XW}{ZW} = \frac{?}{YW}$



17. $\frac{QT}{SQ} = \frac{SQ}{?}$



18. $\frac{?}{EG} = \frac{EG}{EF}$



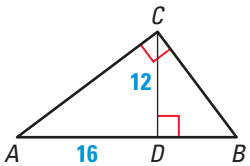
STUDENT HELP

HOMEWORK HELP

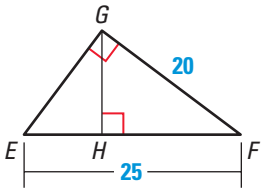
Example 1: Exs. 11–31
Example 2: Exs. 11–31
Example 3: Ex. 32

FINDING LENGTHS **Write similarity statements for three triangles in the diagram. Then find the given length. Round decimals to the nearest tenth.**

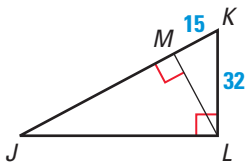
19. Find *DB*.



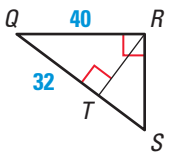
20. Find *HF*.



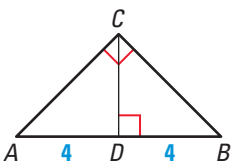
21. Find *JK*.



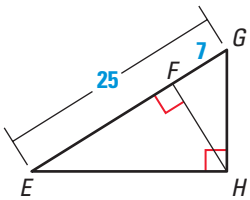
22. Find *QS*.



23. Find *CD*.

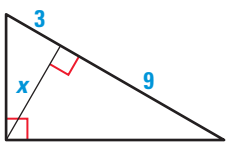


24. Find *FH*.



USING ALGEBRA **Find the value of each variable.**

25.



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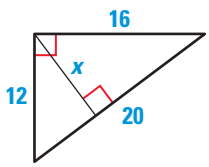

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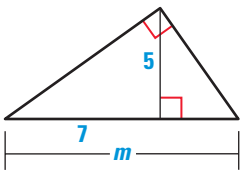
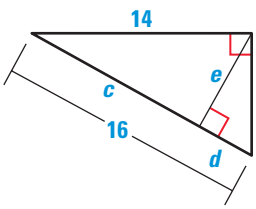


STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 28–30.

28.



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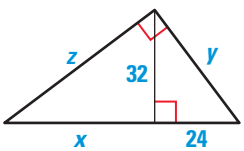

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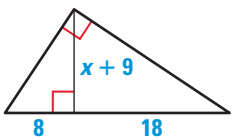
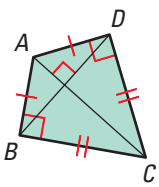
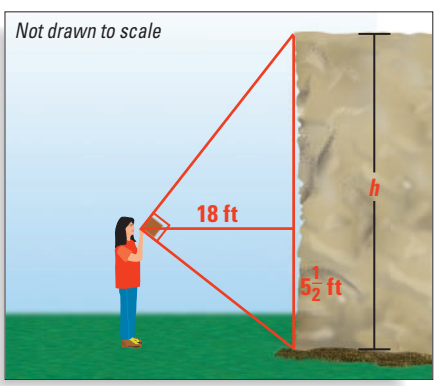


31. **KITE DESIGN** You are designing a diamond-shaped kite. You know that $AD = 44.8$ centimeters, $DC = 72$ centimeters, and $AC = 84.8$ centimeters. You want to use a straight crossbar $\overline{BD}$. About how long should it be? Explain.



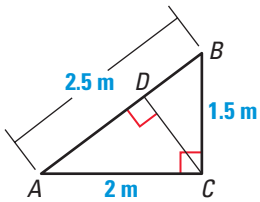
32. **ROCK CLIMBING** You and a friend want to know how much rope you need to climb a large rock. To estimate the height of the rock, you use the method from Example 3 on page 530. As shown at the right, your friend uses a square to line up the top and the bottom of the rock. You measure the vertical distance from the ground to your friend's eye and the distance from your friend to the rock. Estimate the height of the rock.



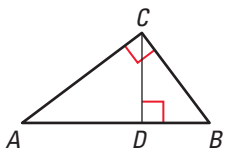
STUDENT HELP

Look Back
For help with finding the area of a triangle, see p. 51.

33. FINDING AREA Write similarity statements for the three similar right triangles in the diagram. Then find the area of each triangle. Explain how you got your answers.



PROVING THEOREMS 9.1, 9.2, AND 9.3 In Exercises 34–36, use the diagram at the right.



34. Use the diagram to prove Theorem 9.1 on page 527. (*Hint:* Look back at the plan for proof on page 528.)

GIVEN ▶ $\triangle ABC$ is a right triangle; altitude $\overline{CD}$ is drawn to hypotenuse $\overline{AB}$.

PROVE ▶ $\triangle CBD \sim \triangle ABC$, $\triangle ACD \sim \triangle ABC$, and $\triangle CBD \sim \triangle ACD$.

35. Use the diagram to prove Theorem 9.2 on page 529.

GIVEN ▶ $\triangle ABC$ is a right triangle; altitude $\overline{CD}$ is drawn to hypotenuse $\overline{AB}$.

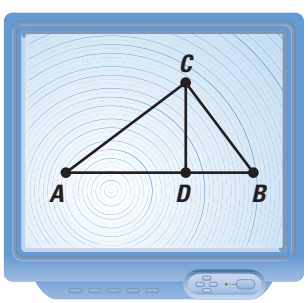
PROVE ▶ $\dfrac{BD}{CD} = \dfrac{CD}{AD}$

36. Use the diagram to prove Theorem 9.3 on page 529.

GIVEN ▶ $\triangle ABC$ is a right triangle; altitude $\overline{CD}$ is drawn to hypotenuse $\overline{AB}$.

PROVE ▶ $\dfrac{AB}{BC} = \dfrac{BC}{BD}$ and $\dfrac{AB}{AC} = \dfrac{AC}{AD}$

USING TECHNOLOGY In Exercises 37–40, use geometry software. You will demonstrate that Theorem 9.2 is true only for a *right* triangle. Follow the steps below to construct a triangle.



1. Draw a triangle and label its vertices A, B, and C. The triangle should *not* be a right triangle.
2. Draw altitude $\overline{CD}$ from point C to side $\overline{AB}$.
3. Measure $\angle C$. Then measure $\overline{AD}$, $\overline{CD}$, and $\overline{BD}$.

37. Calculate the values of the ratios $\dfrac{BD}{CD}$ and $\dfrac{CD}{AD}$. What does Theorem 9.2 say about the values of these ratios?

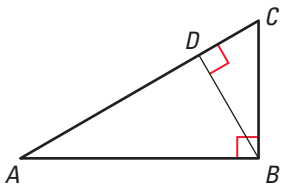
38. Drag point C until $m\angle C = 90°$. What happens to the values of the ratios $\dfrac{BD}{CD}$ and $\dfrac{CD}{AD}$?

39. Explain how your answers to Exercises 37 and 38 support the conclusion that Theorem 9.2 is true only for a right triangle.

40. Use the triangle you constructed to show that Theorem 9.3 is true only for a right triangle. Describe your procedure.

Test Preparation

41. MULTIPLE CHOICE Use the diagram at the right. Decide which proportions are true.



I. $\frac{DB}{DC} = \frac{DA}{DB}$ **II.** $\frac{BA}{CB} = \frac{CB}{BD}$

III. $\frac{CA}{BA} = \frac{BA}{CA}$ **IV.** $\frac{DB}{BC} = \frac{DA}{BA}$

Ⓐ I only Ⓑ II only Ⓒ I and II only Ⓓ I and IV only

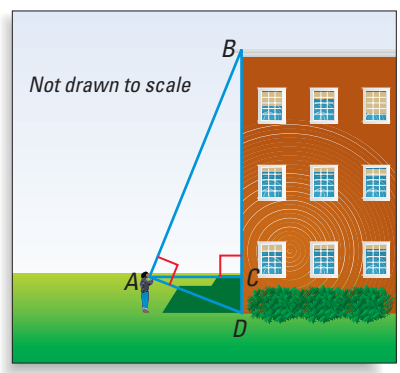
42. MULTIPLE CHOICE In the diagram above, $AC = 24$ and $BC = 12$. Find AD. If necessary, round to the nearest hundredth.

Ⓐ 6 Ⓑ 16.97 Ⓒ 18 Ⓓ 20.78

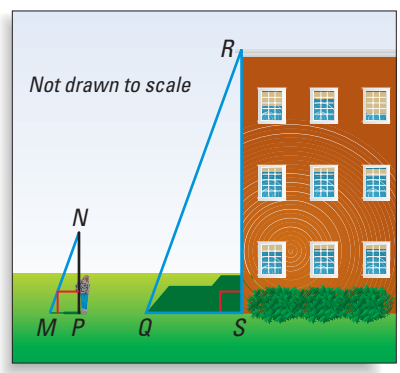
★ Challenge

43. *Writing* Two methods for indirectly measuring the height of a building are shown below. For each method, describe what distances need to be measured directly. Explain how to find the height of the building using these measurements. Describe one advantage and one disadvantage of each method. Copy and label the diagrams as part of your explanations.

Method 1 Use the method described in Example 3 on page 530.



Method 2 Use the method described in Exercises 55 and 56 on page 486.



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

SOLVING EQUATIONS **Solve the equation.** (Skills Review, p. 800, for 9.2)

44. $n^2 = 169$ **45.** $14 + x^2 = 78$ **46.** $d^2 + 18 = 99$

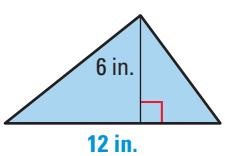
LOGICAL REASONING **Write the converse of the statement. Decide whether the converse is *true* or *false*.** (Review 2.1)

47. If a triangle is obtuse, then one of its angles is greater than 90°.

48. If two triangles are congruent, then their corresponding angles are congruent.

FINDING AREA **Find the area of the figure.** (Review 1.7, 6.7 for 9.2)

49.



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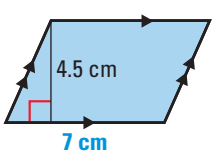
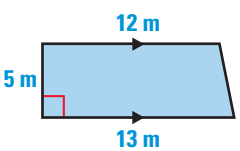


51.



9.2 The Pythagorean Theorem

What you should learn

GOAL 1 Prove the Pythagorean Theorem.

GOAL 2 Use the Pythagorean Theorem to solve **real-life** problems, such as determining how far a ladder will reach in **Ex. 32**.

Why you should learn it

▼ To measure **real-life** lengths indirectly, such as the length of the support beam of a skywalk in **Example 4**.

REAL LIFE

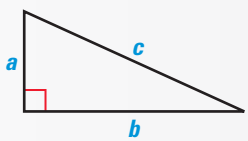
GOAL 1 PROVING THE PYTHAGOREAN THEOREM

In this lesson, you will study one of the most famous theorems in mathematics—the *Pythagorean Theorem*. The relationship it describes has been known for thousands of years.

THEOREM

THEOREM 9.4 ***Pythagorean Theorem***

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.



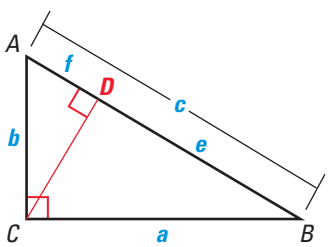
$c^2 = a^2 + b^2$

PROVING THE PYTHAGOREAN THEOREM There are many different proofs of the Pythagorean Theorem. One is shown below. Other proofs are found in Exercises 37 and 38 on page 540, and in the *Math and History* feature on page 557.

GIVEN ▶ In $\triangle ABC$, $\angle BCA$ is a right angle.

PROVE ▶ $a^2 + b^2 = c^2$

Plan for Proof Draw altitude $\overline{CD}$ to the hypotenuse. Then apply Geometric Mean Theorem 9.3, which states that when the altitude is drawn to the hypotenuse of a right triangle, each leg of the right triangle is the geometric mean of the hypotenuse and the segment of the hypotenuse that is adjacent to that leg.



Proof

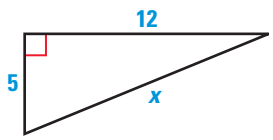
Statements	Reasons
1. Draw a perpendicular from C to $\overline{AB}$.	**1.** Perpendicular Postulate
2. $\frac{c}{a} = \frac{a}{e}$ and $\frac{c}{b} = \frac{b}{f}$	**2.** Geometric Mean Theorem 9.3
3. $ce = a^2$ and $cf = b^2$	**3.** Cross product property
4. $ce + cf = a^2 + b^2$	**4.** Addition property of equality
5. $c(e + f) = a^2 + b^2$	**5.** Distributive property
6. $e + f = c$	**6.** Segment Addition Postulate
7. $c^2 = a^2 + b^2$	**7.** Substitution property of equality

GOAL 2 USING THE PYTHAGOREAN THEOREM

A **Pythagorean triple** is a set of three positive integers a, b, and c that satisfy the equation $c^2 = a^2 + b^2$. For example, the integers 3, 4, and 5 form a Pythagorean triple because $5^2 = 3^2 + 4^2$.

EXAMPLE 1 *Finding the Length of a Hypotenuse*

Find the length of the hypotenuse of the right triangle. Tell whether the side lengths form a Pythagorean triple.



SOLUTION

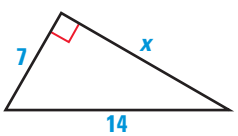
$(\text{hypotenuse})^2 = (\text{leg})^2 + (\text{leg})^2$	**Pythagorean Theorem**
$x^2 = 5^2 + 12^2$	**Substitute.**
$x^2 = 25 + 144$	**Multiply.**
$x^2 = 169$	**Add.**
$x = 13$	**Find the positive square root.**

▶ Because the side lengths 5, 12, and 13 are integers, they form a Pythagorean triple.

Many right triangles have side lengths that do not form a Pythagorean triple, as shown in Example 2.

EXAMPLE 2 *Finding the Length of a Leg*

Find the length of the leg of the right triangle.



SOLUTION

$(\text{hypotenuse})^2 = (\text{leg})^2 + (\text{leg})^2$	**Pythagorean Theorem**
$14^2 = 7^2 + x^2$	**Substitute.**
$196 = 49 + x^2$	**Multiply.**
$147 = x^2$	**Subtract 49 from each side.**
$\sqrt{147} = x$	**Find the positive square root.**
$\sqrt{49} \cdot \sqrt{3} = x$	**Use product property.**
$7\sqrt{3} = x$	**Simplify the radical.**

In Example 2, the side length was written as a radical in simplest form. In real-life problems, it is often more convenient to use a calculator to write a decimal approximation of the side length. For instance, in Example 2, $x = 7 \cdot \sqrt{3} \approx 12.1$.

STUDENT HELP

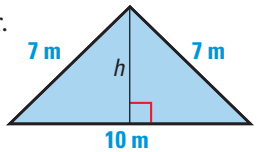
Skills Review
For help with simplifying radicals, see p. 799.

EXAMPLE 3 *Finding the Area of a Triangle*

STUDENT HELP

Look Back
For help with finding the area of a triangle, see p. 51.

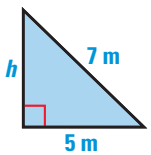
Find the area of the triangle to the nearest tenth of a meter.



SOLUTION

You are given that the base of the triangle is 10 meters, but you do not know the height h.

Because the triangle is isosceles, it can be divided into two congruent right triangles with the given dimensions. Use the Pythagorean Theorem to find the value of h.



$$7^2 = 5^2 + h^2 \quad \textbf{Pythagorean Theorem}$$

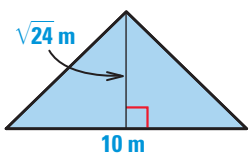
$$49 = 25 + h^2 \quad \textbf{Multiply.}$$

$$24 = h^2 \quad \textbf{Subtract 25 from both sides.}$$

$$\sqrt{24} = h \quad \textbf{Find the positive square root.}$$

Now find the area of the original triangle.

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2}(10)(\sqrt{24}) \\ &\approx 24.5 \text{ m}^2 \end{aligned}$$



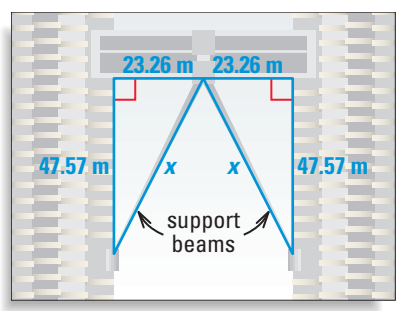
▶ The area of the triangle is about 24.5 m^2.

EXAMPLE 4 *Indirect Measurement*

FOCUS ON PEOPLE

REAL LIFE **CESAR PELLI** is an architect who designed the twin skyscrapers shown on page 535. These 1483 foot buildings tower over the city of Kuala Lumpur, Malaysia.

SUPPORT BEAM The skyscrapers shown on page 535 are connected by a skywalk with support beams. You can use the Pythagorean Theorem to find the approximate length of each support beam.



Each support beam forms the hypotenuse of a right triangle. The right triangles are congruent, so the support beams are the same length.

$$x^2 = (23.26)^2 + (47.57)^2 \quad \textbf{Pythagorean Theorem}$$

$$x = \sqrt{(23.26)^2 + (47.57)^2} \quad \textbf{Find the positive square root.}$$

$$x \approx 52.95 \quad \textbf{Use a calculator to approximate.}$$

▶ The length of each support beam is about 52.95 meters.

GUIDED PRACTICE

Vocabulary Check ✓

1. State the Pythagorean Theorem in your own words.

Concept Check ✓

2. Which equations are true for $\triangle PQR$?

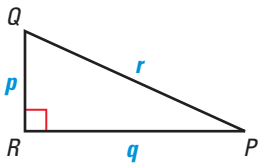
   A. $r^2 = p^2 + q^2$

   B. $q^2 = p^2 + r^2$

   C. $p^2 = r^2 - q^2$

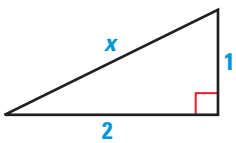
   D. $r^2 = (p + q)^2$

   E. $p^2 = q^2 + r^2$



Skill Check ✓

Find the unknown side length. Tell whether the side lengths form a Pythagorean triple.

3.


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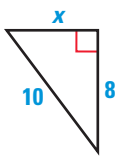

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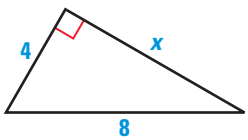
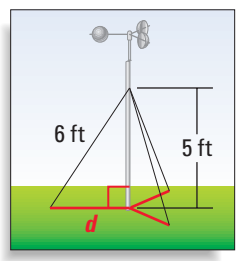


6. **ANEMOMETER** An anemometer (an uh MAHM ih tur) is a device used to measure windspeed. The anemometer shown is attached to the top of a pole. Support wires are attached to the pole 5 feet above the ground. Each support wire is 6 feet long. How far from the base of the pole is each wire attached to the ground?

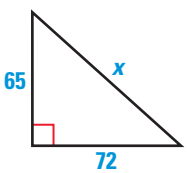


PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 819.

FINDING SIDE LENGTHS **Find the unknown side length. Simplify answers that are radicals. Tell whether the side lengths form a Pythagorean triple.**

7.


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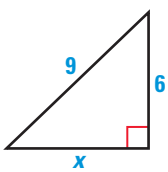

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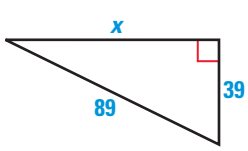

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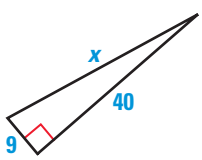

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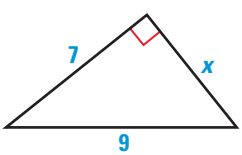

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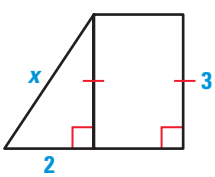

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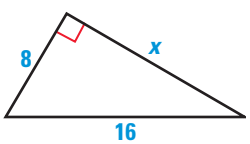
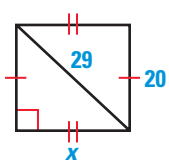


14.


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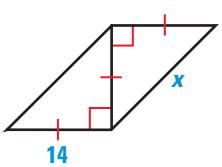


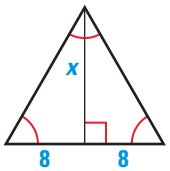
STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 7–24
Example 2: Exs. 7–24
Example 3: Exs. 25–30
Example 4: Exs. 31–36

FINDING LENGTHS **Find the value of *x*. Simplify answers that are radicals.**

16.



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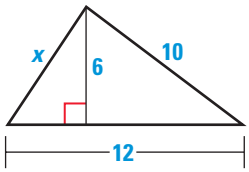

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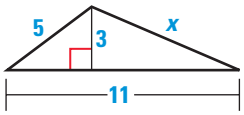


PYTHAGOREAN TRIPLES **The variables *r* and *s* represent the lengths of the legs of a right triangle, and *t* represents the length of the hypotenuse. The values of *r*, *s*, and *t* form a Pythagorean triple. Find the unknown value.**

19. $r = 12, s = 16$ **20.** $r = 9, s = 12$ **21.** $r = 18, t = 30$

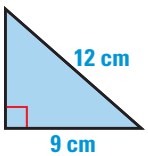
22. $s = 20, t = 101$ **23.** $r = 35, t = 37$ **24.** $t = 757, s = 595$

STUDENT HELP

Look Back
For help with finding areas of quadrilaterals, see pp. 372–375.

FINDING AREA **Find the area of the figure. Round decimal answers to the nearest tenth.**

25.



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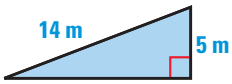

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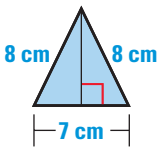

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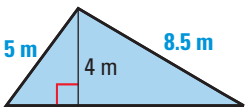

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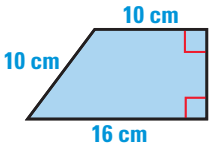

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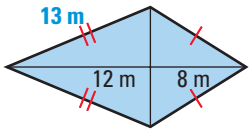
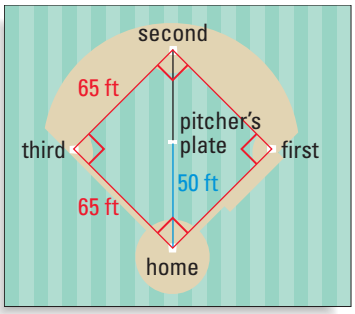
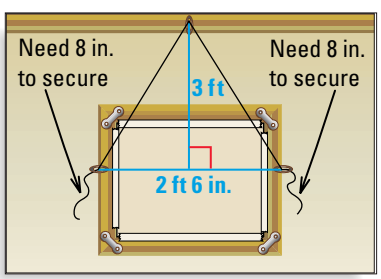


31. **SOFTBALL DIAMOND** In slow-pitch softball, the distance between consecutive bases is 65 feet. The pitcher's plate is located on a line between second base and home plate, 50 feet from home plate. How far is the pitcher's plate from second base? Justify your answer.



32. **SAFETY** The distance of the base of a ladder from the wall it leans against should be at least $\frac{1}{4}$ of the ladder's total length. Suppose a 10 foot ladder is placed according to these guidelines. Give the minimum distance of the base of the ladder from the wall. How far up the wall will the ladder reach? Explain. Include a sketch with your explanation.

33. **ART GALLERY** You want to hang a painting 3 feet from a hook near the ceiling of an art gallery, as shown. In addition to the length of wire needed for hanging, you need 16 inches of wire to secure the wire to the back of the painting. Find the total length of wire needed to hang the painting.

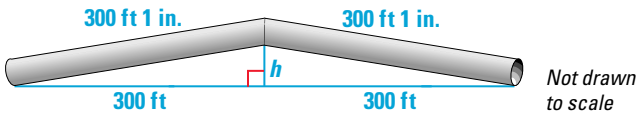


FOCUS ON CAREERS

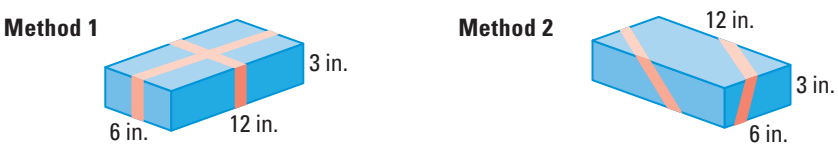
MECHANICAL ENGINEERS use science, mathematics, and engineering principles in their work. They evaluate, install, operate, and maintain mechanical products and systems, such as the Trans-Alaska pipeline.

CAREER LINK www.mcdougallittell.com

34. TRANS-ALASKA PIPELINE Metal expands and contracts with changes in temperature. The Trans-Alaska pipeline was built to accommodate expansion and contraction. Suppose that it had not been built this way. Consider a 600 foot section of pipe that expands 2 inches and buckles, as shown below. Estimate the height h of the buckle.

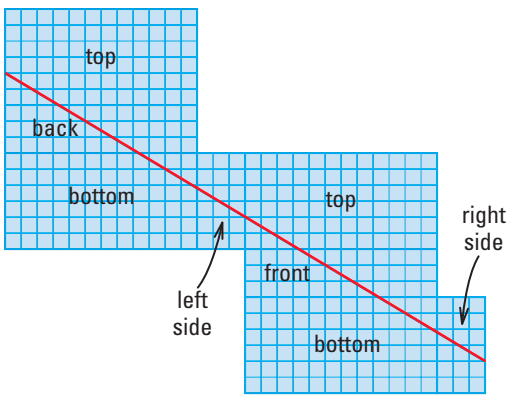


WRAPPING A BOX **In Exercises 35 and 36, two methods are used to wrap ribbon around a rectangular box with the dimensions shown below. The amount of ribbon needed does not include a knot or bow.**



35. How much ribbon is needed to wrap the box using Method 1?

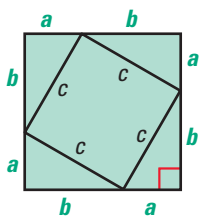
36. The red line on the diagram at the right shows the path the ribbon follows around the box when Method 2 is used. Does Method 2 use more or less ribbon than Method 1? Explain your thinking.



STUDENT HELP

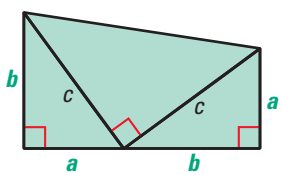
HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with the proofs in Exs. 37 and 38.

37. PROVING THE PYTHAGOREAN THEOREM Explain how the diagram at the right can be used to prove the Pythagorean Theorem algebraically. (*Hint:* Write two different expressions that represent the area of the large square. Then set them equal to each other.)



38. GARFIELD'S PROOF James Abram Garfield, the twentieth president of the United States, discovered a proof of the Pythagorean Theorem in 1876. His proof involved the fact that a trapezoid can be formed from two congruent right triangles and an isosceles right triangle.

Use the diagram to write a paragraph proof showing that $a^2 + b^2 = c^2$. (*Hint:* Write two different expressions that represent the area of the trapezoid. Then set them equal to each other.)



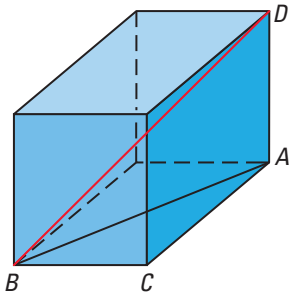
Test Preparation

39. MULTI-STEP PROBLEM To find the length of a diagonal of a rectangular box, you can use the Pythagorean Theorem twice. Use the theorem once with right $\triangle ABC$ to find the length of the diagonal of the base.

$$AB = \sqrt{(AC)^2 + (BC)^2}$$

Then use the theorem with right $\triangle ABD$ to find the length of the diagonal of the box.

$$BD = \sqrt{(AB)^2 + (AD)^2}$$



a. Is it possible to carry a 9 foot piece of lumber in an enclosed rectangular trailer that is 4 feet by 8 feet by 4 feet?

b. Is it possible to store a 20 foot long pipe in a rectangular room that is 10 feet by 12 feet by 8 feet? Explain.

c. *Writing* Write a formula for finding the diagonal d of a rectangular box with length ℓ, width w, and height h. Explain your reasoning.

★ Challenge

PERIMETER OF A RHOMBUS The diagonals of a rhombus have lengths *a* and *b*. Use this information in Exercises 40 and 41.

40. Prove that the perimeter of the rhombus is $2\sqrt{a^2 + b^2}$.

41. The perimeter of a rhombus is 80 centimeters. The lengths of its diagonals are in the ratio 3:4. Find the length of each diagonal.

EXTRA CHALLENGE
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MIXED REVIEW

USING RADICALS Evaluate the expression. (Algebra Review, p. 522, for 9.3)

42. $(\sqrt{6})^2$ **43.** $(\sqrt{9})^2$ **44.** $(\sqrt{14})^2$ **45.** $(2\sqrt{2})^2$

46. $(4\sqrt{13})^2$ **47.** $-(5\sqrt{49})^2$ **48.** $4(\sqrt{9})^2$ **49.** $(-7\sqrt{3})^2$

LOGICAL REASONING Determine whether the true statement can be combined with its converse to form a true biconditional statement. (Review 2.2)

50. If a quadrilateral is a square, then it has four congruent sides.

51. If a quadrilateral is a kite, then it has two pairs of congruent sides.

52. For all real numbers x, if $x \geq 1$, then $x^2 \geq 1$.

53. For all real numbers x, if $x > 1$, then $\frac{1}{x} < 1$.

54. If one interior angle of a triangle is obtuse, then the sum of the other two interior angles is less than 90°.

USING ALGEBRA Prove that the points represent the vertices of a parallelogram. (Review 6.3)

55. $P(4, 3)$, $Q(6, -8)$, $R(10, -3)$, $S(8, 8)$

56. $P(5, 0)$, $Q(2, 9)$, $R(-6, 6)$, $S(-3, -3)$

9.3 The Converse of the Pythagorean Theorem

What you should learn

GOAL 1 Use the Converse of the Pythagorean Theorem to solve problems.

GOAL 2 Use side lengths to classify triangles by their angle measures.

Why you should learn it

▼ To determine whether **real-life** angles are right angles, such as the four angles formed by the foundation of a building in **Example 3**.

REAL LIFE

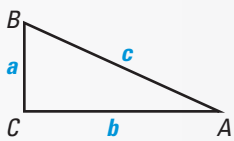
GOAL 1 USING THE CONVERSE

In Lesson 9.2, you learned that if a triangle is a right triangle, then the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. The Converse of the Pythagorean Theorem is also true, as stated below. Exercise 43 asks you to prove the Converse of the Pythagorean Theorem.

THEOREM

THEOREM 9.5 ***Converse of the Pythagorean Theorem***

If the square of the length of the longest side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle.



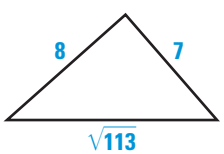
If $c^2 = a^2 + b^2$, then $\triangle ABC$ is a right triangle.

You can use the Converse of the Pythagorean Theorem to verify that a given triangle is a right triangle, as shown in Example 1.

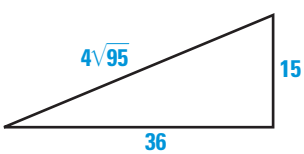
EXAMPLE 1 *Verifying Right Triangles*

The triangles below appear to be right triangles. Tell whether they are right triangles.

a.



b.



SOLUTION

Let c represent the length of the longest side of the triangle. Check to see whether the side lengths satisfy the equation $c^2 = a^2 + b^2$.

a. $(\sqrt{113})^2 \stackrel{?}{=} 7^2 + 8^2$

$113 \stackrel{?}{=} 49 + 64$

$113 = 113$ ✓

The triangle is a right triangle.

b. $(4\sqrt{95})^2 \stackrel{?}{=} 15^2 + 36^2$

$4^2 \cdot (\sqrt{95})^2 \stackrel{?}{=} 15^2 + 36^2$

$16 \cdot 95 \stackrel{?}{=} 225 + 1296$

$1520 \neq 1521$

The triangle is not a right triangle.

GOAL 2 CLASSIFYING TRIANGLES

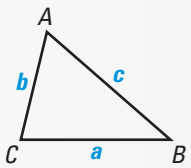
Sometimes it is hard to tell from looking whether a triangle is obtuse or acute. The theorems below can help you tell.

THEOREMS

THEOREM 9.6

If the square of the length of the longest side of a triangle is less than the sum of the squares of the lengths of the other two sides, then the triangle is acute.

If $c^2 < a^2 + b^2$, then $\triangle ABC$ is acute.

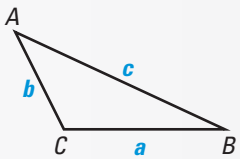


$c^2 < a^2 + b^2$

THEOREM 9.7

If the square of the length of the longest side of a triangle is greater than the sum of the squares of the lengths of the other two sides, then the triangle is obtuse.

If $c^2 > a^2 + b^2$, then $\triangle ABC$ is obtuse.



$c^2 > a^2 + b^2$

EXAMPLE 2 *Classifying Triangles*

Decide whether the set of numbers can represent the side lengths of a triangle. If they can, classify the triangle as *right*, *acute*, or *obtuse*.

a. 38, 77, 86 **b.** 10.5, 36.5, 37.5

SOLUTION

You can use the Triangle Inequality to confirm that each set of numbers can represent the side lengths of a triangle.

STUDENT HELP

Look Back
For help with the Triangle Inequality, see p. 297.

Compare the square of the length of the longest side with the sum of the squares of the lengths of the two shorter sides.

a.

$c^2 \ \underline{?}\ a^2 + b^2$	**Compare c^2 with $a^2 + b^2$.**
$86^2 \ \underline{?}\ 38^2 + 77^2$	**Substitute.**
$7396 \ \underline{?}\ 1444 + 5929$	**Multiply.**
$7396 > 7373$	**c^2 is greater than $a^2 + b^2$.**

▶ Because $c^2 > a^2 + b^2$, the triangle is obtuse.

b.

$c^2 \ \underline{?}\ a^2 + b^2$	**Compare c^2 with $a^2 + b^2$.**
$37.5^2 \ \underline{?}\ 10.5^2 + 36.5^2$	**Substitute.**
$1406.25 \ \underline{?}\ 110.25 + 1332.25$	**Multiply.**
$1406.25 < 1442.5$	**c^2 is less than $a^2 + b^2$.**

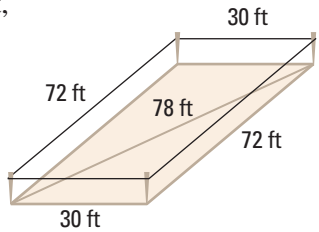
▶ Because $c^2 < a^2 + b^2$, the triangle is acute.

EXAMPLE 3 *Building a Foundation*

CONSTRUCTION You use four stakes and string to mark the foundation of a house. You want to make sure the foundation is rectangular.

a. A friend measures the four sides to be 30 feet, 30 feet, 72 feet, and 72 feet. He says these measurements prove the foundation is rectangular. Is he correct?

b. You measure one of the diagonals to be 78 feet. Explain how you can use this measurement to tell whether the foundation will be rectangular.

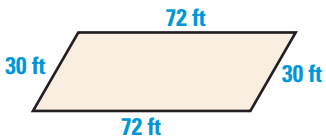


SOLUTION

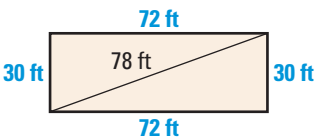
STUDENT HELP

Look Back
For help with classifying quadrilaterals, see Chapter 6.

a. Your friend is not correct. The foundation could be a nonrectangular parallelogram, as shown at the right.



b. The diagonal divides the foundation into two triangles. Compare the square of the length of the longest side with the sum of the squares of the shorter sides of one of these triangles. Because $30^2 + 72^2 = 78^2$, you can conclude that both the triangles are right triangles.



▶ The foundation is a parallelogram with two right angles, which implies that it is rectangular.

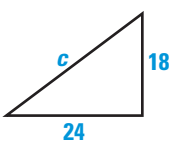
GUIDED PRACTICE

Vocabulary Check ✓

1. State the Converse of the Pythagorean Theorem in your own words.

Concept Check ✓

2. Use the triangle shown at the right. Find values for c so that the triangle is acute, right, and obtuse.

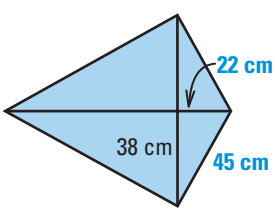


Skill Check ✓

In Exercises 3–6, match the side lengths with the appropriate description.

3. 2, 10, 11	**A.** right triangle
4. 13, 5, 7	**B.** acute triangle
5. 5, 11, 6	**C.** obtuse triangle
6. 6, 8, 10	**D.** not a triangle

7. **KITE DESIGN** You are making the diamond-shaped kite shown at the right. You measure the crossbars to determine whether they are perpendicular. Are they? Explain.



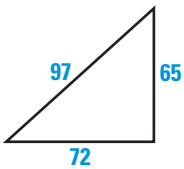
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on pp. 819 and 820.

VERIFYING RIGHT TRIANGLES **Tell whether the triangle is a right triangle.**

8.



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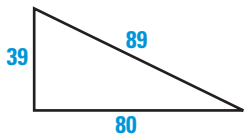

10.

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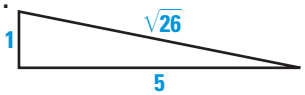

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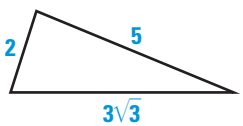

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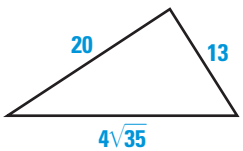


CLASSIFYING TRIANGLES **Decide whether the numbers can represent the side lengths of a triangle. If they can, classify the triangle as *right, acute,* or *obtuse.***

14. 20, 99, 101

15. 21, 28, 35

16. 26, 10, 17

17. 2, 10, 12

18. 4, $\sqrt{67}$, 9

19. $\sqrt{13}$, 6, 7

20. 16, 30, 34

21. 10, 11, 14

22. 4, 5, 5

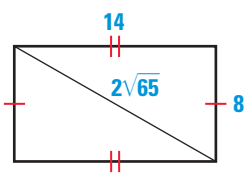
23. 17, 144, 145

24. 10, 49, 50

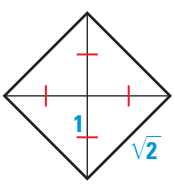
25. $\sqrt{5}$, 5, 5.5

CLASSIFYING QUADRILATERALS **Classify the quadrilateral. Explain how you can prove that the quadrilateral is that type.**

26.



27.



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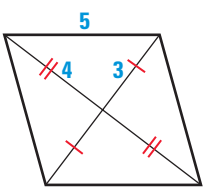
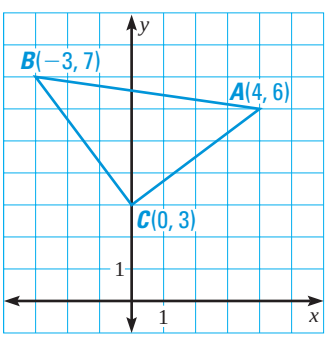


CHOOSING A METHOD **In Exercises 29–31, you will use two different methods for determining whether △*ABC* is a right triangle.**

29. ***Method 1*** Find the slope of $\overline{AC}$ and the slope of $\overline{BC}$. What do the slopes tell you about ∠*ACB*? Is △*ABC* a right triangle? How do you know?

30. ***Method 2*** Use the Distance Formula and the Converse of the Pythagorean Theorem to determine whether △*ABC* is a right triangle.

31. Which method would you use to determine whether a given triangle is right, acute, or obtuse? Explain.



STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 8–13, 30
Example 2: Exs. 14–28, 31–35
Example 3: Exs. 39, 40

USING ALGEBRA **Graph points *P, Q,* and *R*. Connect the points to form △*PQR*. Decide whether △*PQR* is *right, acute,* or *obtuse*.**

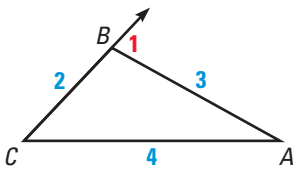
32. $P(-3, 4)$, $Q(5, 0)$, $R(-6, -2)$

33. $P(-1, 2)$, $Q(4, 1)$, $R(0, -1)$

PROOF Write a proof.

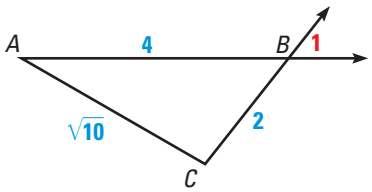
34. GIVEN ▶ $AB = 3$, $BC = 2$, $AC = 4$

PROVE ▶ $\angle 1$ is acute.



35. GIVEN ▶ $AB = 4$, $BC = 2$, $AC = \sqrt{10}$

PROVE ▶ $\angle 1$ is acute.



STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Ex. 36.

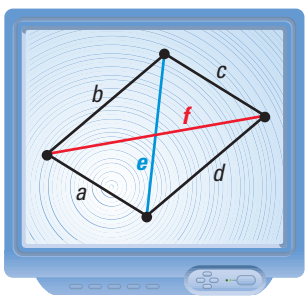
36. PROOF Prove that if a, b, and c are a Pythagorean triple, then ka, kb, and kc (where $k > 0$) represent the side lengths of a right triangle.

37. PYTHAGOREAN TRIPLES Use the results of Exercise 36 and the Pythagorean triple 5, 12, 13. Which sets of numbers can represent the side lengths of a right triangle?

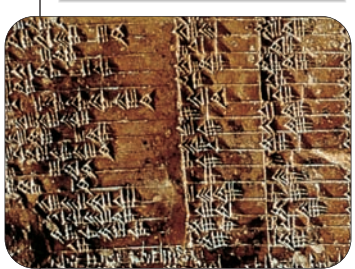
A. 50, 120, 130 **B.** 20, 48, 56 **C.** $1\frac{1}{4}$, 3, $3\frac{1}{4}$ **D.** 1, 2.4, 2.6

38. TECHNOLOGY Use geometry software to construct each of the following figures: a nonspecial quadrilateral, a parallelogram, a rhombus, a square, and a rectangle. Label the sides of each figure a, b, c, and d. Measure each side. Then draw the diagonals of each figure and label them e and f. Measure each diagonal. For which figures does the following statement appear to be true?

$$a^2 + b^2 + c^2 + d^2 = e^2 + f^2$$



FOCUS ON APPLICATIONS



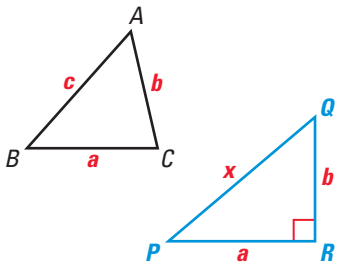
REAL LIFE BABYLONIAN TABLET
This photograph shows part of a Babylonian clay tablet made around 350 B.C. The tablet contains a table of numbers written in cuneiform characters.

39. HISTORY CONNECTION The Babylonian tablet shown at the left contains several sets of triangle side lengths, suggesting that the Babylonians may have been aware of the relationships among the side lengths of right, triangles. The side lengths in the table at the right show several sets of numbers from the tablet. Verify that each set of side lengths forms a Pythagorean triple.

a	b	c
120	119	169
4,800	4,601	6,649
13,500	12,709	18,541

40. AIR TRAVEL You take off in a jet from Cincinnati, Ohio, and fly 403 miles due east to Washington, D.C. You then fly 714 miles to Tallahassee, Florida. Finally, you fly 599 miles back to Cincinnati. Is Cincinnati directly north of Tallahassee? If not, how would you describe its location relative to Tallahassee?

41. **DEVELOPING PROOF** Complete the proof of Theorem 9.6 on page 544.



GIVEN ▶ In $\triangle ABC$, $c^2 < a^2 + b^2$.

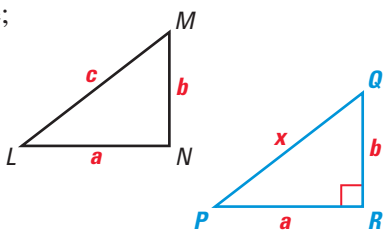
PROVE ▶ $\triangle ABC$ is an acute triangle.

Plan for Proof Draw right $\triangle PQR$ with side lengths a, b, and x. Compare lengths c and x.

Statements	Reasons
1. $x^2 = a^2 + b^2$	**1.** ?
2. $c^2 < a^2 + b^2$	**2.** ?
3. $c^2 < x^2$	**3.** ?
4. $c < x$	**4.** A property of square roots
5. $m\angle C < m\angle R$	**5.** ?
6. $\angle C$ is an acute angle.	**6.** ?
7. $\triangle ABC$ is an acute triangle.	**7.** ?

42. **PROOF** Prove Theorem 9.7 on page 544. Include a diagram and *Given* and *Prove* statements. (*Hint:* Look back at Exercise 41.)

43. **PROOF** Prove the Converse of the Pythagorean Theorem.



GIVEN ▶ In $\triangle LNM$, $\overline{LM}$ is the longest side; $c^2 = a^2 + b^2$.

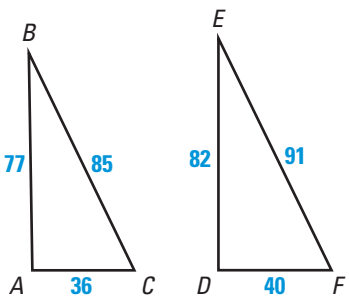
PROVE ▶ $\triangle LNM$ is a right triangle.

Plan for Proof Draw right $\triangle PQR$ with side lengths a, b, and x. Compare lengths c and x.

Test Preparation

QUANTITATIVE COMPARISON **Choose the statement that is true about the given quantities.**

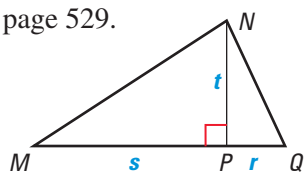
Ⓐ The quantity in column A is greater.
Ⓑ The quantity in column B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the given information.



	Column A	Column B
44.	$m\angle A$	$m\angle D$
45.	$m\angle B + m\angle C$	$m\angle E + m\angle F$

★ Challenge

46. **PROOF** Prove the converse of Theorem 9.2 on page 529.



GIVEN ▶ In $\triangle MQN$, altitude $\overline{NP}$ is drawn to $\overline{MQ}$; t is the geometric mean of r and s.

PROVE ▶ $\triangle MQN$ is a right triangle.

EXTRA CHALLENGE
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MIXED REVIEW

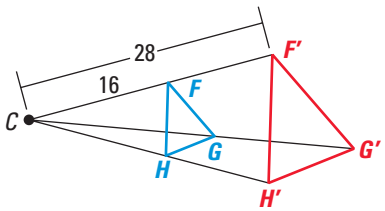
SIMPLIFYING RADICALS **Simplify the expression.** (Skills Review, p. 799, for 9.4)

47. $\sqrt{22} \cdot \sqrt{2}$ **48.** $\sqrt{6} \cdot \sqrt{8}$ **49.** $\sqrt{14} \cdot \sqrt{6}$ **50.** $\sqrt{15} \cdot \sqrt{6}$

51. $\frac{3}{\sqrt{11}}$ **52.** $\frac{4}{\sqrt{5}}$ **53.** $\frac{12}{\sqrt{18}}$ **54.** $\frac{8}{\sqrt{24}}$

DILATIONS **Identify the dilation and find its scale factor.** (Review 8.7)

55.



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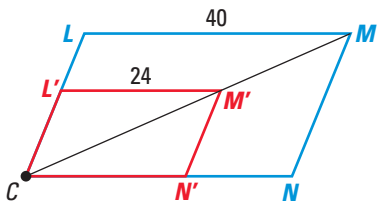
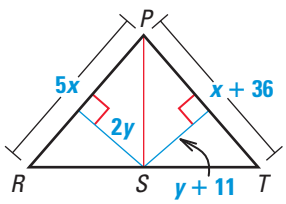


57. **USING ALGEBRA** In the diagram, $\overrightarrow{PS}$ bisects $\angle RPT$, and $\overline{PS}$ is the perpendicular bisector of $\overline{RT}$. Find the values of x and y. (Review 5.1)



QUIZ 1

Self-Test for Lessons 9.1–9.3

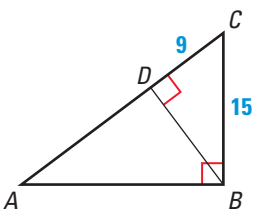
In Exercises 1–4, use the diagram. (Lesson 9.1)

1. Write a similarity statement about the three triangles in the diagram.

2. Which segment's length is the geometric mean of CD and AD?

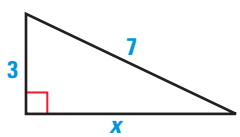
3. Find AC.

4. Find BD.



Find the unknown side length. Simplify answers that are radicals. (Lesson 9.2)

5.



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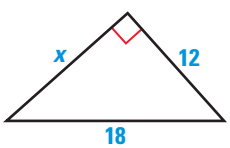

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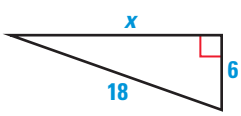
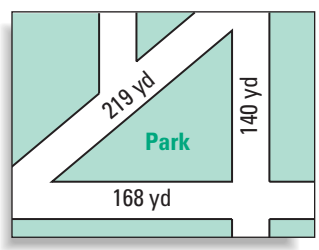


8. **CITY PARK** The diagram shown at the right shows the dimensions of a triangular city park. Does this city park have a right angle? Explain. (Lesson 9.3)



9.4 Special Right Triangles

What you should learn

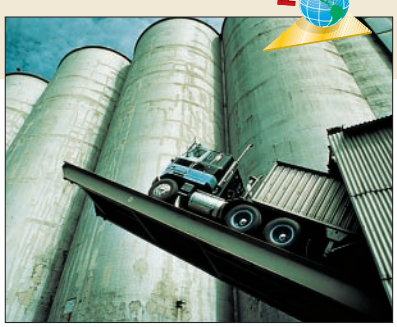
GOAL 1 Find the side lengths of special right triangles.

GOAL 2 Use special right triangles to solve **real-life** problems, such as finding the side lengths of the triangles in the spiral quilt design in **Exs. 31–34**.

Why you should learn it

▼ To use special right triangles to solve **real-life** problems, such as finding the height of a tipping platform in **Example 4**.

REAL LIFE



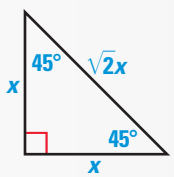
GOAL 1 SIDE LENGTHS OF SPECIAL RIGHT TRIANGLES

Right triangles whose angle measures are 45°-45°-90° or 30°-60°-90° are called **special right triangles**. In the Activity on page 550, you may have noticed certain relationships among the side lengths of each of these special right triangles. The theorems below describe these relationships. Exercises 35 and 36 ask you to prove the theorems.

THEOREMS ABOUT SPECIAL RIGHT TRIANGLES

THEOREM 9.8 ***45°-45°-90° Triangle Theorem***

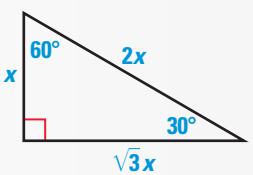
In a 45°-45°-90° triangle, the hypotenuse is $\sqrt{2}$ times as long as each leg.



Hypotenuse = $\sqrt{2}$ · leg

THEOREM 9.9 ***30°-60°-90° Triangle Theorem***

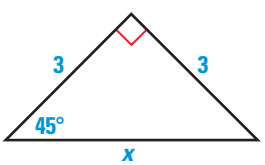
In a 30°-60°-90° triangle, the hypotenuse is twice as long as the shorter leg, and the longer leg is $\sqrt{3}$ times as long as the shorter leg.



Hypotenuse = 2 · shorter leg
Longer leg = $\sqrt{3}$ · shorter leg

EXAMPLE 1 *Finding the Hypotenuse in a 45°-45°-90° Triangle*

Find the value of x.



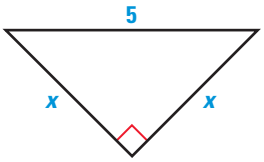
SOLUTION

By the Triangle Sum Theorem, the measure of the third angle is 45°. The triangle is a 45°-45°-90° right triangle, so the length x of the hypotenuse is $\sqrt{2}$ times the length of a leg.

Hypotenuse $= \sqrt{2} \cdot$ leg	**45°-45°-90° Triangle Theorem**
$x = \sqrt{2} \cdot 3$	**Substitute.**
$x = 3\sqrt{2}$	**Simplify.**

EXAMPLE 2 *Finding a Leg in a 45°-45°-90° Triangle*

Find the value of x.



SOLUTION

Because the triangle is an isosceles right triangle, its base angles are congruent. The triangle is a 45°-45°-90° right triangle, so the length of the hypotenuse is $\sqrt{2}$ times the length x of a leg.

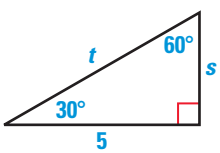
$\text{Hypotenuse} = \sqrt{2} \cdot \text{leg}$	**45°-45°-90° Triangle Theorem**
$5 = \sqrt{2} \cdot x$	**Substitute.**
$\frac{5}{\sqrt{2}} = \frac{\sqrt{2}x}{\sqrt{2}}$	**Divide each side by $\sqrt{2}$.**
$\frac{5}{\sqrt{2}} = x$	**Simplify.**
$\frac{\sqrt{2}}{\sqrt{2}} \cdot \frac{5}{\sqrt{2}} = x$	**Multiply numerator and denominator by $\sqrt{2}$.**
$\frac{5\sqrt{2}}{2} = x$	**Simplify.**

EXAMPLE 3 *Side Lengths in a 30°-60°-90° Triangle*

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

Find the values of s and t.



SOLUTION

Because the triangle is a 30°-60°-90° triangle, the longer leg is $\sqrt{3}$ times the length s of the shorter leg.

$\text{Longer leg} = \sqrt{3} \cdot \text{shorter leg}$	**30°-60°-90° Triangle Theorem**
$5 = \sqrt{3} \cdot s$	**Substitute.**
$\frac{5}{\sqrt{3}} = \frac{\sqrt{3} \cdot s}{\sqrt{3}}$	**Divide each side by $\sqrt{3}$.**
$\frac{5}{\sqrt{3}} = s$	**Simplify.**
$\frac{\sqrt{3}}{\sqrt{3}} \cdot \frac{5}{\sqrt{3}} = s$	**Multiply numerator and denominator by $\sqrt{3}$.**
$\frac{5\sqrt{3}}{3} = s$	**Simplify.**

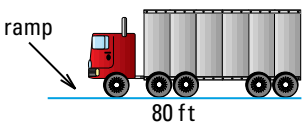
The length t of the hypotenuse is twice the length s of the shorter leg.

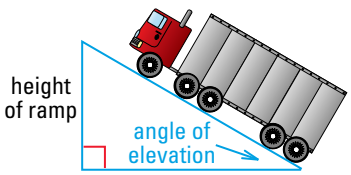
$\text{Hypotenuse} = 2 \cdot \text{shorter leg}$	**30°-60°-90° Triangle Theorem**
$t = 2 \cdot \frac{5\sqrt{3}}{3}$	**Substitute.**
$t = \frac{10\sqrt{3}}{3}$	**Simplify.**

GOAL 2 USING SPECIAL RIGHT TRIANGLES IN REAL LIFE

EXAMPLE 4 *Finding the Height of a Ramp*

TIPPING PLATFORM A tipping platform is a ramp used to unload trucks, as shown on page 551. How high is the end of an 80 foot ramp when it is tipped by a 30° angle? by a 45° angle?





SOLUTION

When the angle of elevation is 30°, the height h of the ramp is the length of the shorter leg of a 30°-60°-90° triangle. The length of the hypotenuse is 80 feet.

$80 = 2h$ **30°-60°-90° Triangle Theorem**

$40 = h$ **Divide each side by 2.**

When the angle of elevation is 45°, the height of the ramp is the length of a leg of a 45°-45°-90° triangle. The length of the hypotenuse is 80 feet.

$80 = \sqrt{2} \cdot h$ **45°-45°-90° Triangle Theorem**

$\frac{80}{\sqrt{2}} = h$ **Divide each side by $\sqrt{2}$.**

$56.6 \approx h$ **Use a calculator to approximate.**

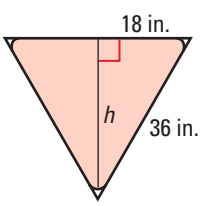
▶ When the angle of elevation is 30°, the ramp height is 40 feet. When the angle of elevation is 45°, the ramp height is about 56 feet 7 inches.

EXAMPLE 5 *Finding the Area of a Sign*

ROAD SIGN The road sign is shaped like an equilateral triangle. Estimate the area of the sign by finding the area of the equilateral triangle.

SOLUTION

First find the height h of the triangle by dividing it into two 30°-60°-90° triangles. The length of the longer leg of one of these triangles is h. The length of the shorter leg is 18 inches.



$h = \sqrt{3} \cdot 18 = 18\sqrt{3}$ **30°-60°-90° Triangle Theorem**

Use $h = 18\sqrt{3}$ to find the area of the equilateral triangle.

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}(36)\left(18\sqrt{3}\right) \approx 561.18$$

▶ The area of the sign is about 561 square inches.

GUIDED PRACTICE

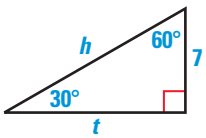
Vocabulary Check ✓ 1. What is meant by the term *special right triangles*?

Concept Check ✓ 2. **CRITICAL THINKING** Explain why any two 30°-60°-90° triangles are similar.

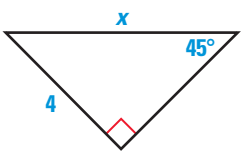
Use the diagram to tell whether the equation is *true* or *false*.

3. $t = 7\sqrt{3}$ 4. $t = \sqrt{3h}$ 5. $h = 2t$

6. $h = 14$ 7. $7 = \frac{h}{2}$ 8. $7 = \frac{t}{\sqrt{3}}$



Skill Check ✓ **Find the value of each variable. Write answers in simplest radical form.**

9.


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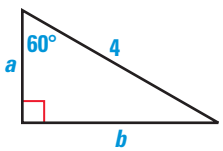

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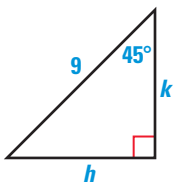
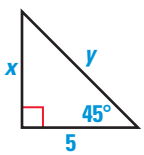


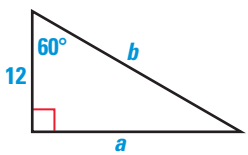
PRACTICE AND APPLICATIONS

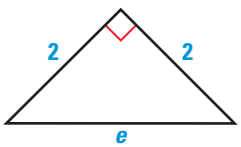
STUDENT HELP

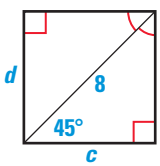
Extra Practice to help you master skills is on p. 820.

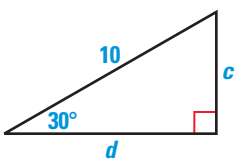
USING ALGEBRA **Find the value of each variable. Write answers in simplest radical form.**

12.


13.


14.


15.


16.


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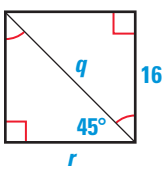
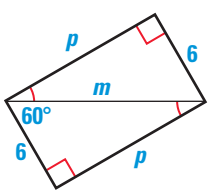


18.


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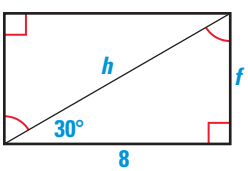
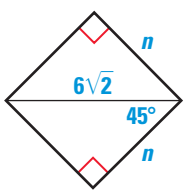


20.


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 12–23
Example 2: Exs. 12–23
Example 3: Exs. 12–23
Example 4: Exs. 28–29, 34
Example 5: Exs. 24–27

FINDING LENGTHS **Sketch the figure that is described. Find the requested length. Round decimals to the nearest tenth.**

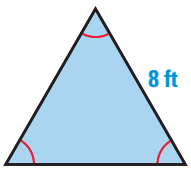
21. The side length of an equilateral triangle is 5 centimeters. Find the length of an altitude of the triangle.

22. The perimeter of a square is 36 inches. Find the length of a diagonal.

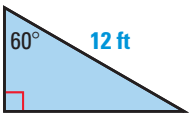
23. The diagonal of a square is 26 inches. Find the length of a side.

FINDING AREA **Find the area of the figure. Round decimal answers to the nearest tenth.**

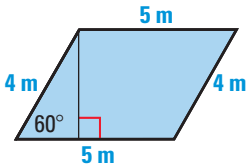
24.



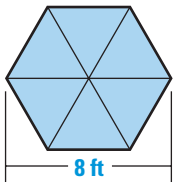
25.



26.



27. **AREA OF A WINDOW** A hexagonal window consists of six congruent panes of glass. Each pane is an equilateral triangle. Find the area of the entire window.

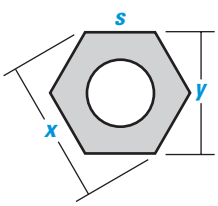


JEWELRY **Estimate the length *x* of each earring.**

28.

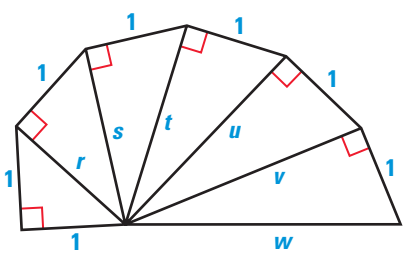
29.

30. **TOOLS** Find the values of x and y for the hexagonal nut shown at the right when $s = 2$ centimeters. (*Hint:* In Exercise 27 above, you saw that a regular hexagon can be divided into six equilateral triangles.)



LOGICAL REASONING **The quilt design in the photo is based on the pattern in the diagram below. Use the diagram in Exercises 31–34.**

Wheel of Theodorus



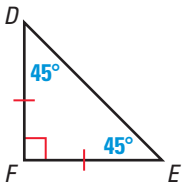
31. Find the values of r, s, t, u, v, and w. Explain the procedure you used to find the values.

32. Which of the triangles, if any, is a 45°-45°-90° triangle?

33. Which of the triangles, if any, is a 30°-60°-90° triangle?

34. **USING ALGEBRA** Suppose there are n triangles in the spiral. Write an expression for the hypotenuse of the nth triangle.

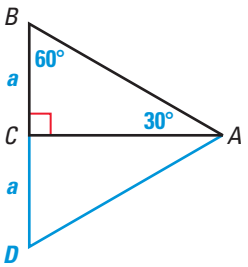
35. **PARAGRAPH PROOF** Write a paragraph proof of Theorem 9.8 on page 551.



GIVEN ▶ $\triangle DEF$ is a 45°-45°-90° triangle.

PROVE ▶ The hypotenuse is $\sqrt{2}$ times as long as each leg.

36. **PARAGRAPH PROOF** Write a paragraph proof of Theorem 9.9 on page 551.



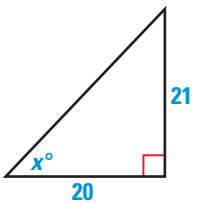
GIVEN ▶ $\triangle ABC$ is a 30°-60°-90° triangle.

PROVE ▶ The hypotenuse is twice as long as the shorter leg and the longer leg is $\sqrt{3}$ times as long as the shorter leg.

Plan for Proof Construct $\triangle ADC$ congruent to $\triangle ABC$. Then prove that $\triangle ABD$ is equilateral. Express the lengths AB and AC in terms of a.

Test Preparation

37. **MULTIPLE CHOICE** Which of the statements below is true about the diagram at the right?

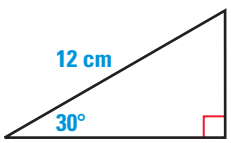


Ⓐ $x < 45$ Ⓑ $x = 45$

Ⓒ $x > 45$ Ⓓ $x \leq 45$

Ⓔ Not enough information is given to determine the value of x.

38. **MULTIPLE CHOICE** Find the perimeter of the triangle shown at the right to the nearest tenth of a centimeter.

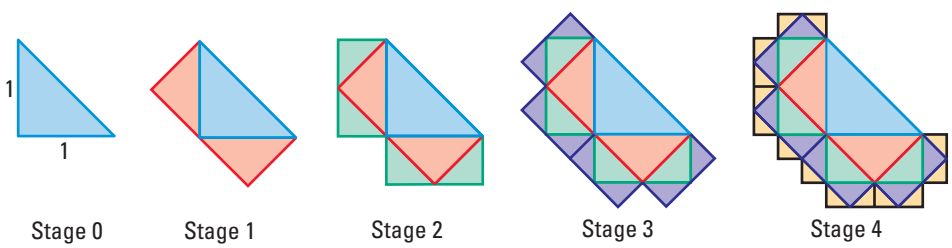


Ⓐ 28.4 cm Ⓑ 30 cm

Ⓒ 31.2 cm Ⓓ 41.6 cm

★ Challenge

VISUAL THINKING **In Exercises 39–41, use the diagram below. Each triangle in the diagram is a 45°-45°-90° triangle. At Stage 0, the legs of the triangle are each 1 unit long.**



39. Find the exact lengths of the legs of the triangles that are added at each stage. Leave radicals in the denominators of fractions.

40. Describe the pattern of the lengths in Exercise 39.

41. Find the length of a leg of a triangle added in Stage 8. Explain how you found your answer.

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

42. FINDING A SIDE LENGTH A triangle has one side of 9 inches and another of 14 inches. Describe the possible lengths of the third side. **(Review 5.5)**

FINDING REFLECTIONS Find the coordinates of the reflection without using a coordinate plane. **(Review 7.2)**

43. $Q(-1, -2)$ reflected in the x-axis

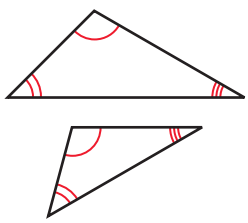
44. $P(8, 3)$ reflected in the y-axis

45. $A(4, -5)$ reflected in the y-axis

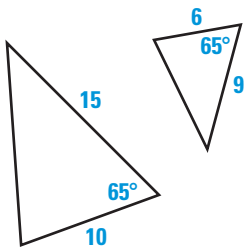
46. $B(0, 10)$ reflected in the x-axis

DEVELOPING PROOF Name a postulate or theorem that can be used to prove that the two triangles are similar. **(Review 8.5 for 9.5)**

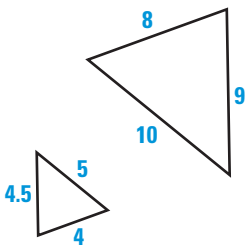
47.



48.



49.



MATH & History

Pythagorean Theorem Proofs

APPLICATION LINK
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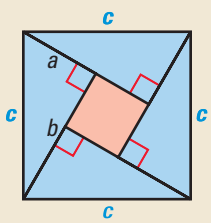
THEN

AROUND THE SIXTH CENTURY B.C., the Greek mathematician Pythagoras founded a school for the study of philosophy, mathematics, and science. Many people believe that an early proof of the Pythagorean Theorem came from this school.

NOW

TODAY, the Pythagorean theorem is one of the most famous theorems in geometry. More than 100 different proofs now exist.

The diagram is based on one drawn by the Hindu mathematician Bhāskara (1114–1185). The four blue right triangles are congruent.

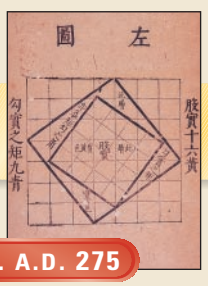


1. Write an expression in terms of a and b for the combined areas of the blue triangles. Then write an expression in terms of a and b for the area of the small red square.
2. Use the diagram to show that $a^2 + b^2 = c^2$. (*Hint:* This proof of the Pythagorean Theorem is similar to the one in Exercise 37 on page 540.)

c. 529 B.C.

School of Pythagoras is founded.

Chinese manuscript includes a diagram that can be used to prove the theorem.



c. A.D. 275

1876

Future U.S. President Garfield discovers a proof of the theorem.

Nicaraguan stamp commemorates the Pythagorean Theorem.

1971

9.5 Trigonometric Ratios

What you should learn

GOAL 1 Find the sine, the cosine, and the tangent of an acute angle.

GOAL 2 Use trigonometric ratios to solve **real-life** problems, such as estimating the height of a tree in **Example 6**.

Why you should learn it

▼ To solve **real-life** problems, such as in finding the height of a water slide in **Ex. 37**.

REAL LIFE

GOAL 1 FINDING TRIGONOMETRIC RATIOS

A **trigonometric ratio** is a ratio of the lengths of two sides of a right triangle. The word *trigonometry* is derived from the ancient Greek language and means measurement of triangles. The three basic trigonometric ratios are **sine**, **cosine**, and **tangent**, which are abbreviated as *sin*, *cos*, and *tan*, respectively.

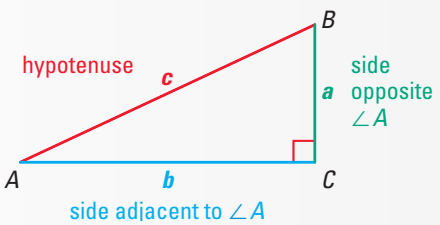
TRIGONOMETRIC RATIOS

Let $\triangle ABC$ be a right triangle. The sine, the cosine, and the tangent of the acute angle $\angle A$ are defined as follows.

$$\sin A = \frac{\text{side opposite } \angle A}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos A = \frac{\text{side adjacent to } \angle A}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{side opposite } \angle A}{\text{side adjacent to } \angle A} = \frac{a}{b}$$



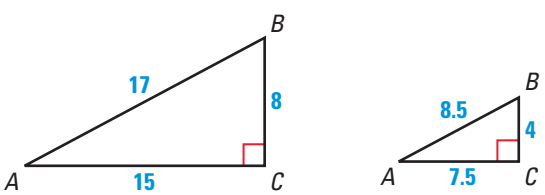
The value of a trigonometric ratio depends only on the measure of the acute angle, not on the particular right triangle that is used to compute the value.

EXAMPLE 1 *Finding Trigonometric Ratios*

Compare the sine, the cosine, and the tangent ratios for $\angle A$ in each triangle below.

SOLUTION

By the SSS Similarity Theorem, the triangles are similar. Their corresponding sides are in proportion, which implies that the trigonometric ratios for $\angle A$ in each triangle are the same.



	Large triangle	Small triangle
$\sin A = \dfrac{\text{opposite}}{\text{hypotenuse}}$	$\dfrac{8}{17} \approx 0.4706$	$\dfrac{4}{8.5} \approx 0.4706$
$\cos A = \dfrac{\text{adjacent}}{\text{hypotenuse}}$	$\dfrac{15}{17} \approx 0.8824$	$\dfrac{7.5}{8.5} \approx 0.8824$
$\tan A = \dfrac{\text{opposite}}{\text{adjacent}}$	$\dfrac{8}{15} \approx 0.5333$	$\dfrac{4}{7.5} \approx 0.5333$

Trigonometric ratios are frequently expressed as decimal approximations.

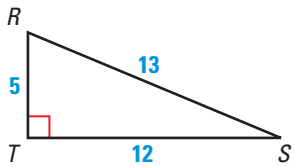
EXAMPLE 2 *Finding Trigonometric Ratios*

STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

Find the sine, the cosine, and the tangent of the indicated angle.

a. $\angle S$ **b.** $\angle R$



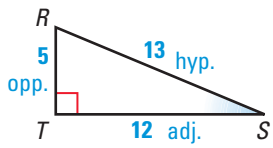
SOLUTION

a. The length of the hypotenuse is 13. For $\angle S$, the length of the opposite side is 5, and the length of the adjacent side is 12.

$$\sin S = \frac{\text{opp.}}{\text{hyp.}} = \frac{5}{13} \approx 0.3846$$

$$\cos S = \frac{\text{adj.}}{\text{hyp.}} = \frac{12}{13} \approx 0.9231$$

$$\tan S = \frac{\text{opp.}}{\text{adj.}} = \frac{5}{12} \approx 0.4167$$

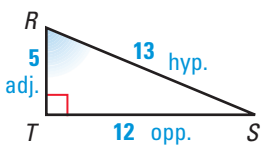


b. The length of the hypotenuse is 13. For $\angle R$, the length of the opposite side is 12, and the length of the adjacent side is 5.

$$\sin R = \frac{\text{opp.}}{\text{hyp.}} = \frac{12}{13} \approx 0.9231$$

$$\cos R = \frac{\text{adj.}}{\text{hyp.}} = \frac{5}{13} \approx 0.3846$$

$$\tan R = \frac{\text{opp.}}{\text{adj.}} = \frac{12}{5} = 2.4$$



. . . . . .

You can find trigonometric ratios for 30°, 45°, and 60° by applying what you know about special right triangles.

EXAMPLE 3 *Trigonometric Ratios for 45°*

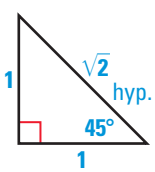
Find the sine, the cosine, and the tangent of 45°.

STUDENT HELP

Study Tip
The expression sin 45° means the sine of an angle whose measure is 45°.

SOLUTION

Begin by sketching a 45°-45°-90° triangle. Because all such triangles are similar, you can make calculations simple by choosing 1 as the length of each leg. From Theorem 9.8 on page 551, it follows that the length of the hypotenuse is $\sqrt{2}$.



$$\sin 45° = \frac{\text{opp.}}{\text{hyp.}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.7071$$

$$\cos 45° = \frac{\text{adj.}}{\text{hyp.}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.7071$$

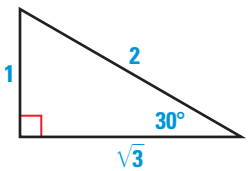
$$\tan 45° = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{1} = 1$$

EXAMPLE 4 *Trigonometric Ratios for 30°*

Find the sine, the cosine, and the tangent of 30°.

SOLUTION

Begin by sketching a 30°-60°-90° triangle. To make the calculations simple, you can choose 1 as the length of the shorter leg. From Theorem 9.9 on page 551, it follows that the length of the longer leg is $\sqrt{3}$ and the length of the hypotenuse is 2.



$$\sin 30° = \frac{\text{opp.}}{\text{hyp.}} = \frac{1}{2} = 0.5$$

$$\cos 30° = \frac{\text{adj.}}{\text{hyp.}} = \frac{\sqrt{3}}{2} \approx 0.8660$$

$$\tan 30° = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.5774$$

EXAMPLE 5 *Using a Calculator*

You can use a calculator to approximate the sine, the cosine, and the tangent of 74°. Make sure your calculator is in *degree mode.* The table shows some sample keystroke sequences accepted by most calculators.

Sample keystroke sequences	Sample calculator display	Rounded approximation
74 SIN or SIN 74 ENTER	0.961261695	0.9613
74 COS or COS 74 ENTER	0.275637355	0.2756
74 TAN or TAN 74 ENTER	3.487414444	3.4874

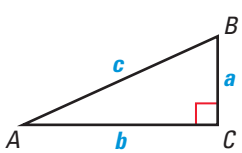
STUDENT HELP

Trig Table
For a table of trigonometric ratios, see p. 845.

If you look back at Examples 1–5, you will notice that the sine or the cosine of an acute angle is always less than 1. The reason is that these trigonometric ratios involve the ratio of a leg of a right triangle to the hypotenuse. The length of a leg of a right triangle is always less than the length of its hypotenuse, so the ratio of these lengths is always less than one.

Because the tangent of an acute angle involves the ratio of one leg to another leg, the tangent of an angle can be less than 1, equal to 1, or greater than 1.

TRIGONOMETRIC IDENTITIES A trigonometric identity is an equation involving trigonometric ratios that is true for all acute angles. You are asked to prove the following identities in Exercises 47 and 52:



$$(\sin A)^2 + (\cos A)^2 = 1$$

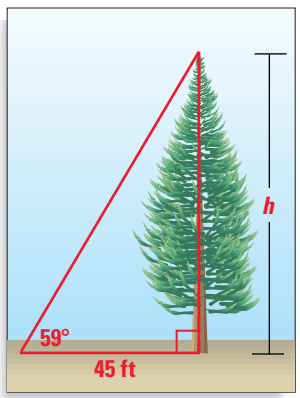
$$\tan A = \frac{\sin A}{\cos A}$$

GOAL 2 USING TRIGONOMETRIC RATIOS IN REAL LIFE

Suppose you stand and look up at a point in the distance, such as the top of the tree in Example 6. The angle that your line of sight makes with a line drawn horizontally is called the **angle of elevation**.

EXAMPLE 6 *Indirect Measurement*

FOREST RY You are measuring the height of a Sitka spruce tree in Alaska. You stand 45 feet from the base of the tree. You measure the angle of elevation from a point on the ground to the top of the tree to be 59°. To estimate the height of the tree, you can write a trigonometric ratio that involves the height h and the known length of 45 feet.



$$\tan 59° = \frac{\text{opposite}}{\text{adjacent}}$$ **Write ratio.**

$$\tan 59° = \frac{h}{45}$$ **Substitute.**

$$45 \tan 59° = h$$ **Multiply each side by 45.**

$$45(1.6643) \approx h$$ **Use a calculator or table to find tan 59°.**

$$74.9 \approx h$$ **Simplify.**

▶ The tree is about 75 feet tall.

FOCUS ON CAREERS

FORESTRY Foresters manage and protect forests. Their work can involve measuring tree heights. Foresters can use an instrument called a *clinometer* to measure the angle of elevation from a point on the ground to the top of a tree.

INTERNET **CAREER LINK**
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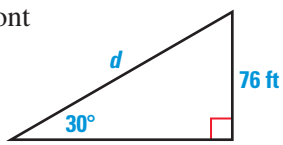
EXAMPLE 7 *Estimating a Distance*

ESCALATORS The escalator at the Wilshire/Vermont Metro Rail Station in Los Angeles rises 76 feet at a 30° angle. To find the distance d a person travels on the escalator stairs, you can write a trigonometric ratio that involves the hypotenuse and the known leg length of 76 feet.



$$\sin 30° = \frac{\text{opposite}}{\text{hypotenuse}}$$ **Write ratio for sine of 30°.**

$$\sin 30° = \frac{76}{d}$$ **Substitute.**

$$d \sin 30° = 76$$ **Multiply each side by *d*.**

$$d = \frac{76}{\sin 30°}$$ **Divide each side by sin 30°.**

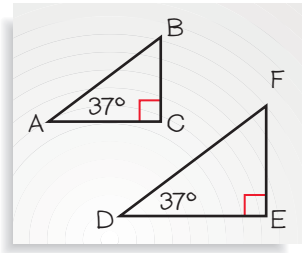
$$d = \frac{76}{0.5}$$ **Substitute 0.5 for sin 30°.**

$$d = 152$$ **Simplify.**

▶ A person travels 152 feet on the escalator stairs.

GUIDED PRACTICE

In Exercises 1 and 2, use the diagram at the right.



Vocabulary Check ✓

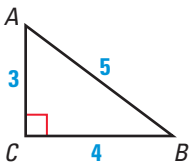
1. Use the diagram to explain what is meant by the *sine*, the *cosine*, and the *tangent* of $\angle A$.

Concept Check ✓

2. **ERROR ANALYSIS** A student says that $\sin D > \sin A$ because the side lengths of $\triangle DEF$ are greater than the side lengths of $\triangle ABC$. Explain why the student is incorrect.

Skill Check ✓

In Exercises 3–8, use the diagram shown at the right to find the trigonometric ratio.

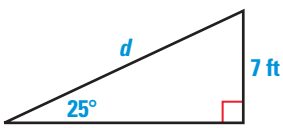


3. $\sin A$ **4.** $\cos A$

5. $\tan A$ **6.** $\sin B$

7. $\cos B$ **8.** $\tan B$

9. **ESCALATORS** One early escalator built in 1896 rose at an angle of 25°. As shown in the diagram at the right, the vertical lift was 7 feet. Estimate the distance d a person traveled on this escalator.



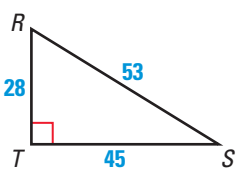
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 820.

FINDING TRIGONOMETRIC RATIOS **Find the sine, the cosine, and the tangent of the acute angles of the triangle. Express each value as a decimal rounded to four places.**

10.



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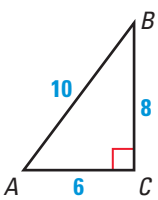

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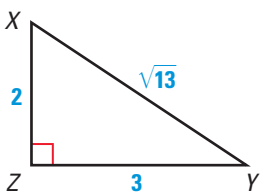

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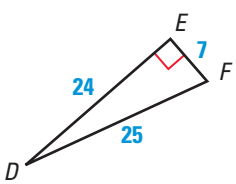

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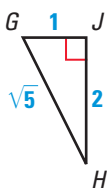

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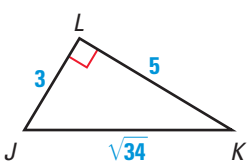


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 10–15, 28–36
Example 2: Exs. 10–15, 28–36
Example 3: Exs. 34–36
Example 4: Exs. 34–36
Example 5: Exs. 16–27
Example 6: Exs. 37–42
Example 7: Exs. 37–42

CALCULATOR **Use a calculator to approximate the given value to four decimal places.**

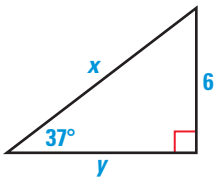
16. $\sin 48°$ **17.** $\cos 13°$ **18.** $\tan 81°$ **19.** $\sin 27°$

20. $\cos 70°$ **21.** $\tan 2°$ **22.** $\sin 78°$ **23.** $\cos 36°$

24. $\tan 23°$ **25.** $\cos 63°$ **26.** $\sin 56°$ **27.** $\tan 66°$

USING TRIGONOMETRIC RATIOS **Find the value of each variable. Round decimals to the nearest tenth.**

28.



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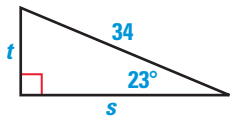

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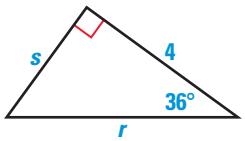

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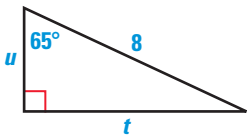

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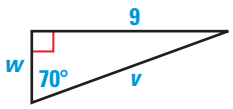

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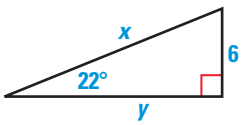
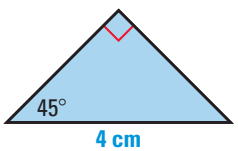


FINDING AREA **Find the area of the triangle. Round decimals to the nearest tenth.**

34.



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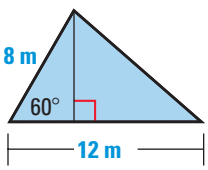

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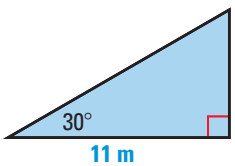
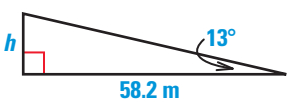


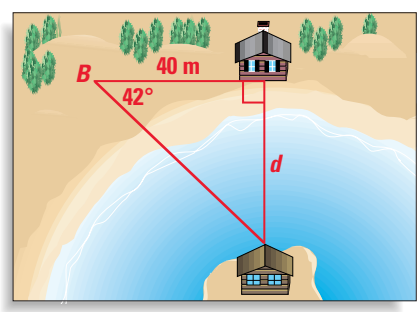
FOCUS ON APPLICATIONS

REAL LIFE WATER SLIDE Even though riders on a water slide may travel at only 20 miles per hour, the curves on the slide can make riders feel as though they are traveling much faster.

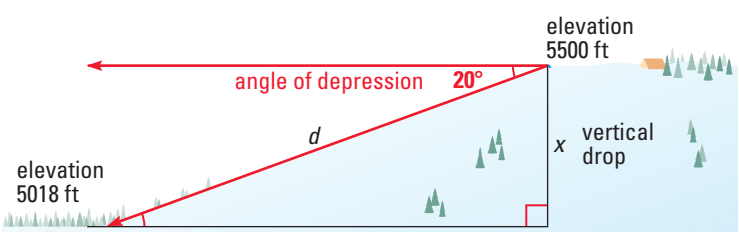
37. **WATER SLIDE** The angle of elevation from the base to the top of a waterslide is about 13°. The slide extends horizontally about 58.2 meters. Estimate the height h of the slide.



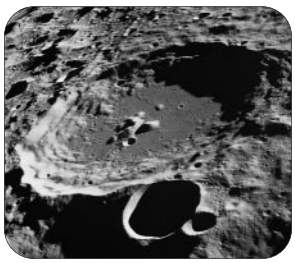
38. **SURVEYING** To find the distance d from a house on shore to a house on an island, a surveyor measures from the house on shore to point B, as shown in the diagram. An instrument called a *transit* is used to find the measure of $\angle B$. Estimate the distance d.



39. **SKI SLOPE** Suppose you stand at the top of a ski slope and look down at the bottom. The angle that your line of sight makes with a line drawn horizontally is called the *angle of depression*, as shown below. The *vertical drop* is the difference in the elevations of the top and the bottom of the slope. Find the vertical drop x of the slope in the diagram. Then estimate the distance d a person skiing would travel on this slope.

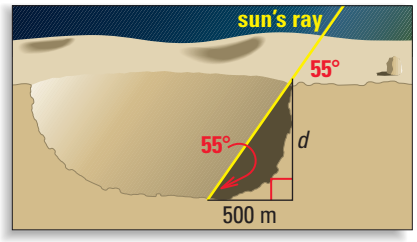


FOCUS ON APPLICATIONS

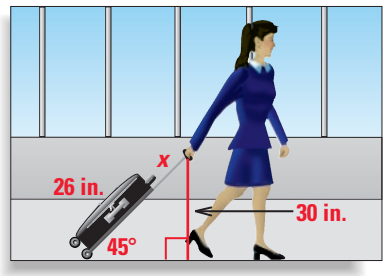


LUNAR CRATERS Because the moon has no atmosphere to protect it from being hit by meteorites, its surface is pitted with craters. There is no wind, so a crater can remain undisturbed for millions of years—unless another meteorite crashes into it.

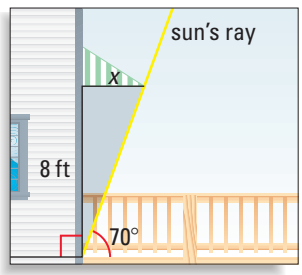
40. **SCIENCE CONNECTION** Scientists can measure the depths of craters on the moon by looking at photos of shadows. The length of the shadow cast by the edge of a crater is about 500 meters. The sun's angle of elevation is 55°. Estimate the depth d of the crater.



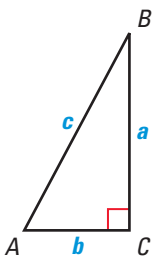
41. **LUGGAGE DESIGN** Some luggage pieces have wheels and a handle so that the luggage can be pulled along the ground. Suppose a person's hand is about 30 inches from the floor. About how long should the handle be on the suitcase shown so that it can roll at a comfortable angle of 45° with the floor?



42. **BUYING AN AWNING** Your family room has a sliding-glass door with a southern exposure. You want to buy an awning for the door that will be just long enough to keep the sun out when it is at its highest point in the sky. The angle of elevation of the sun at this point is 70°, and the height of the door is 8 feet. About how far should the overhang extend?



CRITICAL THINKING **In Exercises 43 and 44, use the diagram.**



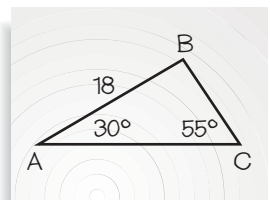
43. Write expressions for the sine, the cosine, and the tangent of each acute angle in the triangle.

44. *Writing* Use your results from Exercise 43 to explain how the tangent of one acute angle of a right triangle is related to the tangent of the other acute angle. How are the sine and the cosine of one acute angle of a right triangle related to the sine and the cosine of the other acute angle?

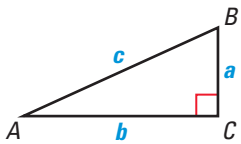
45. **TECHNOLOGY** Use geometry software to construct a right triangle. Use your triangle to explore and answer the questions below. Explain your procedure.

- For what angle measure is the tangent of an acute angle equal to 1?
- For what angle measures is the tangent of an acute angle greater than 1?
- For what angle measures is the tangent of an acute angle less than 1?

46. **ERROR ANALYSIS** To find the length of $\overline{BC}$ in the diagram at the right, a student writes $\tan 55° = \frac{18}{BC}$. What mistake is the student making? Show how the student can find BC. (*Hint:* Begin by drawing an altitude from B to $\overline{AC}$.)



47. **PROOF** Use the diagram of $\triangle ABC$. Complete the proof of the trigonometric identity below.



$$(\sin A)^2 + (\cos A)^2 = 1$$

GIVEN ▶ $\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$

PROVE ▶ $(\sin A)^2 + (\cos A)^2 = 1$

Statements	Reasons
1. $\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$	**1.** ___?___
2. $a^2 + b^2 = c^2$	**2.** ___?___
3. $\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$	**3.** ___?___
4. $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$	**4.** A property of exponents
5. $(\sin A)^2 + (\cos A)^2 = 1$	**5.** ___?___

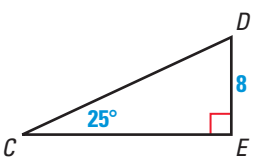
DEMONSTRATING A FORMULA **Show that $(\sin A)^2 + (\cos A)^2 = 1$ for the given angle measure.**

48. $m\angle A = 30°$ **49.** $m\angle A = 45°$ **50.** $m\angle A = 60°$ **51.** $m\angle A = 13°$

52. **PROOF** Use the diagram in Exercise 47. Write a two-column proof of the following trigonometric identity: $\tan A = \frac{\sin A}{\cos A}$.

Test Preparation

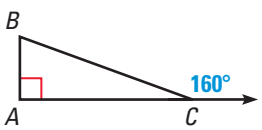
53. **MULTIPLE CHOICE** Use the diagram at the right. Find CD.



Ⓐ $8 \cos 25°$ Ⓑ $8 \sin 25°$ Ⓒ $8 \tan 25°$

Ⓓ $\frac{8}{\sin 25°}$ Ⓔ $\frac{8}{\cos 25°}$

54. **MULTIPLE CHOICE** Use the diagram at the right. Which expression is *not* equivalent to AC?

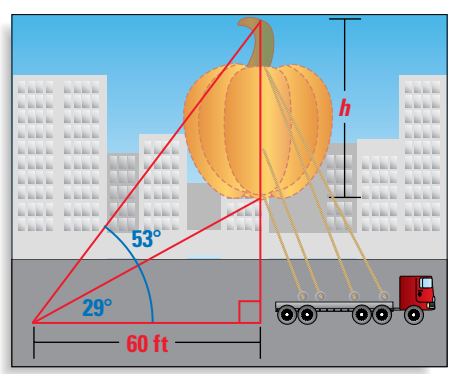


Ⓐ $BC \sin 70°$ Ⓑ $BC \cos 20°$ Ⓒ $\frac{BC}{\tan 20°}$

Ⓓ $\frac{BA}{\tan 20°}$ Ⓔ $BA \tan 70°$

★ Challenge

55. **PARADE** You are at a parade looking up at a large balloon floating directly above the street. You are 60 feet from a point on the street directly beneath the balloon. To see the top of the balloon, you look up at an angle of 53°. To see the bottom of the balloon, you look up at an angle of 29°.



Estimate the height h of the balloon to the nearest foot.

EXTRA CHALLENGE
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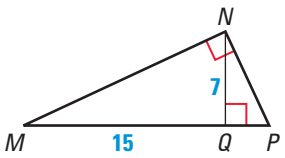
MIXED REVIEW

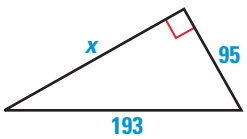
56. SKETCHING A DILATION $\triangle PQR$ is mapped onto $\triangle P'Q'R'$ by a dilation. In $\triangle PQR$, $PQ = 3$, $QR = 5$, and $PR = 4$. In $\triangle P'Q'R'$, $P'Q' = 6$. Sketch the dilation, identify it as a reduction or an enlargement, and find the scale factor. Then find the length of $Q'R'$ and $P'R'$. **(Review 8.7)**

57. FINDING LENGTHS Write similarity statements for the three similar triangles in the diagram. Then find QP and NP. Round decimals to the nearest tenth. **(Review 9.1)**



PYTHAGOREAN THEOREM Find the unknown side length. Simplify answers that are radicals. Tell whether the side lengths form a Pythagorean triple. (Review 9.2 for 9.6)

58.



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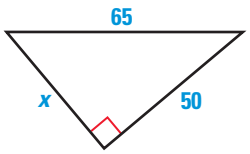

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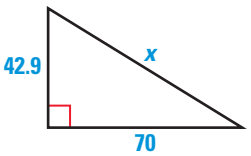


QUIZ 2

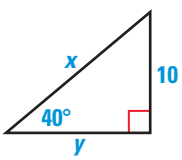
Self-Test for Lessons 9.4 and 9.5

Sketch the figure that is described. Then find the requested information. Round decimals to the nearest tenth. (Lesson 9.4)

1. The side length of an equilateral triangle is 4 meters. Find the length of an altitude of the triangle.
2. The perimeter of a square is 16 inches. Find the length of a diagonal.
3. The side length of an equilateral triangle is 3 inches. Find the area of the triangle.

Find the value of each variable. Round decimals to the nearest tenth. (Lesson 9.5)

4.



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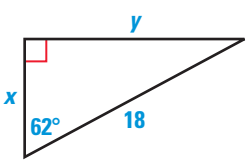

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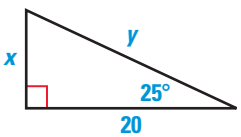
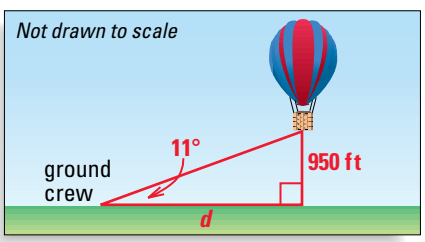


7. HOT-AIR BALLOON The ground crew for a hot-air balloon can see the balloon in the sky at an angle of elevation of 11°. The pilot radios to the crew that the hot-air balloon is 950 feet above the ground. Estimate the horizontal distance d of the hot-air balloon from the ground crew. **(Lesson 9.5)**



9.6 Solving Right Triangles

What you should learn

GOAL 1 Solve a right triangle.

GOAL 2 Use right triangles to solve **real-life** problems, such as finding the glide angle and altitude of a space shuttle in **Example 3**.

Why you should learn it

▼ To solve **real-life** problems such as determining the correct dimensions of a wheel-chair ramp in **Exs. 39–41**.

REAL LIFE

GOAL 1 SOLVING A RIGHT TRIANGLE

Every right triangle has one right angle, two acute angles, one hypotenuse, and two legs. To **solve a right triangle** means to determine the measures of all six parts. You can solve a right triangle if you know either of the following:

- Two side lengths
- One side length and one acute angle measure

As you learned in Lesson 9.5, you can use the side lengths of a right triangle to find trigonometric ratios for the acute angles of the triangle. As you will see in this lesson, once you know the sine, the cosine, or the tangent of an acute angle, you can use a calculator to find the measure of the angle.

In general, for an acute angle A:

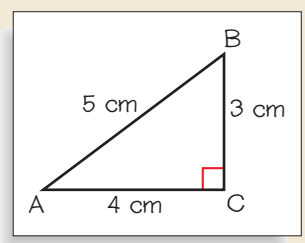
if $\sin A = x$, then $\sin^{-1} x = m\angle A$. ← **The expression $\sin^{-1} x$ is read as "the inverse sine of x."**

if $\cos A = y$, then $\cos^{-1} y = m\angle A$.

if $\tan A = z$, then $\tan^{-1} z = m\angle A$.

ACTIVITY Developing Concepts

Finding Angles in Right Triangles



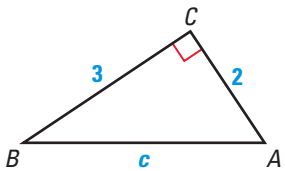
1. Carefully draw right $\triangle ABC$ with side lengths of 3 centimeters, 4 centimeters, and 5 centimeters, as shown.
2. Use trigonometric ratios to find the sine, the cosine, and the tangent of $\angle A$. Express the ratios in decimal form.
3. In Step 2, you found that $\sin A = \frac{3}{5} = 0.6$. You can use a calculator to find $\sin^{-1} 0.6$. Most calculators use one of the keystroke sequences below.

   $\overbrace{\text{2nd SIN}}^{\sin^{-1}}$ 0.6 ENTER or 0.6 $\overbrace{\text{2nd SIN}}^{\sin^{-1}}$

   Make sure your calculator is in degree mode. Then use each of the trigonometric ratios you found in Step 2 to approximate the measure of $\angle A$ to the nearest tenth of a degree.
4. Use a protractor to measure $\angle A$. How does the measured value compare with your calculated values?

EXAMPLE 1 *Solving a Right Triangle*

Solve the right triangle. Round decimals to the nearest tenth.



SOLUTION

Begin by using the Pythagorean Theorem to find the length of the hypotenuse.

$(\text{hypotenuse})^2 = (\text{leg})^2 + (\text{leg})^2$	**Pythagorean Theorem**
$c^2 = 3^2 + 2^2$	**Substitute.**
$c^2 = 13$	**Simplify.**
$c = \sqrt{13}$	**Find the positive square root.**
$c \approx 3.6$	**Use a calculator to approximate.**

Then use a calculator to find the measure of $\angle B$:

(2 ÷ 3) 2nd TAN $\approx 33.7°$

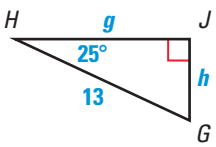
Finally, because $\angle A$ and $\angle B$ are complements, you can write

$$m\angle A = 90° - m\angle B \approx 90° - 33.7° = 56.3°.$$

▶ The side lengths of the triangle are 2, 3, and $\sqrt{13}$, or about 3.6. The triangle has one right angle and two acute angles whose measures are about 33.7° and 56.3°.

EXAMPLE 2 *Solving a Right Triangle*

Solve the right triangle. Round decimals to the nearest tenth.



SOLUTION

Use trigonometric ratios to find the values of g and h.

$\sin H = \dfrac{\text{opp.}}{\text{hyp.}}$	$\cos H = \dfrac{\text{adj.}}{\text{hyp.}}$
$\sin 25° = \dfrac{h}{13}$	$\cos 25° = \dfrac{g}{13}$
$13 \sin 25° = h$	$13 \cos 25° = g$
$13(0.4226) \approx h$	$13(0.9063) \approx g$
$5.5 \approx h$	$11.8 \approx g$

Because $\angle H$ and $\angle G$ are complements, you can write

$$m\angle G = 90° - m\angle H = 90° - 25° = 65°.$$

▶ The side lengths of the triangle are about 5.5, 11.8, and 13. The triangle has one right angle and two acute angles whose measures are 65° and 25°.

STUDENT HELP

Study Tip

There are other ways to find the side lengths in Examples 1 and 2. For instance, in Example 2, you can use a trigonometric ratio to find one side length, and then use the Pythagorean Theorem to find the other side length.

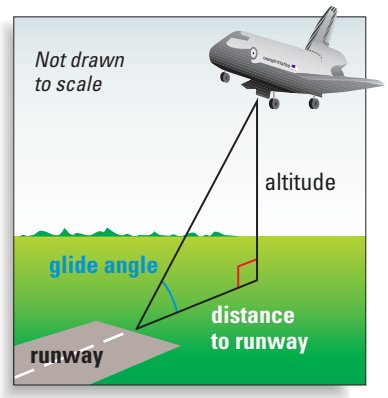
GOAL 2 USING RIGHT TRIANGLES IN REAL LIFE

EXAMPLE 3 *Solving a Right Triangle*

SPACE SHUTTLE During its approach to Earth, the space shuttle's glide angle changes.

a. When the shuttle's altitude is about 15.7 miles, its horizontal distance to the runway is about 59 miles. What is its glide angle? Round your answer to the nearest tenth.

b. When the space shuttle is 5 miles from the runway, its glide angle is about 19°. Find the shuttle's altitude at this point in its descent. Round your answer to the nearest tenth.



FOCUS ON CAREERS

REAL LIFE ASTRONAUT Some astronauts are pilots who are qualified to fly the space shuttle. Some shuttle astronauts are mission specialists whose responsibilities include conducting scientific experiments in space. All astronauts need to have a strong background in science and mathematics.

INTERNET CAREER LINK www.mcdougallittell.com

SOLUTION

a. Sketch a right triangle to model the situation. Let $x°$ = the measure of the shuttle's glide angle. You can use the tangent ratio and a calculator to find the approximate value of x.

$$\tan x° = \frac{\text{opp.}}{\text{adj.}}$$

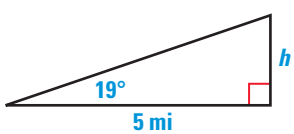
$$\tan x° = \frac{15.7}{59} \qquad \textbf{Substitute.}$$

$$x = \boxed{(}\ 15.7\ \boxed{\div}\ 59\ \boxed{)}\ \boxed{2nd}\ \boxed{TAN} \qquad \textbf{Use a calculator to find } \tan^{-1}\left(\frac{15.7}{59}\right).$$

$$x \approx 14.9$$

▶ When the space shuttle's altitude is about 15.7 miles, the glide angle is about 14.9°.

b. Sketch a right triangle to model the situation. Let h = the altitude of the shuttle. You can use the tangent ratio and a calculator to find the approximate value of h.



$$\tan 19° = \frac{\text{opp.}}{\text{adj.}}$$

$$\tan 19° = \frac{h}{5} \qquad \textbf{Substitute.}$$

$$0.3443 \approx \frac{h}{5} \qquad \textbf{Use a calculator.}$$

$$1.7 \approx h \qquad \textbf{Multiply each side by 5.}$$

▶ The shuttle's altitude is about 1.7 miles.

STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain what is meant by *solving* a right triangle.

Concept Check ✓

Tell whether the statement is *true* or *false*.

2. You can solve a right triangle if you are given the lengths of any two sides.

3. You can solve a right triangle if you know only the measure of one acute angle.

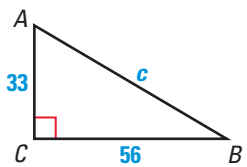
Skill Check ✓

CALCULATOR In Exercises 4–7, $\angle A$ is an acute angle. Use a calculator to approximate the measure of $\angle A$ to the nearest tenth of a degree.

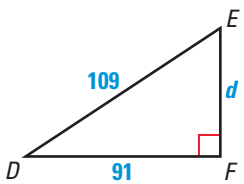
4. $\tan A = 0.7$ 5. $\tan A = 5.4$ 6. $\sin A = 0.9$ 7. $\cos A = 0.1$

Solve the right triangle. Round decimals to the nearest tenth.

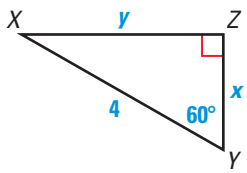
8.



9.



10.

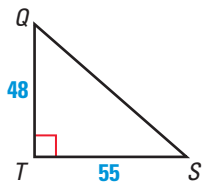


PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 820.

FINDING MEASUREMENTS Use the diagram to find the indicated measurement. Round your answer to the nearest tenth.



11. QS 12. $m\angle Q$ 13. $m\angle S$

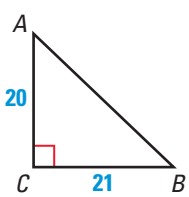
CALCULATOR In Exercises 14–21, $\angle A$ is an acute angle. Use a calculator to approximate the measure of $\angle A$ to the nearest tenth of a degree.

14. $\tan A = 0.5$ 15. $\tan A = 1.0$ 16. $\sin A = 0.5$ 17. $\sin A = 0.35$

18. $\cos A = 0.15$ 19. $\cos A = 0.64$ 20. $\tan A = 2.2$ 21. $\sin A = 0.11$

SOLVING RIGHT TRIANGLES Solve the right triangle. Round decimals to the nearest tenth.

22.



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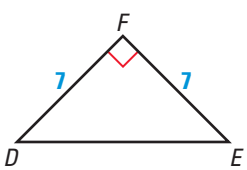

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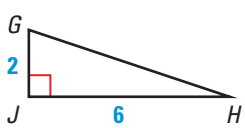

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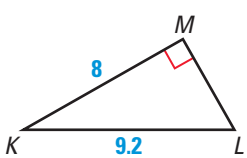

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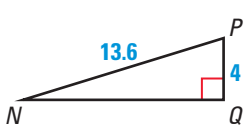

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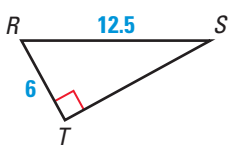


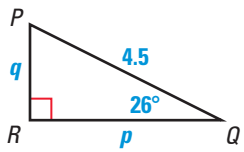
STUDENT HELP

HOMEWORK HELP

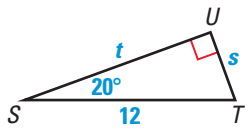
Example 1: Exs. 11–27, 34–37
Example 2: Exs. 28–33
Example 3: Exs. 38–41

SOLVING RIGHT TRIANGLES **Solve the right triangle. Round decimals to the nearest tenth.**

28.



29.



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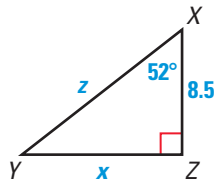
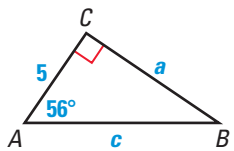
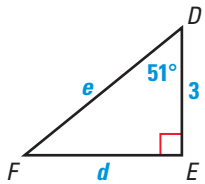


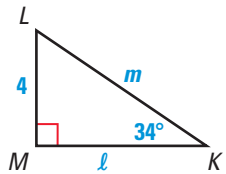
31.



32.



33.



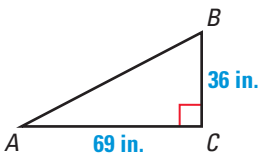
NATIONAL AQUARIUM **Use the diagram of one of the triangular windowpanes at the National Aquarium in Baltimore, Maryland, to find the indicated value.**

34. $\tan B \approx$?

35. $m\angle B \approx$?

36. $AB \approx$?

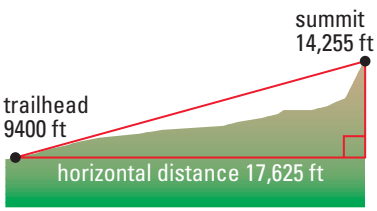
37. $\sin A \approx$?



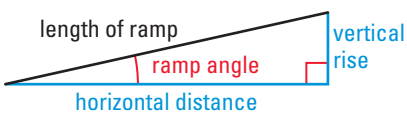
FOCUS ON APPLICATIONS

REAL LIFE BENCHMARKS If you hike to the top of a mountain you may find a brass plate called a *benchmark.* A benchmark gives an official elevation for the point that can be used by surveyors as a reference for surveying elevations of other landmarks.

38. **HIKING** You are hiking up a mountain peak. You begin hiking at a trailhead whose elevation is about 9400 feet. The trail ends near the summit at 14,255 feet. The horizontal distance between these two points is about 17,625 feet. Estimate the angle of elevation from the trailhead to the summit.



RAMPS **In Exercises 39–41, use the information about wheelchair ramps.** The Uniform Federal Accessibility Standards specify that the ramp angle used for a wheelchair ramp must be less than or equal to 4.76°.



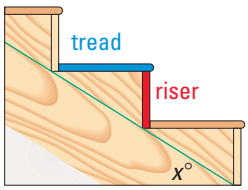
39. The length of one ramp is 20 feet. The vertical rise is 17 inches. Estimate the ramp's horizontal distance and its ramp angle.

40. You want to build a ramp with a vertical rise of 8 inches. You want to minimize the horizontal distance taken up by the ramp. Draw a sketch showing the approximate dimensions of your ramp.

41. *Writing* Measure the horizontal distance and the vertical rise of a ramp near your home or school. Find the ramp angle. Does the ramp meet the specifications described above? Explain.

Test Preparation

MULTI-STEP PROBLEM In Exercises 42–45, use the diagram and the information below.
The horizontal part of a step is called the *tread*. The vertical part is called the *riser*. The ratio of the riser length to the tread length affects the safety of a staircase. Traditionally, builders have used a riser-to-tread ratio of about $8\frac{1}{4}$ inches : 9 inches. A newly recommended ratio is 7 inches : 11 inches.



42. Find the value of x for stairs built using the new riser-to-tread ratio.

43. Find the value of x for stairs built using the old riser-to-tread ratio.

44. Suppose you want to build a stairway that is less steep than either of the ones in Exercises 42 and 43. Give an example of a riser-to-tread ratio that you could use. Find the value of x for your stairway.

45. *Writing* Explain how the riser-to-tread ratio that is used for a stairway could affect the safety of the stairway.

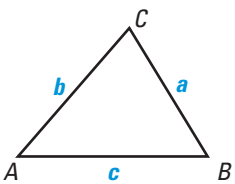
★ Challenge

46. **PROOF** Write a proof.

GIVEN ▶ $\angle A$ and $\angle B$ are acute angles.

PROVE ▶ $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

(*Hint:* Draw an altitude from C to $\overline{AB}$. Label it h.)



EXTRA CHALLENGE
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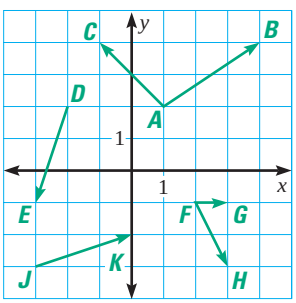
MIXED REVIEW

USING VECTORS Write the component form of the vector. (Review 7.4 for 9.7)

47. $\overrightarrow{AB}$ **48.** $\overrightarrow{AC}$

49. $\overrightarrow{DE}$ **50.** $\overrightarrow{FG}$

51. $\overrightarrow{FH}$ **52.** $\overrightarrow{JK}$



SOLVING PROPORTIONS Solve the proportion. (Review 8.1)

53. $\dfrac{x}{30} = \dfrac{5}{6}$ **54.** $\dfrac{7}{16} = \dfrac{49}{y}$ **55.** $\dfrac{3}{10} = \dfrac{g}{42}$

56. $\dfrac{7}{18} = \dfrac{84}{k}$ **57.** $\dfrac{m}{2} = \dfrac{7}{1}$ **58.** $\dfrac{8}{t} = \dfrac{4}{11}$

CLASSIFYING TRIANGLES Decide whether the numbers can represent the side lengths of a triangle. If they can, classify the triangle as *right, acute,* or *obtuse*. (Review 9.3)

59. 18, 14, 2 **60.** 60, 228, 220 **61.** 8.5, 7.7, 3.6

62. 250, 263, 80 **63.** 113, 15, 112 **64.** 15, 75, 59

9.7 Vectors

What you should learn

GOAL 1 Find the magnitude and the direction of a vector.

GOAL 2 Add vectors.

Why you should learn it

▼ To solve **real-life** problems, such as describing the velocity of a skydiver in **Exs. 41–45.**

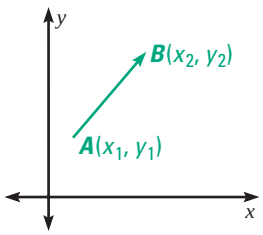
REAL LIFE

GOAL 1 FINDING THE MAGNITUDE OF A VECTOR

As defined in Lesson 7.4, a *vector* is a quantity that has both magnitude and direction. In this lesson, you will learn how to find the *magnitude of a vector* and the *direction of a vector*. You will also learn how to add vectors.

The **magnitude of a vector** $\overrightarrow{AB}$ is the distance from the initial point A to the terminal point B, and is written $|\overrightarrow{AB}|$. If a vector is drawn in a coordinate plane, you can use the Distance Formula to find its magnitude.

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



EXAMPLE 1 *Finding the Magnitude of a Vector*

Points P and Q are the initial and terminal points of the vector $\overrightarrow{PQ}$. Draw $\overrightarrow{PQ}$ in a coordinate plane. Write the component form of the vector and find its magnitude.

a. $P(0, 0)$, $Q(-6, 3)$ **b.** $P(0, 2)$, $Q(5, 4)$ **c.** $P(3, 4)$, $Q(-2, -1)$

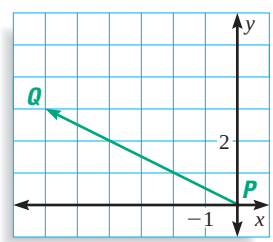
SOLUTION

a. Component form $= \langle x_2 - x_1, y_2 - y_1 \rangle$

$$\overrightarrow{PQ} = \langle -6 - 0, 3 - 0 \rangle$$
$$= \langle -6, 3 \rangle$$

Use the Distance Formula to find the magnitude.

$$|\overrightarrow{PQ}| = \sqrt{(-6 - 0)^2 + (3 - 0)^2} = \sqrt{45} \approx 6.7$$



STUDENT HELP

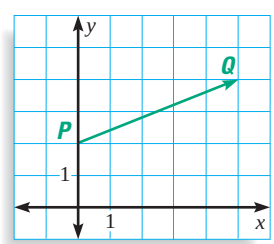
Look Back
For help with the component form of a vector, see p. 423.

b. Component form $= \langle x_2 - x_1, y_2 - y_1 \rangle$

$$\overrightarrow{PQ} = \langle 5 - 0, 4 - 2 \rangle$$
$$= \langle 5, 2 \rangle$$

Use the Distance Formula to find the magnitude.

$$|\overrightarrow{PQ}| = \sqrt{(5 - 0)^2 + (4 - 2)^2} = \sqrt{29} \approx 5.4$$

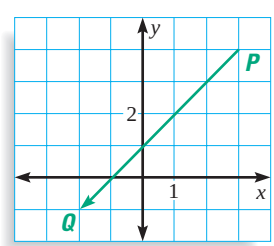


c. Component form $= \langle x_2 - x_1, y_2 - y_1 \rangle$

$$\overrightarrow{PQ} = \langle -2 - 3, -1 - 4 \rangle$$
$$= \langle -5, -5 \rangle$$

Use the Distance Formula to find the magnitude.

$$|\overrightarrow{PQ}| = \sqrt{(-2 - 3)^2 + (-1 - 4)^2} = \sqrt{50} \approx 7.1$$



FOCUS ON APPLICATIONS

NAVIGATION One of the most common vector quantities in real life is the velocity of a moving object. A velocity vector is used in navigation to describe both the speed and the direction of a moving object.

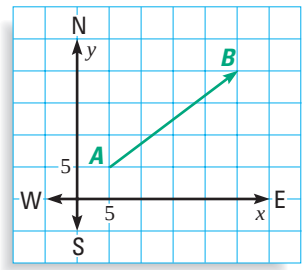
The **direction of a vector** is determined by the angle it makes with a horizontal line. In real-life applications, the direction angle is described relative to the directions north, east, south, and west. In a coordinate plane, the x-axis represents an east-west line. The y-axis represents a north-south line.

EXAMPLE 2 *Describing the Direction of a Vector*

The vector $\overrightarrow{AB}$ describes the velocity of a moving ship. The scale on each axis is in miles per hour.

a. Find the speed of the ship.

b. Find the direction it is traveling relative to east.



SOLUTION

a. The magnitude of the vector $\overrightarrow{AB}$ represents the ship's speed. Use the Distance Formula.

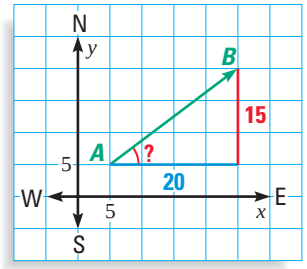
$$|\overrightarrow{AB}| = \sqrt{(25-5)^2 + (20-5)^2}$$
$$= \sqrt{20^2 + 15^2}$$
$$= 25$$

▶ The speed of the ship is 25 miles per hour.

b. The tangent of the angle formed by the vector and a line drawn parallel to the x-axis is $\frac{15}{20}$, or 0.75. Use a calculator to find the angle measure.

$$0.75 \ \boxed{\text{2nd}} \ \boxed{\text{TAN}} \approx 36.9°$$

▶ The ship is traveling in a direction about 37° north of east.



STUDENT HELP

Look Back For help with using trigonometric ratios to find angle measures, see pp. 567 and 568.

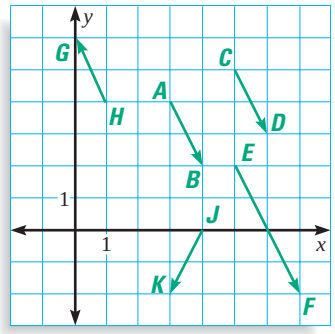
Two vectors are **equal** if they have the same magnitude and direction. They do *not* have to have the same initial and terminal points. Two vectors are **parallel** if they have the same or opposite directions.

EXAMPLE 3 *Identifying Equal and Parallel Vectors*

In the diagram, these vectors have the same direction: $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$.

These vectors are equal: $\overrightarrow{AB}$, $\overrightarrow{CD}$.

These vectors are parallel: $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$, $\overrightarrow{HG}$.

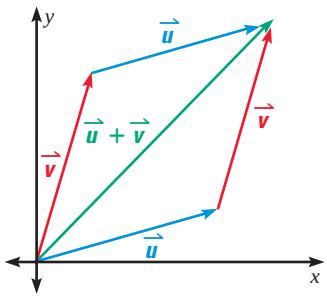


GOAL 2 ADDING VECTORS

STUDENT HELP

Study Tip
A single letter with an arrow over it, such as $\vec{u}$, can be used to denote a vector.

Two vectors can be added to form a new vector. To add $\vec{u}$ and $\vec{v}$ geometrically, place the initial point of $\vec{v}$ on the terminal point of $\vec{u}$, (or place the initial point of $\vec{u}$ on the terminal point of $\vec{v}$). The sum is the vector that joins the initial point of the first vector and the terminal point of the second vector.



This method of adding vectors is often called the *parallelogram rule* because the sum vector is the diagonal of a parallelogram. You can also add vectors algebraically.

ADDING VECTORS

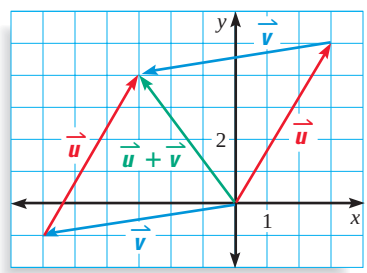
SUM OF TWO VECTORS

The **sum** of $\vec{u} = \langle a_1, b_1 \rangle$ and $\vec{v} = \langle a_2, b_2 \rangle$ is $\vec{u} + \vec{v} = \langle a_1 + a_2, b_1 + b_2 \rangle$.

EXAMPLE 4 *Finding the Sum of Two Vectors*

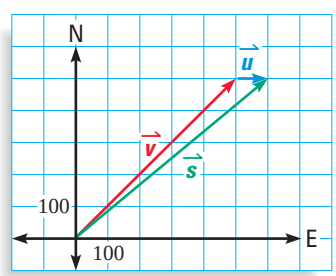
Let $\vec{u} = \langle 3, 5 \rangle$ and $\vec{v} = \langle -6, -1 \rangle$. To find the sum vector $\vec{u} + \vec{v}$, add the horizontal components and add the vertical components of $\vec{u}$ and $\vec{v}$.

$$\vec{u} + \vec{v} = \langle 3 + (-6), 5 + (-1) \rangle$$
$$= \langle -3, 4 \rangle$$



EXAMPLE 5 *Velocity of a Jet*

AVIATION A jet is flying northeast at about 707 miles per hour. Its velocity is represented by the vector $\vec{v} = \langle 500, 500 \rangle$.



The jet encounters a wind blowing from the west at 100 miles per hour. The wind velocity is represented by $\vec{u} = \langle 100, 0 \rangle$. The jet's new velocity vector $\vec{s}$ is the sum of its original velocity vector and the wind's velocity vector.

$$\vec{s} = \vec{v} + \vec{u}$$
$$= \langle 500 + 100, 500 + 0 \rangle$$
$$= \langle 600, 500 \rangle$$

The magnitude of the sum vector $\vec{s}$ represents the new speed of the jet.

$$\text{New speed} = |\vec{s}| = \sqrt{(600 - 0)^2 + (500 - 0)^2} \approx 781 \text{ mi/h}$$

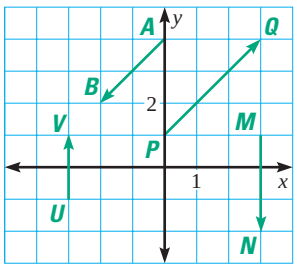
GUIDED PRACTICE

Vocabulary Check ✓

1. What is meant by the *magnitude of a vector* and the *direction of a vector*?

Concept Check ✓

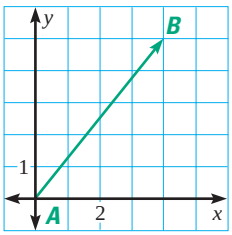
In Exercises 2–4, use the diagram.

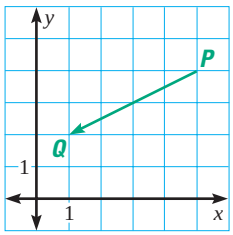


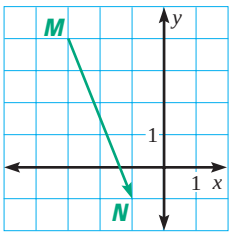
2. Write the component form of each vector.
3. Identify any parallel vectors.
4. Vectors $\overrightarrow{PQ}$ and $\overrightarrow{ST}$ are equal vectors. Although $\overrightarrow{ST}$ is not shown, the coordinates of its initial point are $(-1, -1)$. Give the coordinates of its terminal point.

Skill Check ✓

Write the vector in component form. Find the magnitude of the vector. Round your answer to the nearest tenth.

5.


6.


7.


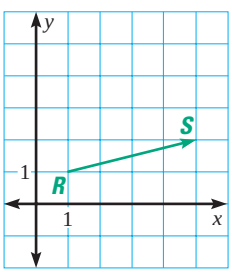
8. Use the vector in Exercise 5. Find the direction of the vector relative to east.
9. Find the sum of the vectors in Exercises 5 and 6.

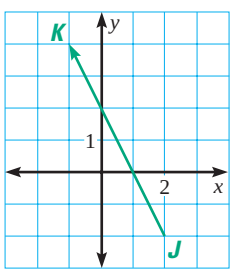
PRACTICE AND APPLICATIONS

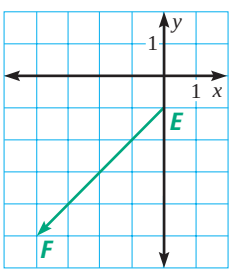
STUDENT HELP

Extra Practice to help you master skills is on p. 820.

FINDING MAGNITUDE **Write the vector in component form. Find the magnitude of the vector. Round your answer to the nearest tenth.**

10.


11.


12.


FINDING MAGNITUDE **Draw vector $\overrightarrow{PQ}$ in a coordinate plane. Write the component form of the vector and find its magnitude. Round your answer to the nearest tenth.**

13. $P(0, 0)$, $Q(2, 7)$
14. $P(5, 1)$, $Q(2, 6)$
15. $P(-3, 2)$, $Q(7, 6)$
16. $P(-4, -3)$, $Q(2, -7)$
17. $P(5, 0)$, $Q(-1, -4)$
18. $P(6, 3)$, $Q(-2, 1)$
19. $P(-6, 0)$, $Q(-5, -4)$
20. $P(0, 5)$, $Q(3, 5)$

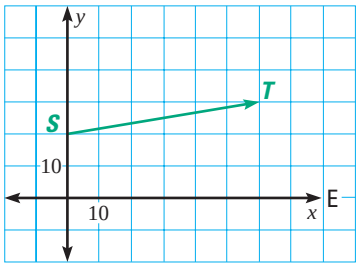
STUDENT HELP

HOMEWORK HELP

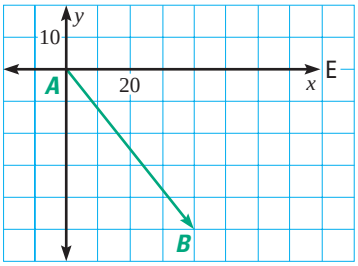
Example 1: Exs. 10–20
Example 2: Exs. 21–24
Example 3: Exs. 25–29
Example 4: Exs. 31–40
Example 5: Exs. 41–45

NAVIGATION **The given vector represents the velocity of a ship at sea. Find the ship's speed, rounded to the nearest mile per hour. Then find the direction the ship is traveling relative to the given direction.**

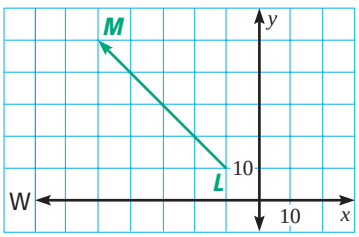
21. Find direction relative to east.



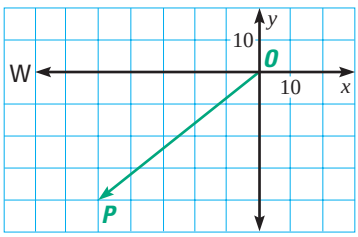
22. Find direction relative to east.



23. Find direction relative to west.

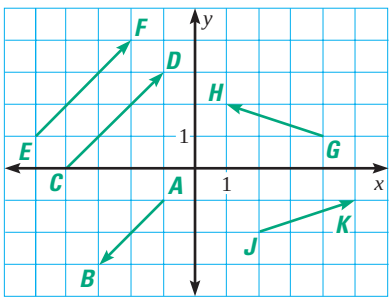


24. Find direction relative to west.



PARALLEL AND EQUAL VECTORS

In Exercises 25–28, use the diagram shown at the right.



25. Which vectors are parallel?

26. Which vectors have the same direction?

27. Which vectors are equal?

28. Name two vectors that have the same magnitude but different directions.

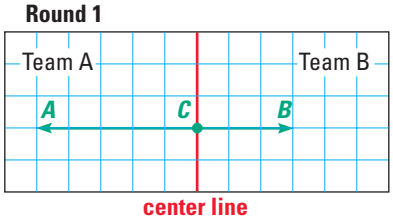
FOCUS ON APPLICATIONS

TUG-OF-WAR
In the game *tug-of-war*, two teams pull on opposite ends of a rope. The team that succeeds in pulling a member of the other team across a center line wins.

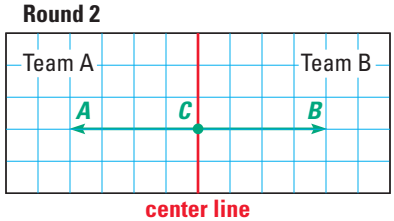
TUG-OF-WAR GAME **In Exercises 29 and 30, use the information below.**

The forces applied in a game of *tug-of-war* can be represented by vectors. The magnitude of the vector represents the amount of force with which the rope is pulled. The direction of the vector represents the direction of the pull. The diagrams below show the forces applied in two different rounds of tug-of-war.

Round 1



Round 2

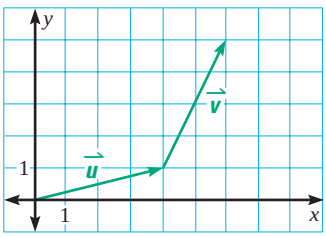


29. In Round 2, are $\overrightarrow{CA}$ and $\overrightarrow{CB}$ parallel vectors? Are they equal vectors?

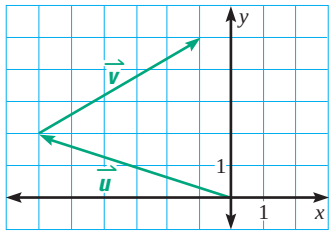
30. In which round was the outcome a tie? How do you know? Describe the outcome in the other round. Explain your reasoning.

PARALLELOGRAM RULE **Copy the vectors $\vec{u}$ and $\vec{v}$. Write the component form of each vector. Then find the sum $\vec{u} + \vec{v}$ and draw the vector $\vec{u} + \vec{v}$.**

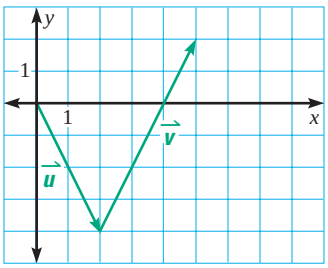
31.



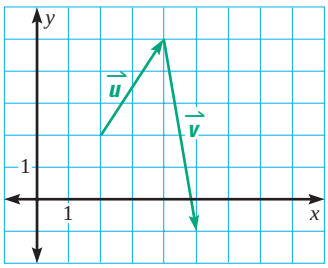
32.



33.



34.



ADDING VECTORS **Let $\vec{u} = \langle 7, 3\rangle$, $\vec{v} = \langle 1, 4\rangle$, $\vec{w} = \langle 3, 7\rangle$, and $\vec{z} = \langle -3, -7\rangle$. Find the given sum.**

35. $\vec{v} + \vec{w}$ **36.** $\vec{u} + \vec{v}$ **37.** $\vec{u} + \vec{w}$

38. $\vec{v} + \vec{z}$ **39.** $\vec{u} + \vec{z}$ **40.** $\vec{w} + \vec{z}$

FOCUS ON APPLICATIONS

SKYDIVING A skydiver who has not yet opened his or her parachute is in *free fall.* During free fall, the skydiver accelerates at first. Air resistance eventually stops this acceleration, and the skydiver falls at *terminal velocity.*

SKYDIVING **In Exercises 41–45, use the information and diagram below.** A skydiver is falling at a constant downward velocity of 120 miles per hour. In the diagram, vector $\vec{u}$ represents the skydiver's velocity. A steady breeze pushes the skydiver to the east at 40 miles per hour. Vector $\vec{v}$ represents the wind velocity. The scales on the axes of the graph are in miles per hour.

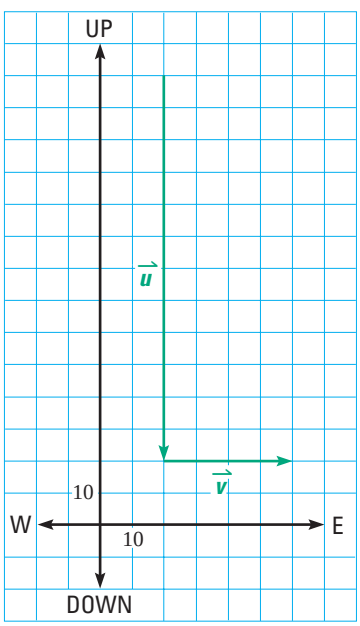
41. Write the vectors $\vec{u}$ and $\vec{v}$ in component form.

42. Let $\vec{s} = \vec{u} + \vec{v}$. Copy the diagram and draw vector $\vec{s}$.

43. Find the magnitude of $\vec{s}$. What information does the magnitude give you about the skydiver's fall?

44. If there were no wind, the skydiver would fall in a path that was straight down. At what angle to the ground is the path of the skydiver when the skydiver is affected by the 40 mile per hour wind from the west?

45. Suppose the skydiver was blown to the west at 30 miles per hour. Sketch a new diagram and find the skydiver's new velocity.

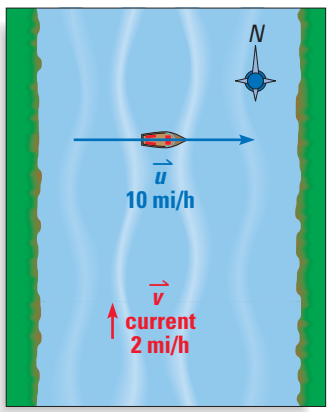


46. *Writing* Write the component form of a vector with the same magnitude as $\overrightarrow{JK} = \langle 1, 3 \rangle$ but a different direction. Explain how you found the vector.

47. **LOGICAL REASONING** Let vector $\vec{u} = \langle r, s \rangle$. Suppose the horizontal and the vertical components of $\vec{u}$ are multiplied by a constant k. The resulting vector is $\vec{v} = \langle kr, ks \rangle$. How are the magnitudes and the directions of $\vec{u}$ and $\vec{v}$ related when k is positive? when k is negative? Justify your answers.

Test Preparation

48. **MULTI-STEP PROBLEM** A motorboat heads due east across a river at a speed of 10 miles per hour. Vector $\vec{u} = \langle 10, 0 \rangle$ represents the velocity of the motorboat. The current of the river is flowing due north at a speed of 2 miles per hour. Vector $\vec{v} = \langle 0, 2 \rangle$ represents the velocity of the current.



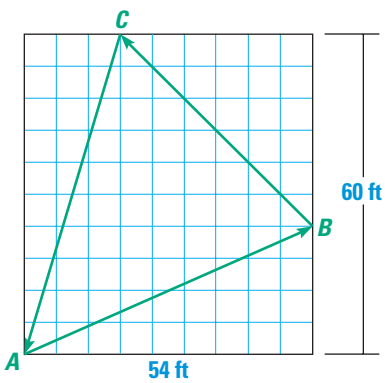
a. Let $\vec{s} = \vec{u} + \vec{v}$. Draw the vectors $\vec{u}$, $\vec{v}$, and $\vec{s}$ in a coordinate plane.

b. Find the speed and the direction of the motorboat as it is affected by the current.

c. Suppose the speed of the motorboat is greater than 10 miles per hour, and the speed of the current is less than 2 miles per hour. Describe one possible set of vectors $\vec{u}$ and $\vec{v}$ that could represent the velocity of the motorboat and the velocity of the current. Write and solve a word problem that can be solved by finding the sum of the two vectors.

★ Challenge

BUMPER CARS **In Exercises 49–52, use the information below.**
As shown in the diagram below, a bumper car moves from point A to point B to point C and back to point A. The car follows the path shown by the vectors. The magnitude of each vector represents the distance traveled by the car from the initial point to the terminal point.



49. Find the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Write the sum vector in component form.

50. Add vector $\overrightarrow{CA}$ to the sum vector from Exercise 49.

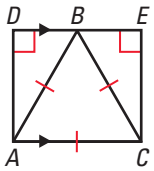
51. Find the total distance traveled by the car.

52. Compare your answers to Exercises 50 and 51. Why are they different?

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

53. **PROOF** Use the information and the diagram to write a proof. **(Review 4.5)**

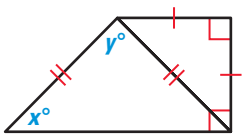


GIVEN ▶ $\angle D$ and $\angle E$ are right angles; $\triangle ABC$ is equilateral; $\overline{DE} \parallel \overline{AC}$

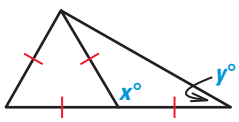
PROVE ▶ B is the midpoint of $\overline{DE}$.

USING ALGEBRA **Find the values of *x* and *y*.** **(Review 4.6)**

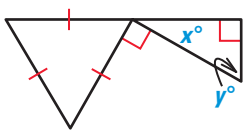
54.



55.



56.



USING ALGEBRA **Find the product.** **(Skills Review, p. 798, for 10.1)**

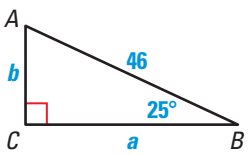
57. $(x + 1)^2$ **58.** $(x + 7)^2$ **59.** $(x + 11)^2$ **60.** $(7 + x)^2$

QUIZ 3

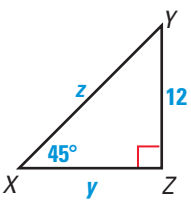
Self-Test for Lessons 9.6 and 9.7

Solve the right triangle. Round decimals to the nearest tenth. **(Lesson 9.6)**

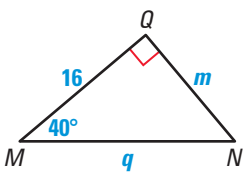
1.



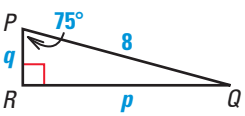
2.



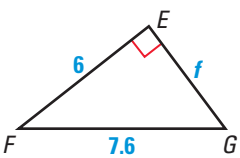
3.



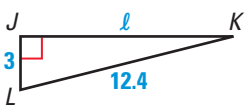
4.



5.



6.



Draw vector $\overrightarrow{PQ}$ in a coordinate plane. Write the component form of the vector and find its magnitude. Round your answer to the nearest tenth. **(Lesson 9.7)**

7. $P(3, 4)$, $Q(-2, 3)$ **8.** $P(-2, 2)$, $Q(4, -3)$

9. $P(0, -1)$, $Q(3, 4)$ **10.** $P(2, 6)$, $Q(-5, -5)$

11. Vector $\overrightarrow{ST} = \langle 3, 8 \rangle$. Draw $\overrightarrow{ST}$ in a coordinate plane and find its direction relative to east. **(Lesson 9.7)**

Let $\vec{u} = \langle 0, -5 \rangle$, $\vec{v} = \langle 4, 7 \rangle$, $\vec{w} = \langle -2, -3 \rangle$, and $\vec{z} = \langle 2, 6 \rangle$. Find the given sum. **(Lesson 9.7)**

12. $\vec{u} + \vec{v}$ **13.** $\vec{v} + \vec{w}$ **14.** $\vec{u} + \vec{w}$

15. $\vec{u} + \vec{z}$ **16.** $\vec{v} + \vec{z}$ **17.** $\vec{w} + \vec{z}$

CHAPTER 9

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Solve problems involving similar right triangles formed by the altitude drawn to the hypotenuse of a right triangle. (9.1)	Find a height in a real-life structure, such as the height of a triangular roof. (p. 528)
Use the Pythagorean Theorem. (9.2)	Solve real-life problems, such as finding the length of a skywalk support beam. (p. 537)
Use the Converse of the Pythagorean Theorem. (9.3)	Use in construction methods, such as verifying whether a foundation is rectangular. (p. 545)
Use side lengths to classify triangles by their angle measures. (9.3)	Write proofs about triangles. (p. 547)
Find side lengths of special right triangles. (9.4)	Solve real-life problems, such as finding the height of a loading platform. (p. 553)
Find trigonometric ratios of an acute angle. (9.5)	Measure distances indirectly, such as the depth of a crater on the moon. (p. 564)
Solve a right triangle. (9.6)	Solve real-life problems, such as finding the glide angle and altitude of the space shuttle. (p. 569)
Find the magnitude and the direction of a vector. (9.7)	Describe physical quantities, such as the speed and direction of a ship. (p. 574)
Find the sum of two vectors. (9.7)	Model real-life motion, such as the path of a skydiver. (p. 578)

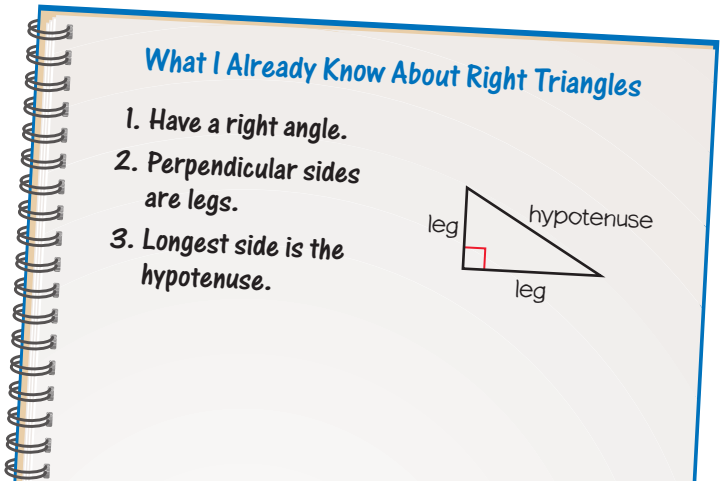
How does Chapter 9 fit into the BIGGER PICTURE of geometry?

In this chapter, you studied two of the most important theorems in mathematics—the Pythagorean Theorem and its converse. You were also introduced to a branch of mathematics called *trigonometry*. Properties of right triangles allow you to estimate distances and angle measures that cannot be measured directly. These properties are important tools in areas such as surveying, construction, and navigation.

STUDY STRATEGY

What did you learn about right triangles?

Your lists about what you knew and what you expected to learn about right triangles, following the study strategy on page 526, may resemble this one.



CHAPTER 9

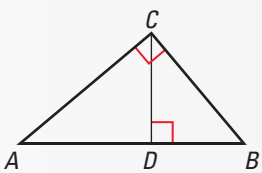
Chapter Review

- Pythagorean triple, p. 536
- special right triangles, p. 551
- trigonometric ratio, p. 558
- sine, p. 558
- cosine, p. 558
- tangent, p. 558
- angle of elevation, p. 561
- solve a right triangle, p. 567
- magnitude of a vector, p. 573
- direction of a vector, p. 574
- equal vectors, p. 574
- parallel vectors, p. 574
- sum of two vectors, p. 575

9.1 SIMILAR RIGHT TRIANGLES

Examples on pp. 528–530

EXAMPLES



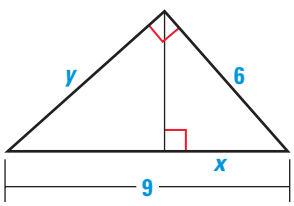
$\triangle ACB \sim \triangle CDB$, so $\frac{DB}{CB} = \frac{CB}{AB}$. CB is the geometric mean of DB and AB.

$\triangle ADC \sim \triangle ACB$, so $\frac{AD}{AC} = \frac{AC}{AB}$. AC is the geometric mean of AD and AB.

$\triangle CDB \sim \triangle ADC$, so $\frac{DA}{DC} = \frac{DC}{DB}$. DC is the geometric mean of DA and DB.

Find the value of each variable.

1.



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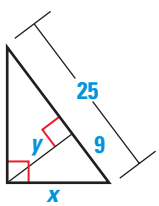

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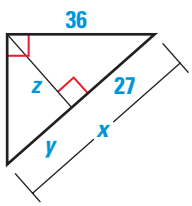
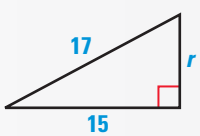


9.2 THE PYTHAGOREAN THEOREM

Examples on pp. 536–537

EXAMPLE You can use the Pythagorean Theorem to find the value of r. $17^2 = r^2 + 15^2$, or $289 = r^2 + 225$. Then $64 = r^2$, so $r = 8$.

The side lengths 8, 15, and 17 form a Pythagorean triple because they are integers.



The variables *r* and *s* represent the lengths of the legs of a right triangle, and *t* represents the length of the hypotenuse. Find the unknown value. Then tell whether the lengths form a Pythagorean triple.

4. $r = 12, s = 16$ **5.** $r = 8, t = 12$ **6.** $s = 16, t = 34$ **7.** $r = 4, s = 6$

9.3 THE CONVERSE OF THE PYTHAGOREAN THEOREM

Examples on pp. 543–545

EXAMPLES You can use side lengths to classify a triangle by its angle measures. Let a, b, and c represent the side lengths of a triangle, with c as the length of the longest side.

If $c^2 = a^2 + b^2$, the triangle is a right triangle: $8^2 = (2\sqrt{7})^2 + 6^2$, so $2\sqrt{7}$, 6, and 8 are the side lengths of a right triangle.

If $c^2 < a^2 + b^2$, the triangle is an acute triangle: $12^2 < 8^2 + 9^2$, so 8, 9, and 12 are the side lengths of an acute triangle.

If $c^2 > a^2 + b^2$, the triangle is an obtuse triangle: $8^2 > 5^2 + 6^2$, so 5, 6, and 8 are the side lengths of an obtuse triangle.

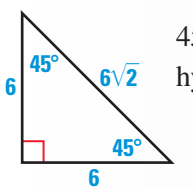
Decide whether the numbers can represent the side lengths of a triangle. If they can, classify the triangle as *acute, right,* or *obtuse*.

8. 6, 7, 10 **9.** 9, 40, 41 **10.** 8, 12, 20 **11.** 3, $4\sqrt{5}$, 9

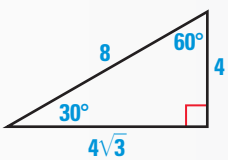
9.4 SPECIAL RIGHT TRIANGLES

Examples on pp. 551–553

EXAMPLES Triangles whose angle measures are 45°-45°-90° or 30°-60°-90° are called *special right triangles*.



45°-45°-90° triangle
hypotenuse = $\sqrt{2}$ • leg



30°-60°-90° triangle
hypotenuse = 2 • shorter leg
longer leg = $\sqrt{3}$ • shorter leg

12. An isosceles right triangle has legs of length $3\sqrt{2}$. Find the length of the hypotenuse.

13. A diagonal of a square is 6 inches long. Find its perimeter and its area.

14. A 30°-60°-90° triangle has a hypotenuse of length 12 inches. What are the lengths of the legs?

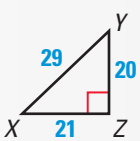
15. An equilateral triangle has sides of length 18 centimeters. Find the length of an altitude of the triangle. Then find the area of the triangle.

9.5 TRIGONOMETRIC RATIOS

Examples on pp. 558–561

EXAMPLE A trigonometric ratio is a ratio of the lengths of two sides of a right triangle.

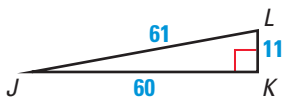
$\sin X = \frac{\text{opp.}}{\text{hyp.}} = \frac{20}{29}$ $\cos X = \frac{\text{adj.}}{\text{hyp.}} = \frac{21}{29}$ $\tan X = \frac{\text{opp.}}{\text{adj.}} = \frac{20}{21}$



9.5 continued

Find the sine, the cosine, and the tangent of the acute angles of the triangle. Express each value as a decimal rounded to four places.

16.



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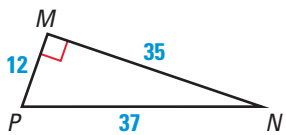

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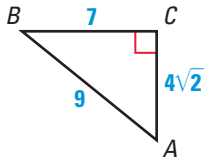


9.6 SOLVING RIGHT TRIANGLES

Examples on pp. 568–569

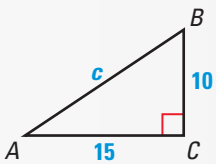
EXAMPLE To solve $\triangle ABC$, begin by using the Pythagorean Theorem to find the length of the hypotenuse.

$c^2 = 10^2 + 15^2 = 325$. So, $c = \sqrt{325} = 5\sqrt{13}$.

Then find $m\angle A$ and $m\angle B$.

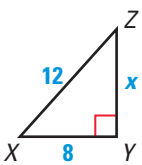
$\tan A = \frac{10}{15} = \frac{2}{3}$. Use a calculator to find that $m\angle A \approx 33.7°$.

Then $m\angle B = 90° - m\angle A \approx 90° - 33.7° = 56.3°$.



Solve the right triangle. Round decimals to the nearest tenth.

19.



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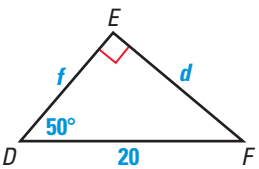

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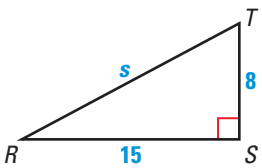


9.7 VECTORS

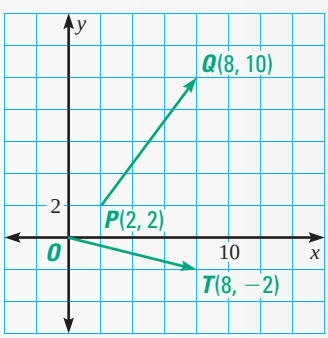
Examples on pp. 573–575

EXAMPLES You can use the Distance Formula to find the magnitude of $\overrightarrow{PQ}$.

$$|\overrightarrow{PQ}| = \sqrt{(8-2)^2 + (10-2)^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$$

To add vectors, find the sum of their horizontal components and the sum of their vertical components.

$$\overrightarrow{PQ} + \overrightarrow{OT} = \langle 6, 8\rangle + \langle 8, -2\rangle = \langle 6 + 8, 8 + (-2)\rangle = \langle 14, 6\rangle$$



Draw vector $\overrightarrow{PQ}$ in a coordinate plane. Write the component form of the vector and find its magnitude. Round decimals to the nearest tenth.

22. $P(2, 3)$, $Q(1, -1)$

23. $P(-6, 3)$, $Q(6, -2)$

24. $P(-2, 0)$, $Q(1, 2)$

25. Let $\vec{u} = \langle 1, 2\rangle$ and $\vec{v} = \langle 13, 7\rangle$. Find $\vec{u} + \vec{v}$. Find the magnitude of the sum vector and its direction relative to east.

CHAPTER 9

Chapter Test

Use the diagram at the right to match the angle or segment with its measure. (Some measures are rounded to two decimal places.)

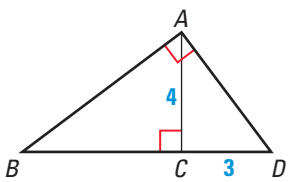
1. $\overline{AB}$ **A.** 5.33

2. $\overline{BC}$ **B.** 36.87°

3. $\overline{AD}$ **C.** 5

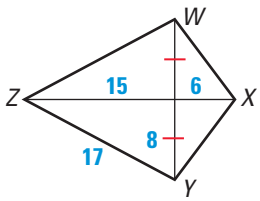
4. $\angle BAC$ **D.** 53.13°

5. $\angle CAD$ **E.** 6.67



6. Refer to the diagram above. Complete the following statement: $\triangle ABC \sim \triangle$ __?__ $\sim \triangle$ __?__.

7. Classify quadrilateral *WXYZ* in the diagram at the right. Explain your reasoning.



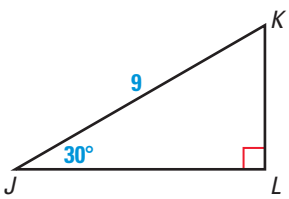
8. The vertices of $\triangle PQR$ are $P(-2, 3)$, $Q(3, 1)$, and $R(0, -3)$. Decide whether $\triangle PQR$ is *right*, *acute*, or *obtuse*.

9. Complete the following statement: 15, __?__, and 113 form a Pythagorean triple.

10. The measure of one angle of a rhombus is 60°. The perimeter of the rhombus is 24 inches. Sketch the rhombus and give its side lengths. Then find its area.

Solve the right triangle. Round decimals to the nearest tenth.

11.



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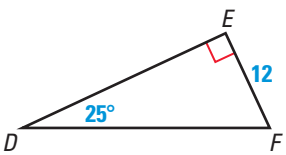

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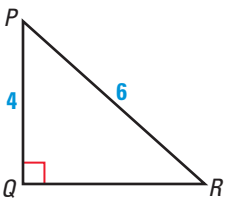
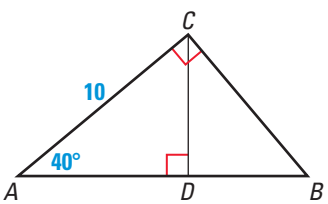
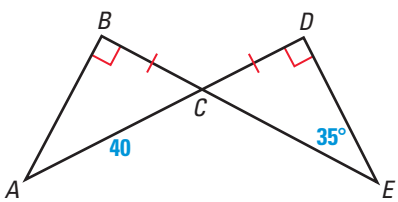


14. $L = (3, 7)$ and $M = (7, 4)$ are the initial and the terminal points of $\overrightarrow{LM}$. Draw $\overrightarrow{LM}$ in a coordinate plane. Write the component form of the vector. Then find its magnitude and direction relative to east.

15. Find the lengths of $\overline{CD}$ and $\overline{AB}$.



16. Find the measure of $\angle BCA$ and the length of $\overline{DE}$.



Let $\vec{u} = \langle 0, -5\rangle$, $\vec{v} = \langle -2, -3\rangle$, and $\vec{w} = \langle 4, 6\rangle$. Find the given sum.

17. $\vec{u} + \vec{v}$ **18.** $\vec{u} + \vec{w}$ **19.** $\vec{v} + \vec{w}$

10.1 Tangents to Circles

What you should learn

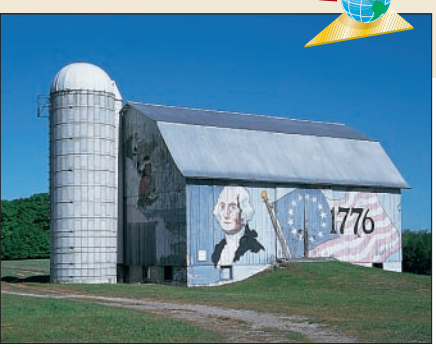
GOAL 1 Identify segments and lines related to circles.

GOAL 2 Use properties of a tangent to a circle.

Why you should learn it

▼ You can use properties of tangents of circles to find **real-life** distances, such as the radius of the silo in **Example 5**.

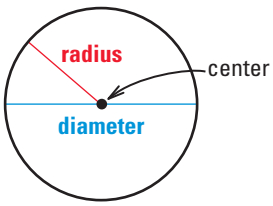
REAL LIFE



GOAL 1 COMMUNICATING ABOUT CIRCLES

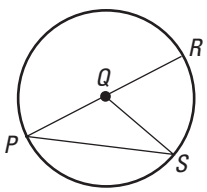
A **circle** is the set of all points in a plane that are equidistant from a given point, called the **center** of the circle. A circle with center P is called "circle P", or $\odot P$.

The distance from the center to a point on the circle is the **radius** of the circle. Two circles are **congruent** if they have the same radius.



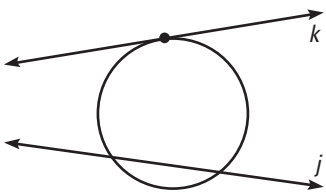
The distance across the circle, through its center, is the **diameter** of the circle. The diameter is twice the radius.

The terms *radius* and *diameter* describe segments as well as measures. A **radius** is a segment whose endpoints are the center of the circle and a point on the circle. $\overline{QP}$, $\overline{QR}$, and $\overline{QS}$ are radii of $\odot Q$ below. All radii of a circle are congruent.



A **chord** is a segment whose endpoints are points on the circle. $\overline{PS}$ and $\overline{PR}$ are chords.

A **diameter** is a chord that passes through the center of the circle. $\overline{PR}$ is a diameter.



A **secant** is a line that intersects a circle in two points. Line j is a secant.

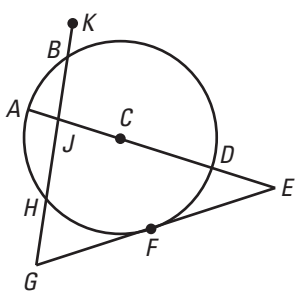
A **tangent** is a line in the plane of a circle that intersects the circle in exactly one point. Line k is a tangent.

EXAMPLE 1 *Identifying Special Segments and Lines*

Tell whether the line or segment is best described as a *chord*, a *secant*, a *tangent*, a *diameter*, or a *radius* of $\odot C$.

a. $\overline{AD}$ **b.** $\overline{CD}$

c. $\overleftrightarrow{EG}$ **d.** $\overline{HB}$



SOLUTION

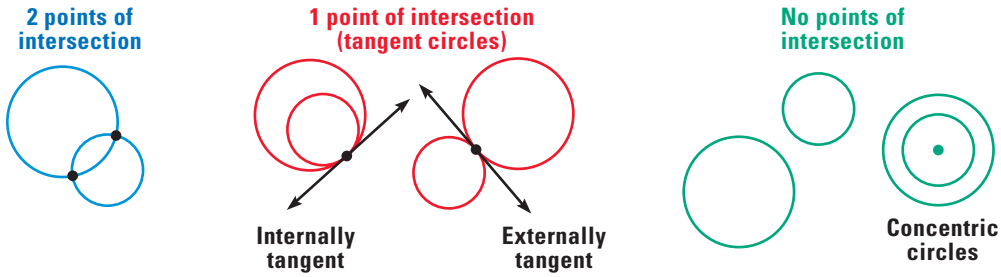
a. $\overline{AD}$ is a diameter because it contains the center C.

b. $\overline{CD}$ is a radius because C is the center and D is a point on the circle.

c. $\overleftrightarrow{EG}$ is a tangent because it intersects the circle in one point.

d. $\overline{HB}$ is a chord because its endpoints are on the circle.

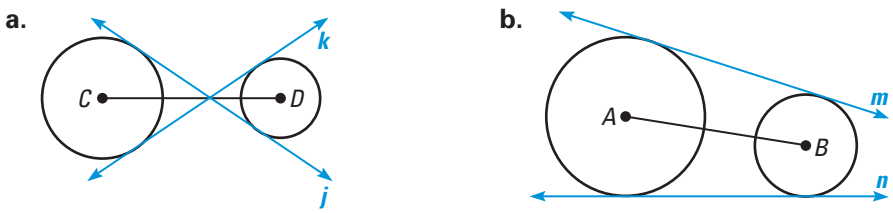
In a plane, two circles can intersect in two points, one point, or no points. Coplanar circles that intersect in one point are called **tangent circles**. Coplanar circles that have a common center are called **concentric**.



A line or segment that is tangent to two coplanar circles is called a **common tangent**. A *common internal tangent* intersects the segment that joins the centers of the two circles. A *common external tangent* does not intersect the segment that joins the centers of the two circles.

EXAMPLE 2 *Identifying Common Tangents*

Tell whether the **common tangents** are *internal* or *external*.



SOLUTION

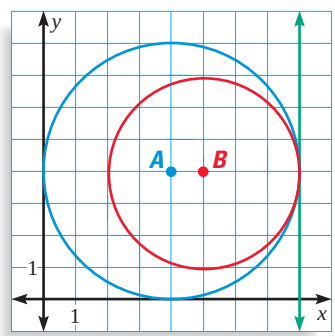
a. The lines j and k intersect $\overline{CD}$, so they are common internal tangents.

b. The lines m and n do not intersect $\overline{AB}$, so they are common external tangents.

In a plane, the **interior of a circle** consists of the points that are inside the circle. The **exterior of a circle** consists of the points that are outside the circle.

EXAMPLE 3 *Circles in Coordinate Geometry*

Give the center and the radius of each circle. Describe the intersection of the two circles and describe all common tangents.



SOLUTION

The center of $\odot A$ is $A(4, 4)$ and its radius is 4. The center of $\odot B$ is $B(5, 4)$ and its radius is 3. The two circles have only one point of intersection. It is the point $(8, 4)$. The vertical line $x = 8$ is the only common tangent of the two circles.

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 USING PROPERTIES OF TANGENTS

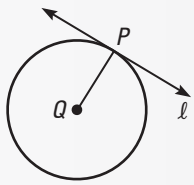
The point at which a tangent line intersects the circle to which it is tangent is the **point of tangency**. You will justify the following theorems in the exercises.

THEOREMS

THEOREM 10.1

If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency.

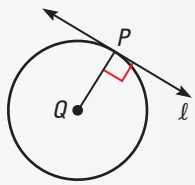
If ℓ is tangent to $\odot Q$ at P, then $\ell \perp \overline{QP}$.



THEOREM 10.2

In a plane, if a line is perpendicular to a radius of a circle at its endpoint on the circle, then the line is tangent to the circle.

If $\ell \perp \overline{QP}$ at P, then ℓ is tangent to $\odot Q$.

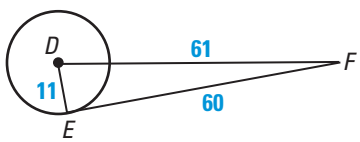


STUDENT HELP

Study Tip
A secant can look like a tangent if it intersects the circle in two points that are close together.

EXAMPLE 4 *Verifying a Tangent to a Circle*

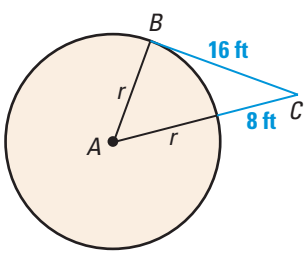
You can use the Converse of the Pythagorean Theorem to tell whether $\overleftrightarrow{EF}$ is tangent to $\odot D$.



Because $11^2 + 60^2 = 61^2$, $\triangle DEF$ is a right triangle and $\overline{DE}$ is perpendicular to $\overline{EF}$. So, by Theorem 10.2, $\overleftrightarrow{EF}$ is tangent to $\odot D$.

EXAMPLE 5 *Finding the Radius of a Circle*

You are standing at C, 8 feet from a grain silo. The distance from you to a point of tangency on the tank is 16 feet. What is the radius of the silo?



SOLUTION

Tangent $\overleftrightarrow{BC}$ is perpendicular to radius $\overline{AB}$ at B, so $\triangle ABC$ is a right triangle. So, you can use the Pythagorean Theorem.

$(r + 8)^2 = r^2 + 16^2$	**Pythagorean Theorem**
$r^2 + 16r + 64 = r^2 + 256$	**Square of binomial**
$16r + 64 = 256$	**Subtract r^2 from each side.**
$16r = 192$	**Subtract 64 from each side.**
$r = 12$	**Divide.**

▶ The radius of the silo is 12 feet.

STUDENT HELP

Skills Review
For help squaring a binomial, see p. 798.

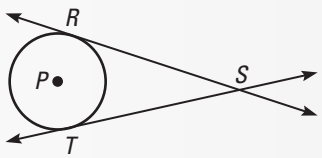
From a point in a circle's exterior, you can draw exactly two different tangents to the circle. The following theorem tells you that the segments joining the external point to the two points of tangency are congruent.

THEOREM

THEOREM 10.3

If two segments from the same exterior point are tangent to a circle, then they are congruent.

If $\overleftrightarrow{SR}$ and $\overleftrightarrow{ST}$ are tangent to $\odot P$, then $\overline{SR} \cong \overline{ST}$.

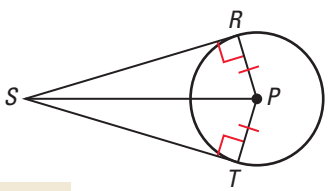


EXAMPLE 6 *Proof of Theorem 10.3*

Proof

GIVEN ▶ $\overleftrightarrow{SR}$ is tangent to $\odot P$ at R.
$\overleftrightarrow{ST}$ is tangent to $\odot P$ at T.

PROVE ▶ $\overline{SR} \cong \overline{ST}$



$\overleftrightarrow{SR}$ and $\overleftrightarrow{ST}$ are both tangent to $\odot P$. — Given

$\overline{SR} \perp \overline{RP}$, $\overline{ST} \perp \overline{TP}$ — Tangent and radius are $\perp$.

$RP = TP$ — Def. of circle

$\overline{RP} \cong \overline{TP}$ — Def. of congruence

$\overline{PS} \cong \overline{PS}$ — Reflexive Property

$\triangle PRS \cong \triangle PTS$ — HL Congruence Theorem

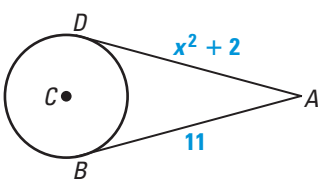
$\overline{SR} \cong \overline{ST}$ — Corresp. parts of $\cong$ $\triangle$ are $\cong$.

EXAMPLE 7 *Using Properties of Tangents*

Using Algebra

$\overleftrightarrow{AB}$ is tangent to $\odot C$ at B.
$\overleftrightarrow{AD}$ is tangent to $\odot C$ at D.

Find the value of x.



SOLUTION

$AB = AD$	Two tangent segments from the same point are $\cong$.
$11 = x^2 + 2$	Substitute.
$9 = x^2$	Subtract 2 from each side.
$\pm 3 = x$	Find the square roots of 9.

▶ The value of x is 3 or -3.

GUIDED PRACTICE

Vocabulary Check ✓

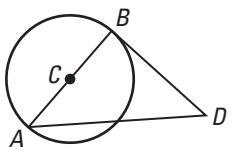
1. Sketch a circle. Then sketch and label a *radius*, a *diameter*, and a *chord*.
2. How are chords and secants of circles alike? How are they different?

Concept Check ✓

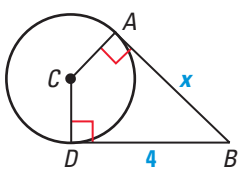
3. $\overleftrightarrow{XY}$ is tangent to $\odot C$ at point P. What is $m\angle CPX$? Explain.
4. The diameter of a circle is 13 cm. What is the radius of the circle?

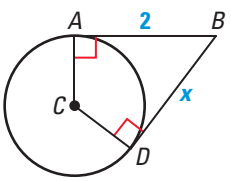
Skill Check ✓

5. In the diagram at the right, $AB = BD = 5$ and $AD = 7$. Is $\overleftrightarrow{BD}$ tangent to $\odot C$? Explain.



$\overleftrightarrow{AB}$ is tangent to $\odot C$ at A and $\overleftrightarrow{DB}$ is tangent to $\odot C$ at D. Find the value of x.

6.


7.


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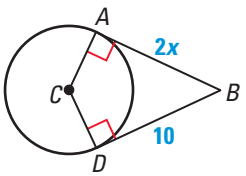


PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 821.

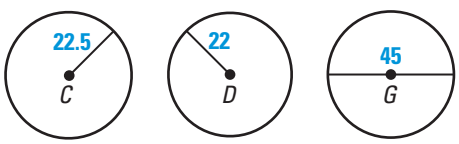
FINDING RADII **The diameter of a circle is given. Find the radius.**

9. $d = 15$ cm
10. $d = 6.7$ in.
11. $d = 3$ ft
12. $d = 8$ cm

FINDING DIAMETERS **The radius of $\odot C$ is given. Find the diameter of $\odot C$.**

13. $r = 26$ in.
14. $r = 62$ ft
15. $r = 8.7$ in.
16. $r = 4.4$ cm

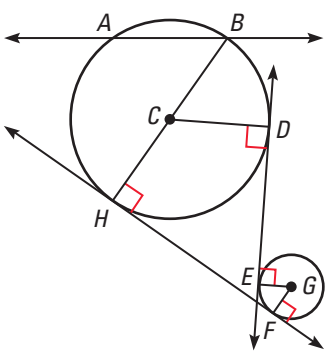
17. **CONGRUENT CIRCLES** Which two circles below are congruent? Explain your reasoning.



MATCHING TERMS **Match the notation with the term that best describes it.**

18. $\overline{AB}$
19. H
20. $\overleftrightarrow{HF}$
21. $\overline{CH}$
22. C
23. $\overline{HB}$
24. $\overleftrightarrow{AB}$
25. $\overleftrightarrow{DE}$

A. Center
B. Chord
C. Diameter
D. Radius
E. Point of tangency
F. Common external tangent
G. Common internal tangent
H. Secant



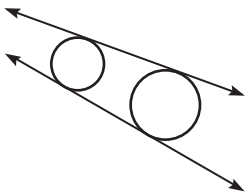
STUDENT HELP

HOMEWORK HELP

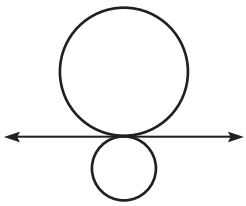
Example 1: Exs. 18–25, 42–45
Example 2: Exs. 26–31
Example 3: Exs. 32–35
Example 4: Exs. 36–39
Example 5: Exs. 40, 41
Example 6: Exs. 49–53
Example 7: Exs. 46–48

IDENTIFYING TANGENTS **Tell whether the common tangent(s) are *internal* or *external*.**

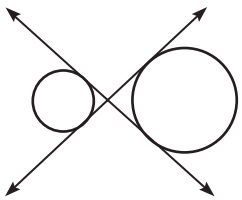
26.



27.

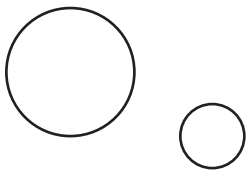


28.

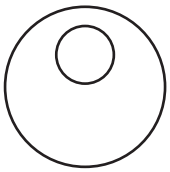


DRAWING TANGENTS **Copy the diagram. Tell how many common tangents the circles have. Then sketch the tangents.**

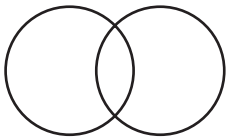
29.



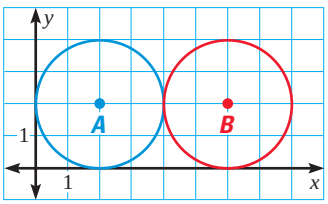
30.



31.



COORDINATE GEOMETRY **Use the diagram at the right.**



32. What are the center and radius of $\odot A$?

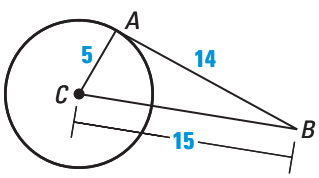
33. What are the center and radius of $\odot B$?

34. Describe the intersection of the two circles.

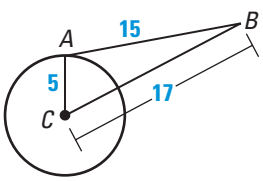
35. Describe all the common tangents of the two circles.

DETERMINING TANGENCY **Tell whether $\overleftrightarrow{AB}$ is tangent to $\odot C$. Explain your reasoning.**

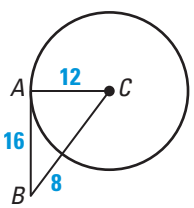
36.



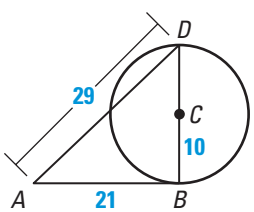
37.



38.

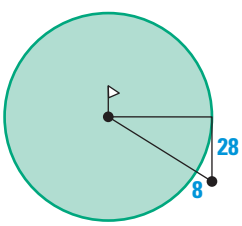


39.



GOLF **In Exercises 40 and 41, use the following information.**

A green on a golf course is in the shape of a circle. A golf ball is 8 feet from the edge of the green and 28 feet from a point of tangency on the green, as shown at the right. Assume that the green is flat.



40. What is the radius of the green?

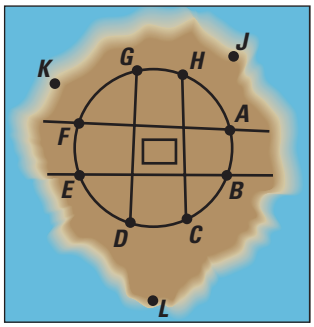
41. How far is the golf ball from the cup at the center?

FOCUS ON PEOPLE

TIGER WOODS At age 15 Tiger Woods became the youngest golfer ever to win the U.S. Junior Amateur Championship, and at age 21 he became the youngest Masters champion ever.

Mexcaltitlán Island, Mexico

MEXCALTITLÁN **The diagram shows the layout of the streets on Mexcaltitlán Island.**



42. Name two secants.

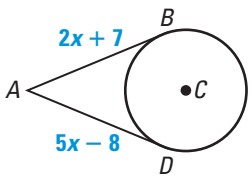
43. Name two chords.

44. Is the diameter of the circle greater than HC? Explain.

45. If $\triangle LJK$ were drawn, one of its sides would be tangent to the circle. Which side is it?

USING ALGEBRA **$\overleftrightarrow{AB}$ and $\overleftrightarrow{AD}$ are tangent to $\odot C$. Find the value of x.**

46.



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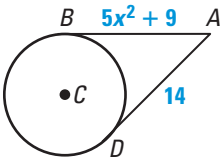
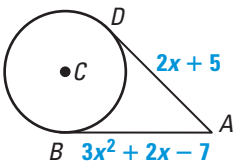
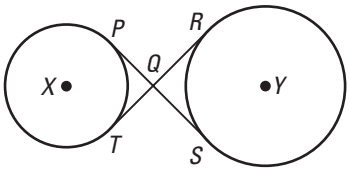


48.



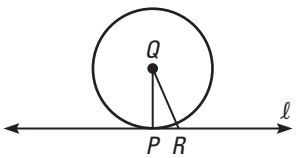
49. **PROOF** Write a proof.



GIVEN ▶ $\overleftrightarrow{PS}$ is tangent to $\odot X$ at P.
$\overleftrightarrow{PS}$ is tangent to $\odot Y$ at S.
$\overleftrightarrow{RT}$ is tangent to $\odot X$ at T.
$\overleftrightarrow{RT}$ is tangent to $\odot Y$ at R.

PROVE ▶ $\overline{PS} \cong \overline{RT}$

PROVING THEOREM 10.1 **In Exercises 50–52, you will use an indirect argument to prove Theorem 10.1.**



GIVEN ▶ ℓ is tangent to $\odot Q$ at P.

PROVE ▶ $\ell \perp \overline{QP}$

50. Assume ℓ and $\overline{QP}$ are not perpendicular. Then the perpendicular segment from Q to ℓ intersects ℓ at some other point R. Because ℓ is a tangent, R cannot be in the interior of $\odot Q$. So, how does QR compare to QP? Write an inequality.

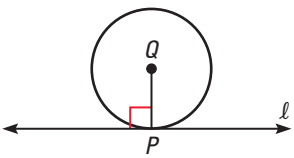
51. $\overline{QR}$ is the perpendicular segment from Q to ℓ, so $\overline{QR}$ is the shortest segment from Q to ℓ. Write another inequality comparing QR to QP.

52. Use your results from Exercises 50 and 51 to complete the indirect proof of Theorem 10.1.

53. **PROVING THEOREM 10.2** Write an indirect proof of Theorem 10.2. (*Hint:* The proof is like the one in Exercises 50–52.)

GIVEN ▶ ℓ is in the plane of $\odot Q$.
$\ell \perp$ radius $\overline{QP}$ at P.

PROVE ▶ ℓ is tangent to $\odot Q$.



LOGICAL REASONING **In ⊙*C*, radii $\overline{CA}$ and $\overline{CB}$ are perpendicular. $\overleftrightarrow{BD}$ and $\overleftrightarrow{AD}$ are tangent to ⊙*C*.**

54. Sketch ⊙*C*, $\overline{CA}$, $\overline{CB}$, $\overline{BD}$, and $\overline{AD}$.

55. What type of quadrilateral is *CADB*? Explain.

Test Preparation

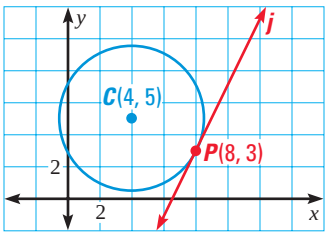
56. MULTI-STEP PROBLEM In the diagram, line *j* is tangent to ⊙*C* at *P*.

a. What is the slope of radius $\overline{CP}$?

b. What is the slope of *j*? Explain.

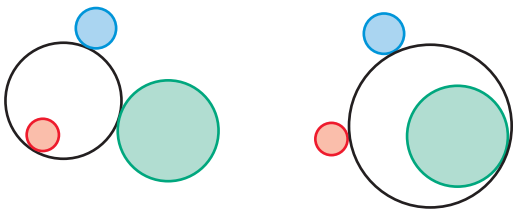
c. Write an equation for *j*.

d. *Writing* Explain how to find an equation for a line tangent to ⊙*C* at a point other than *P*.



★ Challenge

57. CIRCLES OF APOLLONIUS The Greek mathematician Apollonius (c. 200 B.C.) proved that for any three circles with no common points or common interiors, there are eight ways to draw a circle that is tangent to the given three circles. The red, blue, and green circles are given. Two ways to draw a circle that is tangent to the given three circles are shown below. Sketch the other six ways.



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

58. TRIANGLE INEQUALITIES The lengths of two sides of a triangle are 4 and 10. Use an inequality to describe the length of the third side. **(Review 5.5)**

PARALLELOGRAMS **Show that the vertices represent the vertices of a parallelogram. Use a different method for each proof.** **(Review 6.3)**

59. $P(5, 0)$, $Q(2, 9)$, $R(-6, 6)$, $S(-3, -3)$

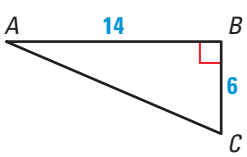
60. $P(4, 3)$, $Q(6, -8)$, $R(10, -3)$, $S(8, 8)$

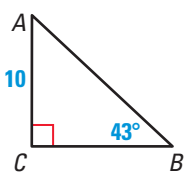
SOLVING PROPORTIONS **Solve the proportion.** **(Review 8.1)**

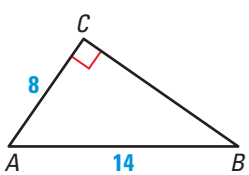
61. $\frac{x}{11} = \frac{3}{5}$ **62.** $\frac{x}{6} = \frac{9}{2}$ **63.** $\frac{x}{7} = \frac{12}{3}$ **64.** $\frac{33}{x} = \frac{18}{42}$

65. $\frac{10}{3} = \frac{8}{x}$ **66.** $\frac{3}{x+2} = \frac{4}{x}$ **67.** $\frac{2}{x-3} = \frac{3}{x}$ **68.** $\frac{5}{x-1} = \frac{9}{2x}$

SOLVING TRIANGLES **Solve the right triangle. Round decimals to the nearest tenth.** **(Review 9.6)**

69.


70.


71.


10.2 Arcs and Chords

What you should learn

GOAL 1 Use properties of arcs of circles, as applied in **Exs. 49–51**.

GOAL 2 Use properties of chords of circles, as applied in **Ex. 52**.

Why you should learn it

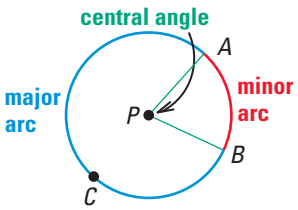
▼ To find the centers of **real-life** arcs, such as the arc of an ax swing in **Example 6**.

REAL LIFE

GOAL 1 USING ARCS OF CIRCLES

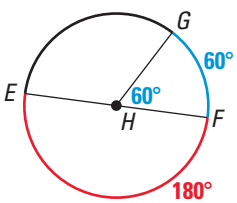
In a plane, an angle whose vertex is the center of a circle is a **central angle** of the circle.

If the measure of a central angle, $\angle APB$, is less than 180°, then A and B and the points of $\odot P$ in the interior of $\angle APB$ form a **minor arc** of the circle. The points A and B and the points of $\odot P$ in the *exterior* of $\angle APB$ form a **major arc** of the circle. If the endpoints of an arc are the endpoints of a diameter, then the arc is a **semicircle**.



NAMING ARCS Arcs are named by their endpoints. For example, the minor arc associated with $\angle APB$ above is $\widehat{AB}$. Major arcs and semicircles are named by their endpoints and by a point on the arc. For example, the major arc associated with $\angle APB$ above is $\widehat{ACB}$. $\widehat{EGF}$ below is a semicircle.

MEASURING ARCS The **measure of a minor arc** is defined to be the measure of its central angle. For instance, $m\widehat{GF} = m\angle GHF = 60°$. "$m\widehat{GF}$" is read "the measure of arc GF." You can write the measure of an arc next to the arc. The measure of a semicircle is 180°.



The **measure of a major arc** is defined as the difference between 360° and the measure of its associated minor arc. For example, $m\widehat{GEF} = 360° - 60° = 300°$. The measure of a whole circle is 360°.

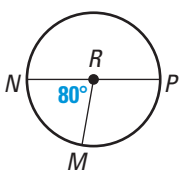
EXAMPLE 1 *Finding Measures of Arcs*

Find the measure of each arc of $\odot R$.

a. $\widehat{MN}$

b. $\widehat{MPN}$

c. $\widehat{PMN}$



SOLUTION

a. $\widehat{MN}$ is a minor arc, so $m\widehat{MN} = m\angle MRN = 80°$

b. $\widehat{MPN}$ is a major arc, so $m\widehat{MPN} = 360° - 80° = 280°$

c. $\widehat{PMN}$ is a semicircle, so $m\widehat{PMN} = 180°$

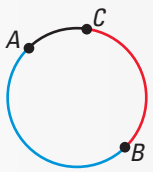
Two arcs of the same circle are *adjacent* if they intersect at exactly one point. You can add the measures of adjacent arcs.

POSTULATE

POSTULATE 26 ***Arc Addition Postulate***

The measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs.

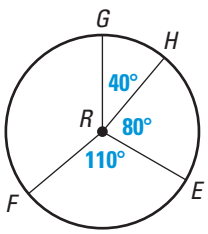
$$m\widehat{ABC} = m\widehat{AB} + m\widehat{BC}$$



EXAMPLE 2 *Finding Measures of Arcs*

Find the measure of each arc.

a. $\widehat{GE}$ **b.** $\widehat{GEF}$ **c.** $\widehat{GF}$



SOLUTION

a. $m\widehat{GE} = m\widehat{GH} + m\widehat{HE} = 40° + 80° = 120°$

b. $m\widehat{GEF} = m\widehat{GE} + m\widehat{EF} = 120° + 110° = 230°$

c. $m\widehat{GF} = 360° - m\widehat{GEF} = 360° - 230° = 130°$

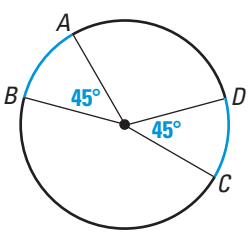
Two arcs of the same circle or of congruent circles are **congruent arcs** if they have the same measure. So, two minor arcs of the same circle or of congruent circles are congruent if their central angles are congruent.

EXAMPLE 3 *Identifying Congruent Arcs*

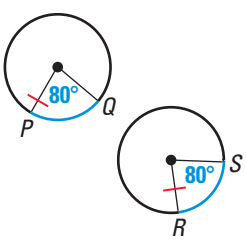
Logical Reasoning

Find the measures of the blue arcs. Are the arcs congruent?

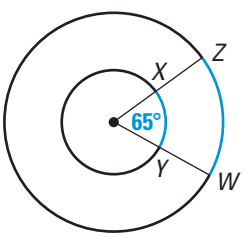
a.



b.



c.



SOLUTION

a. $\widehat{AB}$ and $\widehat{DC}$ are in the same circle and $m\widehat{AB} = m\widehat{DC} = 45°$. So, $\widehat{AB} \cong \widehat{DC}$.

b. $\widehat{PQ}$ and $\widehat{RS}$ are in congruent circles and $m\widehat{PQ} = m\widehat{RS} = 80°$. So, $\widehat{PQ} \cong \widehat{RS}$.

c. $m\widehat{XY} = m\widehat{ZW} = 65°$, but $\widehat{XY}$ and $\widehat{ZW}$ are not arcs of the same circle or of congruent circles, so $\widehat{XY}$ and $\widehat{ZW}$ are *not* congruent.

GOAL 2 USING CHORDS OF CIRCLES

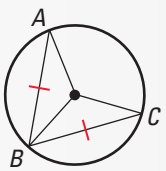
A point Y is called the *midpoint* of $\widehat{XYZ}$ if $\widehat{XY} \cong \widehat{YZ}$. Any line, segment, or ray that contains Y *bisects* $\widehat{XYZ}$. You will prove Theorems 10.4–10.6 in the exercises.

THEOREMS ABOUT CHORDS OF CIRCLES

THEOREM 10.4

In the same circle, or in congruent circles, two minor arcs are congruent if and only if their corresponding chords are congruent.

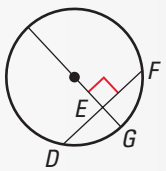
$\widehat{AB} \cong \widehat{BC}$ if and only if $\overline{AB} \cong \overline{BC}$.



THEOREM 10.5

If a diameter of a circle is perpendicular to a chord, then the diameter bisects the chord and its arc.

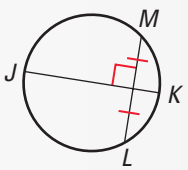
$\overline{DE} \cong \overline{EF}$, $\widehat{DG} \cong \widehat{GF}$



THEOREM 10.6

If one chord is a perpendicular bisector of another chord, then the first chord is a diameter.

$\overline{JK}$ is a diameter of the circle.



EXAMPLE 4 *Using Theorem 10.4*

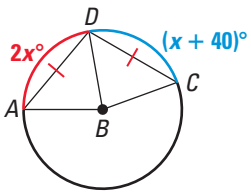
Using Algebra

You can use Theorem 10.4 to find $m\widehat{AD}$.

Because $\overline{AD} \cong \overline{DC}$, $\widehat{AD} \cong \widehat{DC}$. So, $m\widehat{AD} = m\widehat{DC}$.

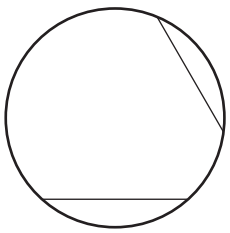
$2x = x + 40$ **Substitute.**

$x = 40$ **Subtract x from each side.**

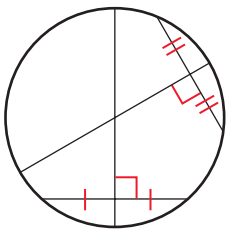


EXAMPLE 5 *Finding the Center of a Circle*

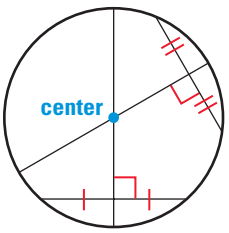
Theorem 10.6 can be used to locate a circle's center, as shown below.



1 Draw any two chords that are not parallel to each other.



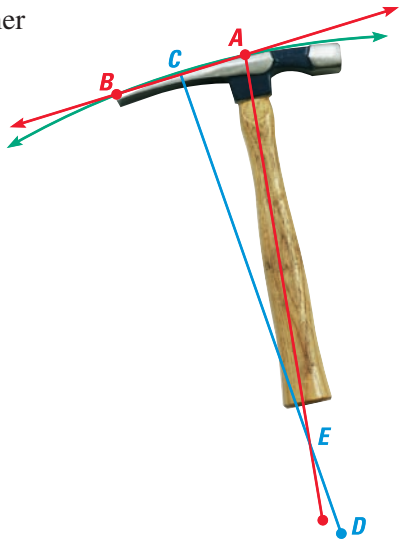
2 Draw the perpendicular bisector of each chord. These are diameters.



3 The perpendicular bisectors intersect at the circle's center.

EXAMPLE 6 *Using Properties of Chords*

MASONRY HAMMER A masonry hammer has a hammer on one end and a curved pick on the other. The pick works best if you swing it along a circular curve that matches the shape of the pick. Find the center of the circular swing.



SOLUTION

Draw a segment $\overline{AB}$, from the top of the masonry hammer to the end of the pick. Find the midpoint C, and draw a perpendicular bisector $\overline{CD}$. Find the intersection of $\overline{CD}$ with the line formed by the handle.

▶ So, the center of the swing lies at E.

You are asked to prove Theorem 10.7 in Exercises 61 and 62.

STUDENT HELP

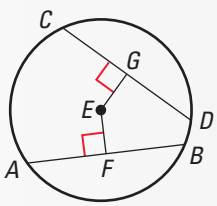
Look Back
Remember that the distance from a point to a line is the length of the perpendicular segment from the point to the line. (p. 266)

THEOREM

THEOREM 10.7

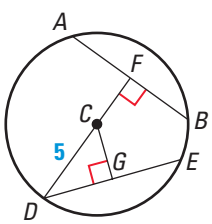
In the same circle, or in congruent circles, two chords are congruent if and only if they are equidistant from the center.

$\overline{AB} \cong \overline{CD}$ if and only if $\overline{EF} \cong \overline{EG}$.



EXAMPLE 7 *Using Theorem 10.7*

$AB = 8$, $DE = 8$, and $CD = 5$. Find CF.



SOLUTION

Because $\overline{AB}$ and $\overline{DE}$ are congruent chords, they are equidistant from the center. So, $\overline{CF} \cong \overline{CG}$. To find CG, first find DG.

$\overline{CG} \perp \overline{DE}$, so $\overline{CG}$ bisects $\overline{DE}$. Because $DE = 8$, $DG = \frac{8}{2} = 4$.

Then use DG to find CG.

$DG = 4$ and $CD = 5$, so $\triangle CGD$ is a 3-4-5 right triangle. So, $CG = 3$.

Finally, use CG to find CF.

▶ Because $\overline{CF} \cong \overline{CG}$, $CF = CG = 3$.

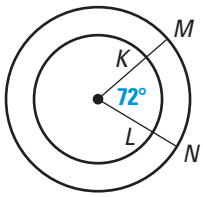
GUIDED PRACTICE

Vocabulary Check ✓

1. The measure of an arc is 170°. Is the arc a *major arc*, a *minor arc*, or a *semicircle*?

Concept Check ✓

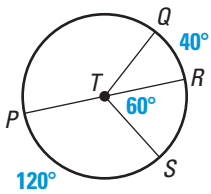
2. In the figure at the right, what is $m\widehat{KL}$? What is $m\widehat{MN}$? Are $\widehat{KL}$ and $\widehat{MN}$ congruent? Explain.



Skill Check ✓

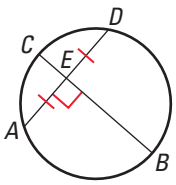
Find the measure in ⊙*T*.

3. $m\widehat{RS}$
4. $m\widehat{RPS}$
5. $m\widehat{PQR}$
6. $m\widehat{QS}$
7. $m\widehat{QSP}$
8. $m\angle QTR$



What can you conclude about the diagram? State a postulate or theorem that justifies your answer.

9.



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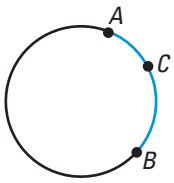
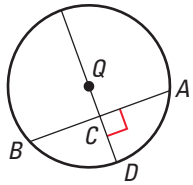


11.



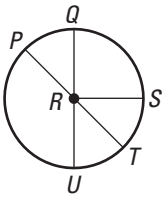
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 821.

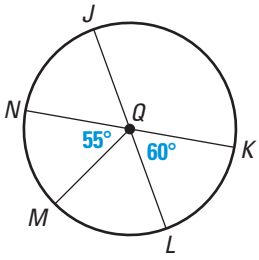
UNDERSTANDING THE CONCEPT **Determine whether the arc is a *minor arc*, a *major arc*, or a *semicircle* of ⊙*R*.**

12. $\widehat{PQ}$
13. $\widehat{SU}$
14. $\widehat{PQT}$
15. $\widehat{QT}$
16. $\widehat{TUQ}$
17. $\widehat{TUP}$
18. $\widehat{QUT}$
19. $\widehat{PUQ}$



MEASURING ARCS AND CENTRAL ANGLES **$\overline{KN}$ and $\overline{JL}$ are diameters. Copy the diagram. Find the indicated measure.**

20. $m\widehat{KL}$
21. $m\widehat{MN}$
22. $m\widehat{LNK}$
23. $m\widehat{MKN}$
24. $m\widehat{NJK}$
25. $m\widehat{JML}$
26. $m\angle JQN$
27. $m\angle MQL$
28. $m\widehat{JN}$
29. $m\widehat{ML}$
30. $m\widehat{JM}$
31. $m\widehat{LN}$



STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 12–29
Example 2: Exs. 30–34, 49, 50
Example 3: Ex. 35

continued on p. 608

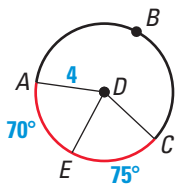
STUDENT HELP

HOMEWORK HELP
continued from p. 607

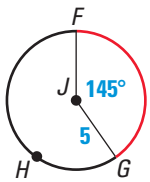
Example 4: Exs. 36–38
Example 5: Exs. 52, 54
Example 6: Exs. 52, 54
Example 7: Exs. 39–47

FINDING ARC MEASURES **Find the measure of the red arc.**

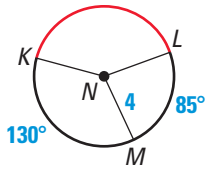
32.



33.



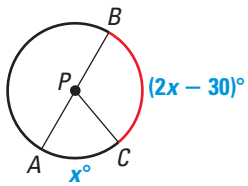
34.



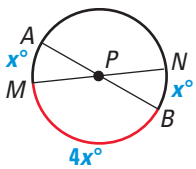
35. Name two pairs of congruent arcs in Exercises 32–34. Explain your reasoning.

USING ALGEBRA **Use ⊙*P* to find the value of *x*. Then find the measure of the red arc.**

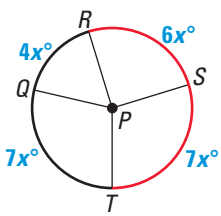
36.



37.

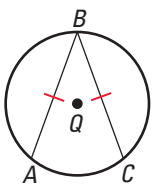


38.



LOGICAL REASONING **What can you conclude about the diagram? State a postulate or theorem that justifies your answer.**

39.



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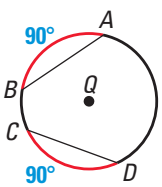
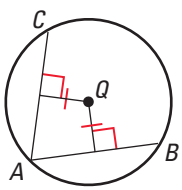
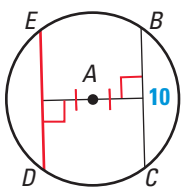


41.

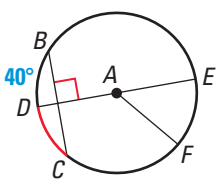


MEASURING ARCS AND CHORDS **Find the measure of the red arc or chord in ⊙*A*. Explain your reasoning.**

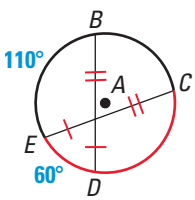
42.



43.

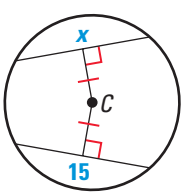


44.



MEASURING ARCS AND CHORDS **Find the value of *x* in ⊙*C*. Explain your reasoning.**

45.



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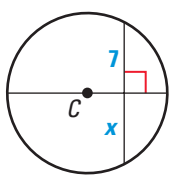
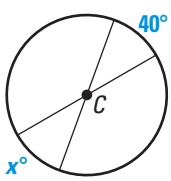


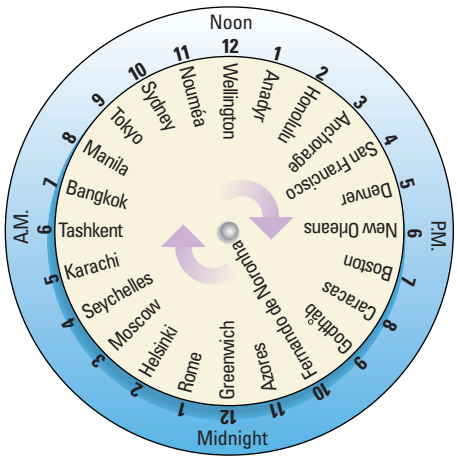
47.



48. SKETCHING Draw a circle with two noncongruent chords. Is the shorter chord's midpoint farther from the center or closer to the center than the longer chord's midpoint?

TIME ZONE WHEEL **In Exercises 49–51, use the following information.**

The time zone wheel shown at the right consists of two concentric circular pieces of cardboard fastened at the center so the smaller wheel can rotate. To find the time in Tashkent when it is 4 P.M. in San Francisco, you rotate the small wheel until 4 P.M. and San Francisco line up as shown. Then look at Tashkent to see that it is 6 A.M. there. The arcs between cities are congruent.

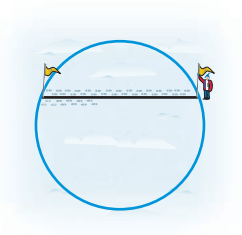


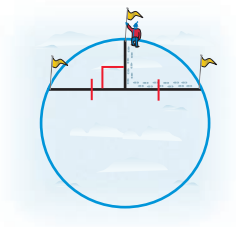
49. What is the arc measure for each time zone on the wheel?

50. What is the measure of the minor arc from the Tokyo zone to the Anchorage zone?

51. If two cities differ by 180° on the wheel, then it is 3:00 P.M. in one city if and only if it is ? in the other city.

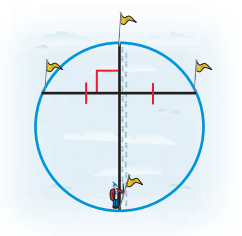
52. **AVALANCHE RESCUE BEACON** An avalanche rescue beacon is a small device carried by backcountry skiers that gives off a signal that can be picked up only within a circle of a certain radius. During a practice drill, a ski patrol uses steps similar to the following to locate a beacon buried in the snow. Write a paragraph explaining why this procedure works. ▶ Source: The Mountaineers

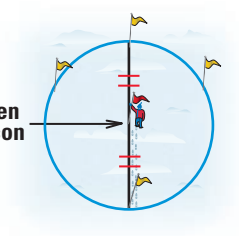




1. Walk until the signal disappears, turn around, and pace the distance in a straight line until the signal disappears again.

2. Pace back to the halfway point, and walk away from the line at a 90° angle until the signal disappears.





3. Turn around and pace the distance in a straight line until the signal disappears again.

4. Pace back to the halfway point. You will be at or near the center of the circle. The beacon is underneath you.

53. **LOGICAL REASONING** Explain why two minor arcs of the same circle or of congruent circles are congruent if and only if their central angles are congruent.

FOCUS ON CAREERS

EMTS Some Emergency Medical Technicians (EMTs) train specifically for wilderness emergencies. These EMTs must be able to improvise with materials they have on hand.

CAREER LINK
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54. **CONSTRUCTION** Trace a circular object like a cup or can. Then use a compass and straightedge to find the center of the circle. Explain your steps.

55. **CONSTRUCTION** Construct a large circle with two congruent chords. Are the chords the same distance from the center? How can you tell?

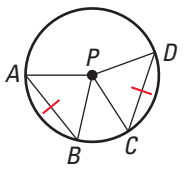
STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 56–62.

PROVING THEOREM 10.4 **In Exercises 56 and 57, you will prove Theorem 10.4 for the case in which the two chords are in the same circle. Write a plan for a proof.**

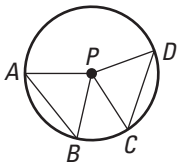
56. **GIVEN** ▶ $\overline{AB}$ and $\overline{DC}$ are in $\odot P$. $\overline{AB} \cong \overline{DC}$

PROVE ▶ $\widehat{AB} \cong \widehat{DC}$



57. **GIVEN** ▶ $\overline{AB}$ and $\overline{DC}$ are in $\odot P$. $\widehat{AB} \cong \widehat{DC}$

PROVE ▶ $\overline{AB} \cong \overline{DC}$



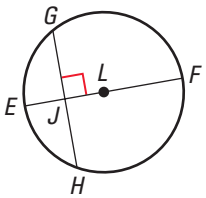
58. **JUSTIFYING THEOREM 10.4** Explain how the proofs in Exercises 56 and 57 would be different if $\overline{AB}$ and $\overline{DC}$ were in congruent circles rather than the same circle.

PROVING THEOREMS 10.5 AND 10.6 **Write a proof.**

59. **GIVEN** ▶ $\overline{EF}$ is a diameter of $\odot L$. $\overline{EF} \perp \overline{GH}$

PROVE ▶ $\overline{GJ} \cong \overline{JH}$, $\widehat{GE} \cong \widehat{EH}$

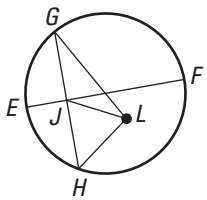
Plan for Proof Draw $\overline{LG}$ and $\overline{LH}$. Use congruent triangles to show $\overline{GJ} \cong \overline{JH}$ and $\angle GLE \cong \angle HLE$. Then show $\widehat{GE} \cong \widehat{EH}$.



60. **GIVEN** ▶ $\overline{EF}$ is the $\perp$ bisector of $\overline{GH}$.

PROVE ▶ $\overline{EF}$ is a diameter of $\odot L$.

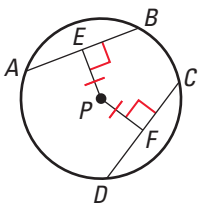
Plan for Proof Use indirect reasoning. Assume center L is not on $\overline{EF}$. Prove that $\triangle GLJ \cong \triangle HLJ$, so $\overline{JL} \perp \overline{GH}$. Then use the Perpendicular Postulate.



PROVING THEOREM 10.7 **Write a proof.**

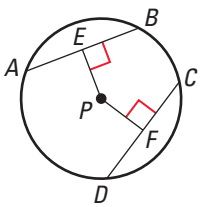
61. **GIVEN** ▶ $\overline{PE} \perp \overline{AB}$, $\overline{PF} \perp \overline{DC}$, $\overline{PE} \cong \overline{PF}$

PROVE ▶ $\overline{AB} \cong \overline{DC}$



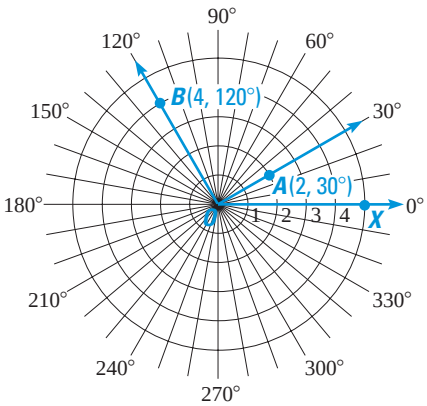
62. **GIVEN** ▶ $\overline{PE} \perp \overline{AB}$, $\overline{PF} \perp \overline{DC}$, $\overline{AB} \cong \overline{DC}$

PROVE ▶ $\overline{PE} \cong \overline{PF}$



POLAR COORDINATES **In Exercises 63–67, use the following information.**

A *polar coordinate system* locates a point in a plane by its distance from the origin O and by the measure of a central angle. For instance, the point $A(2, 30°)$ at the right is 2 units from the origin and $m\angle XOA = 30°$. Similarly, the point $B(4, 120°)$ is 4 units from the origin and $m\angle XOB = 120°$.



63. Use polar graph paper or a protractor and a ruler to graph points A and B. Also graph $C(4, 210°)$, $D(4, 330°)$, and $E(2, 150°)$.

64. Find $m\widehat{AE}$. **65.** Find $m\widehat{BC}$. **66.** Find $m\widehat{BD}$. **67.** Find $m\widehat{BCD}$.

Test Preparation

68. **MULTI-STEP PROBLEM** You want to find the radius of a circular object. First you trace the object on a piece of paper.

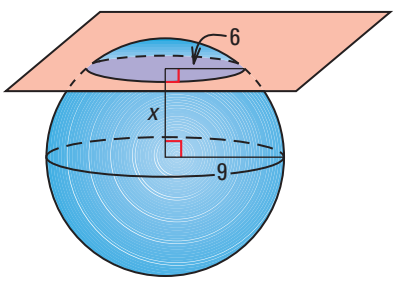
a. Explain how to use two chords that are not parallel to each other to find the radius of the circle.

b. Explain how to use two tangent lines that are not parallel to each other to find the radius of the circle.

c. *Writing* Would the methods in parts (a) and (b) work better for small objects or for large objects? Explain your reasoning.

★ Challenge

69. The plane at the right intersects the sphere in a circle that has a diameter of 12. If the diameter of the sphere is 18, what is the value of x? Give your answer in simplified radical form.



EXTRA CHALLENGE
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MIXED REVIEW

INTERIOR OF AN ANGLE **Plot the points in a coordinate plane and sketch $\angle ABC$. Write the coordinates of a point that lies in the interior and a point that lies in the exterior of $\angle ABC$.** (Review 1.4 for 10.3)

70. $A(4, 2)$, $B(0, 2)$, $C(3, 0)$

71. $A(-2, 3)$, $B(0, 0)$, $C(4, -1)$

72. $A(-2, -3)$, $B(0, -1)$, $C(2, -3)$

73. $A(-3, 2)$, $B(0, 0)$, $C(3, 2)$

COORDINATE GEOMETRY **The coordinates of the vertices of parallelogram *PQRS* are given. Decide whether ▱*PQRS* is best described as a *rhombus*, a *rectangle*, or a *square*. Explain your reasoning.** (Review 6.4 for 10.3)

74. $P(-2, 1)$, $Q(-1, 4)$, $R(0, 1)$, $S(-1, -2)$

75. $P(-1, 2)$, $Q(2, 5)$, $R(5, 2)$, $S(2, -1)$

GEOMETRIC MEAN **Find the geometric mean of the numbers.** (Review 8.2)

76. 9, 16 **77.** 8, 32 **78.** 4, 49 **79.** 9, 36

10.3 Inscribed Angles

What you should learn

GOAL 1 Use inscribed angles to solve problems.

GOAL 2 Use properties of inscribed polygons.

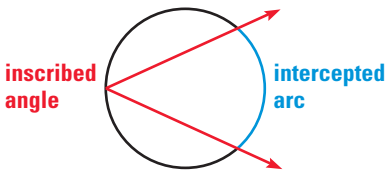
Why you should learn it

▼ To solve **real-life** problems, such as finding the different seats in a theater that will give you the same viewing angle, as in **Example 4**.

REAL LIFE

GOAL 1 USING INSCRIBED ANGLES

An **inscribed angle** is an angle whose vertex is on a circle and whose sides contain chords of the circle. The arc that lies in the interior of an inscribed angle and has endpoints on the angle is called the **intercepted arc** of the angle.

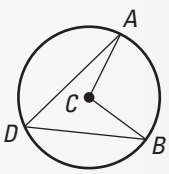


THEOREM

THEOREM 10.8 ***Measure of an Inscribed Angle***

If an angle is inscribed in a circle, then its measure is half the measure of its intercepted arc.

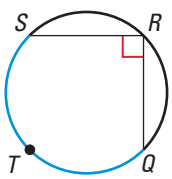
$$m\angle ADB = \frac{1}{2}m\widehat{AB}$$



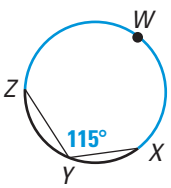
EXAMPLE 1 *Finding Measures of Arcs and Inscribed Angles*

Find the measure of the blue arc or angle.

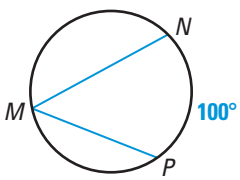
a.



b.



c.



SOLUTION

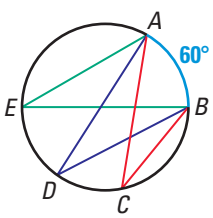
a. $m\widehat{QTS} = 2m\angle QRS = 2(90°) = 180°$

b. $m\widehat{ZWX} = 2m\angle ZYX = 2(115°) = 230°$

c. $m\angle NMP = \frac{1}{2}m\widehat{NP} = \frac{1}{2}(100°) = 50°$

EXAMPLE 2 *Comparing Measures of Inscribed Angles*

Find $m\angle ACB$, $m\angle ADB$, and $m\angle AEB$.



SOLUTION

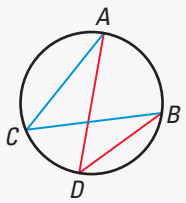
The measure of each angle is half the measure of $\widehat{AB}$. $m\widehat{AB} = 60°$, so the measure of each angle is 30°.

Example 2 suggests the following theorem. You are asked to prove Theorem 10.8 and Theorem 10.9 in Exercises 35–38.

THEOREM

THEOREM 10.9

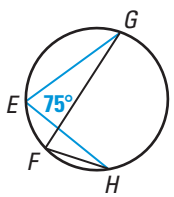
If two inscribed angles of a circle intercept the same arc, then the angles are congruent.



$\angle C \cong \angle D$

EXAMPLE 3 *Finding the Measure of an Angle*

It is given that $m\angle E = 75°$. What is $m\angle F$?



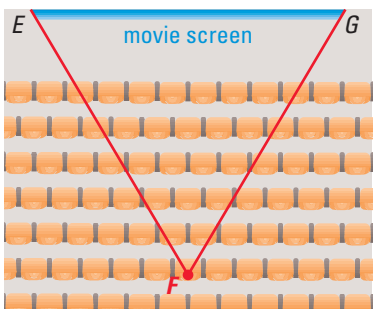
SOLUTION

$\angle E$ and $\angle F$ both intercept $\widehat{GH}$, so $\angle E \cong \angle F$.

▶ So, $m\angle F = m\angle E = 75°$.

EXAMPLE 4 *Using the Measure of an Inscribed Angle*

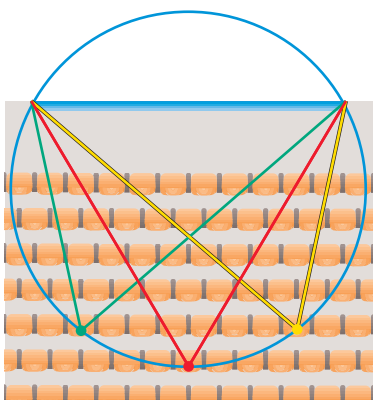
THEATER DESIGN When you go to the movies, you want to be close to the movie screen, but you don't want to have to move your eyes too much to see the edges of the picture. If E and G are the ends of the screen and you are at F, $m\angle EFG$ is called your *viewing angle*.



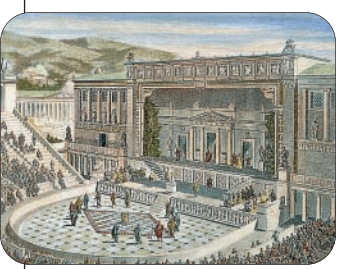
You decide that the middle of the sixth row has the best viewing angle. If someone is sitting there, where else can you sit to have the same viewing angle?

SOLUTION

Draw the circle that is determined by the endpoints of the screen and the sixth row center seat. Any other location on the circle will have the same viewing angle.



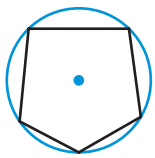
FOCUS ON APPLICATIONS



REAL LIFE THEATER DESIGN In Ancient Greece, stages were often part of a circle and the seats were on concentric circles.

GOAL 2 USING PROPERTIES OF INSCRIBED POLYGONS

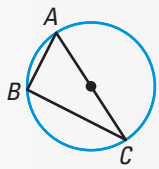
If all of the vertices of a polygon lie on a circle, the polygon is **inscribed** in the circle and the circle is **circumscribed** about the polygon. The polygon is an *inscribed polygon* and the circle is a *circumscribed circle*. You are asked to justify Theorem 10.10 and part of Theorem 10.11 in Exercises 39 and 40. A complete proof of Theorem 10.11 appears on page 840.



THEOREMS ABOUT INSCRIBED POLYGONS

THEOREM 10.10

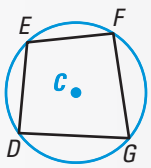
If a right triangle is inscribed in a circle, then the hypotenuse is a diameter of the circle. Conversely, if one side of an inscribed triangle is a diameter of the circle, then the triangle is a right triangle and the angle opposite the diameter is the right angle.



$\angle B$ is a right angle if and only if $\overline{AC}$ is a diameter of the circle.

THEOREM 10.11

A quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary.



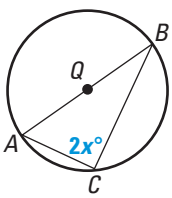
D, E, F, and G lie on some circle, $\odot C$, if and only if $m\angle D + m\angle F = 180°$ and $m\angle E + m\angle G = 180°$.

EXAMPLE 5 *Using Theorems 10.10 and 10.11*

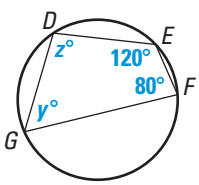
Using Algebra

Find the value of each variable.

a.



b.



SOLUTION

a. $\overline{AB}$ is a diameter. So, $\angle C$ is a right angle and $m\angle C = 90°$.

$$2x° = 90°$$

$$x = 45$$

b. $DEFG$ is inscribed in a circle, so opposite angles are supplementary.

$$m\angle D + m\angle F = 180° \qquad m\angle E + m\angle G = 180°$$

$$z + 80 = 180 \qquad 120 + y = 180$$

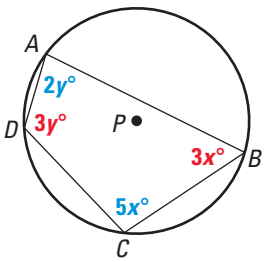
$$z = 100 \qquad y = 60$$

STUDENT HELP

Skills Review
For help with solving systems of equations, see p. 796.

EXAMPLE 6 *Using an Inscribed Quadrilateral*

In the diagram, $ABCD$ is inscribed in $\odot P$. Find the measure of each angle.



SOLUTION

$ABCD$ is inscribed in a circle, so opposite angles are supplementary.

$$3x + 3y = 180 \qquad 5x + 2y = 180$$

To solve this system of linear equations, you can solve the first equation for y to get $y = 60 - x$. Substitute this expression into the second equation.

$5x + 2y = 180$	**Write second equation.**
$5x + 2(60 - x) = 180$	**Substitute $60 - x$ for y.**
$5x + 120 - 2x = 180$	**Distributive property**
$3x = 60$	**Subtract 120 from each side.**
$x = 20$	**Divide each side by 3.**
$y = 60 - 20 = 40$	**Substitute and solve for y.**

▶ $x = 20$ and $y = 40$, so $m\angle A = 80°$, $m\angle B = 60°$, $m\angle C = 100°$, and $m\angle D = 120°$.

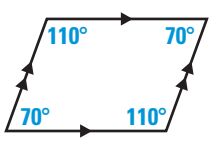
GUIDED PRACTICE

Vocabulary Check ✓

1. Draw a circle and an inscribed angle, $\angle ABC$. Name the intercepted arc of $\angle ABC$. Label additional points on your sketch if you need to.

Concept Check ✓

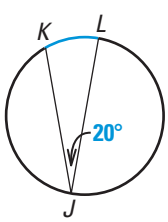
2. Determine whether the quadrilateral can be inscribed in a circle. Explain your reasoning.



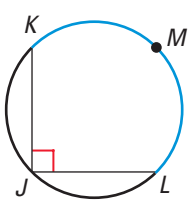
Skill Check ✓

Find the measure of the blue arc.

3.



4.



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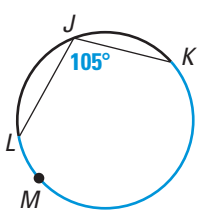
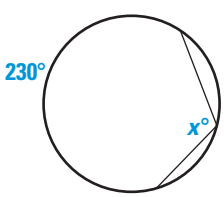


Find the value of each variable.

6.



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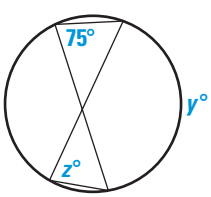

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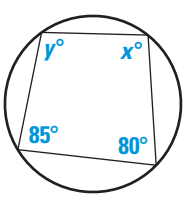


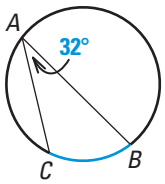
PRACTICE AND APPLICATIONS

STUDENT HELP

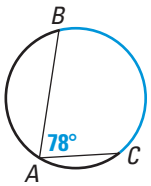
Extra Practice to help you master skills is on p. 821.

ARC AND ANGLE MEASURES **Find the measure of the blue arc or angle.**

9.



10.



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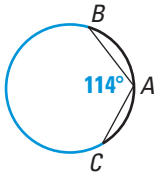

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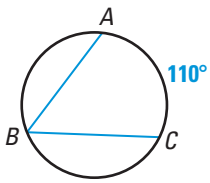

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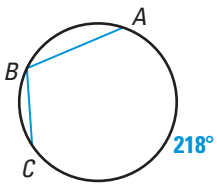

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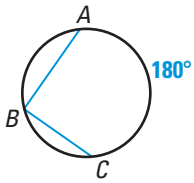
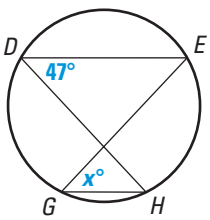
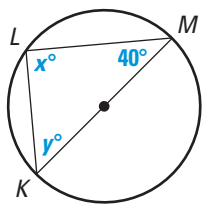


USING ALGEBRA **Find the value of each variable. Explain.**

15.



16.



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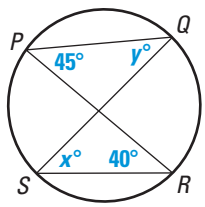
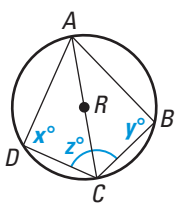
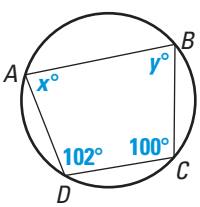


USING ALGEBRA **Find the values of *x*, *y*, and *z*.**

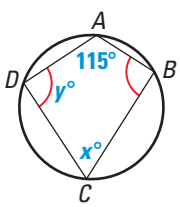
18. $m\widehat{BCD} = 136°$



19. $m\widehat{BCD} = z°$

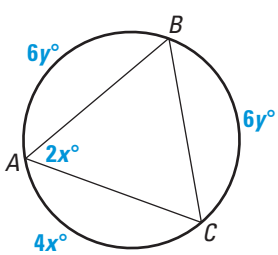


20. $m\widehat{ABC} = z°$

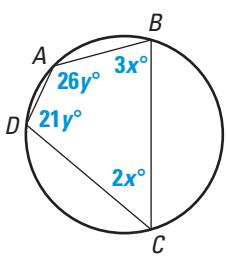


USING ALGEBRA **Find the values of *x* and *y*. Then find the measures of the interior angles of the polygon.**

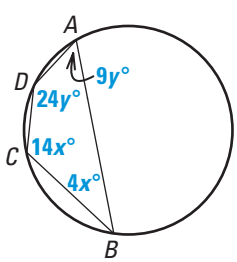
21.



22.



23.



LOGICAL REASONING **Can the quadrilateral always be inscribed in a circle? Explain your reasoning.**

24. square

25. rectangle

26. parallelogram

27. kite

28. rhombus

29. isosceles trapezoid

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 9–14, 19–21
Example 2: Exs. 15, 17
Example 3: Exs. 15, 17
Example 4: Exs. 15, 17
Example 5: Exs. 15–20, 24–29, 31–34
Example 6: Exs. 21–23

30. **CONSTRUCTION** Construct a $\odot C$ and a point A on $\odot C$. Construct the tangent to $\odot C$ at A. Explain why your construction works.

CONSTRUCTION In Exercises 31–33, you will construct a tangent to a circle from a point outside the circle.

31. Construct a $\odot C$ and a point outside the circle, A. Draw $\overline{AC}$ and construct its midpoint M. Construct $\odot M$ with radius MC. What kind of chord is $\overline{AC}$?

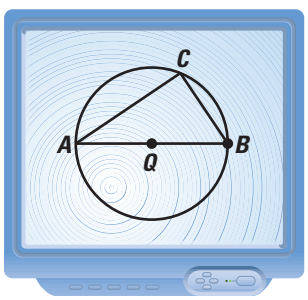
32. $\odot C$ and $\odot M$ have two points of intersection. Label one of the points B. Draw $\overline{AB}$ and $\overline{CB}$. What is $m\angle CBA$? How do you know?

33. Which segment is tangent to $\odot C$ from A? Explain.

STUDENT HELP

SOFTWARE HELP Visit our Web site www.mcdougallittell.com to see instructions for several software applications.

34. **USING TECHNOLOGY** Use geometry software to construct $\odot Q$, diameter $\overline{AB}$, and point C on $\odot Q$. Construct $\overline{AC}$ and $\overline{CB}$. Measure the angles of $\triangle ABC$. Drag point C along $\odot Q$. Record and explain your observations.

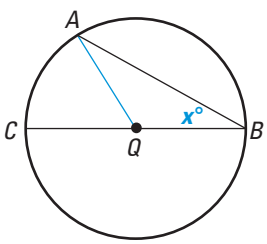


PROVING THEOREM 10.8 If an angle is inscribed in $\odot Q$, the center Q can be on a side of the angle, in the interior of the angle, or in the exterior of the angle. To prove Theorem 10.8, you must prove each of these cases.

35. Fill in the blanks to complete the proof.

GIVEN ▶ $\angle ABC$ is inscribed in $\odot Q$. Point Q lies on $\overline{BC}$.

PROVE ▶ $m\angle ABC = \frac{1}{2}m\widehat{AC}$



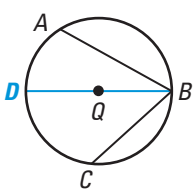
Paragraph Proof Let $m\angle ABC = x°$. Because $\overline{QA}$ and $\overline{QB}$ are both radii of $\odot Q$, $\overline{QA} \cong$ __?__ and $\triangle AQB$ is __?__. Because $\angle A$ and $\angle B$ are __?__ of an isosceles triangle, __?__. So, by substitution, $m\angle A = x°$.

By the __?__ Theorem, $m\angle AQC = m\angle A + m\angle B =$ __?__. So, by the definition of the measure of a minor arc, $m\widehat{AC} =$ __?__. Divide each side by __?__ to show that $x° =$ __?__. Then, by substitution, $m\angle ABC =$ __?__.

36. Write a plan for a proof.

GIVEN ▶ $\angle ABC$ is inscribed in $\odot Q$. Point Q is in the interior of $\angle ABC$.

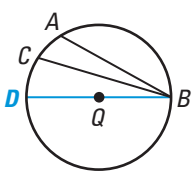
PROVE ▶ $m\angle ABC = \frac{1}{2}m\widehat{AC}$



37. Write a plan for a proof.

GIVEN ▶ $\angle ABC$ is inscribed in $\odot Q$. Point Q is in the exterior of $\angle ABC$.

PROVE ▶ $m\angle ABC = \frac{1}{2}m\widehat{AC}$



38. **PROVING THEOREM 10.9** Write a proof of Theorem 10.9. First draw a diagram and write GIVEN and PROVE statements.

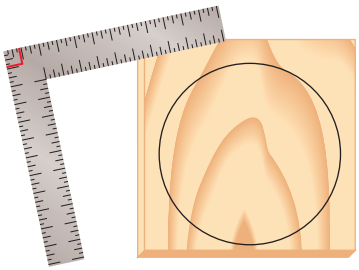
39. **PROVING THEOREM 10.10** Theorem 10.10 is written as a conditional statement and its converse. Write a plan for a proof of each statement.

40. **PROVING THEOREM 10.11** Draw a diagram and write a proof of part of Theorem 10.11.

GIVEN ▶ $DEFG$ is inscribed in a circle.

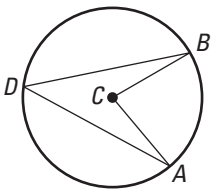
PROVE ▶ $m\angle D + m\angle F = 180°$, $m\angle E + m\angle G = 180°$

41. **CARPENTER'S SQUARE** A carpenter's square is an L-shaped tool used to draw right angles. Suppose you are making a copy of a wooden plate. You trace the plate on a piece of wood. How could you use a carpenter's square to find the center of the circle?



Test Preparation

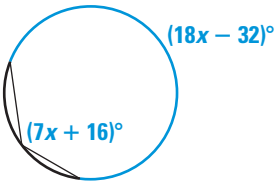
42. **MULTIPLE CHOICE** In the diagram at the right, if $\angle ACB$ is a central angle and $m\angle ACB = 80°$, what is $m\angle ADB$?



Ⓐ 20° Ⓑ 40° Ⓒ 80°

Ⓓ 100° Ⓔ 160°

43. **MULTIPLE CHOICE** In the diagram at the right, what is the value of x?



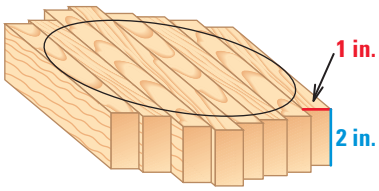
Ⓐ $\frac{48}{11}$ Ⓑ 12 Ⓒ 16

Ⓓ 18 Ⓔ 24

★ Challenge

CUTTING BOARD **In Exercises 44–47, use the following information.**

You are making a circular cutting board. To begin, you glue eight 1 inch by 2 inch boards together, as shown at the right. Then you draw and cut a circle with an 8 inch diameter from the boards.

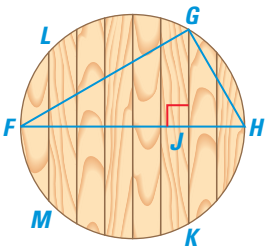


44. $\overline{FH}$ is a diameter of the circular cutting board. What kind of triangle is $\triangle FGH$?

45. How is GJ related to FJ and JH? State a theorem to justify your answer.

46. Find FJ, JH, and JG. What is the length of the seam of the cutting board that is labeled $\overline{GK}$?

47. Find the length of $\overline{LM}$.



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

WRITING EQUATIONS **Write an equation in slope-intercept form of the line that passes through the given point and has the given slope.** (Review 3.6)

48. $(-2, -6)$, $m = -1$ **49.** $(5, 1)$, $m = 2$ **50.** $(3, 3)$, $m = 0$

51. $(0, 7)$, $m = \frac{4}{3}$ **52.** $(-8, 4)$, $m = -\frac{1}{2}$ **53.** $(-5, -12)$, $m = -\frac{4}{5}$

SKETCHING IMAGES **Sketch the image of $\triangle PQR$ after a composition using the given transformations in the order in which they appear. $\triangle PQR$ has vertices $P(-5, 4)$, $Q(-2, 1)$, and $R(-1, 3)$.** (Review 7.5)

54. **translation:** $(x, y) \rightarrow (x + 6, y)$
reflection: in the x-axis

55. **translation:** $(x, y) \rightarrow (x + 8, y + 1)$
reflection: in the line $y = 1$

56. **reflection:** in the line $x = 3$
translation: $(x, y) \rightarrow (x - 1, y - 7)$

57. **reflection:** in the y-axis
rotation: 90° clockwise about the origin

58. What is the length of an altitude of an equilateral triangle whose sides have lengths of $26\sqrt{2}$? (Review 9.4)

FINDING TRIGONOMETRIC RATIOS **$\triangle ABC$ is a right triangle in which $AB = 4\sqrt{3}$, $BC = 4$, and $AC = 8$.** (Review 9.5 for 10.4)

59. $\sin A =$? **60.** $\cos A =$?

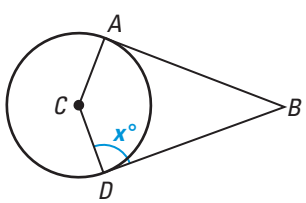
61. $\sin C =$? **62.** $\tan C =$?

QUIZ 1

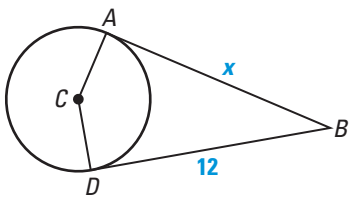
Self-Test for Lessons 10.1–10.3

$\overleftrightarrow{AB}$ is tangent to $\odot C$ at A and $\overleftrightarrow{DB}$ is tangent to $\odot C$ at D. Find the value of x. Write the postulate or theorem that justifies your answer. (Lesson 10.1)

1.



2.

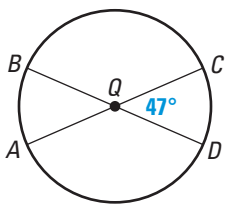


Find the measure of the arc of $\odot Q$. (Lesson 10.2)

3. $\widehat{AB}$ **4.** $\widehat{BC}$

5. $\widehat{ABD}$ **6.** $\widehat{BCA}$

7. $\widehat{ADC}$ **8.** $\widehat{CD}$



9. If an angle that has a measure of 42.6° is inscribed in a circle, what is the measure of its intercepted arc? (Lesson 10.3)

10.4 Other Angle Relationships in Circles

What you should learn

GOAL 1 Use angles formed by tangents and chords to solve problems in geometry.

GOAL 2 Use angles formed by lines that intersect a circle to solve problems.

Why you should learn it

▼ To solve **real-life** problems, such as finding from how far away you can see fireworks, as in **Ex. 35**.

REAL LIFE

GOAL 1 USING TANGENTS AND CHORDS

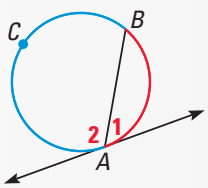
You know that the measure of an angle inscribed in a circle is half the measure of its intercepted arc. This is true even if one side of the angle is tangent to the circle. You will be asked to prove Theorem 10.12 in Exercises 37–39.

THEOREM

THEOREM 10.12

If a tangent and a chord intersect at a point on a circle, then the measure of each angle formed is one half the measure of its intercepted arc.

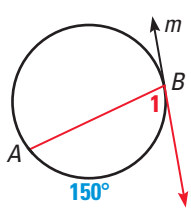
$m\angle 1 = \frac{1}{2}m\widehat{AB}$ $\qquad$ $m\angle 2 = \frac{1}{2}m\widehat{BCA}$



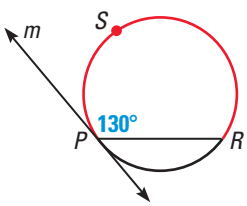
EXAMPLE 1 *Finding Angle and Arc Measures*

Line m is tangent to the circle. Find the measure of the red angle or arc.

a.



b.



SOLUTION

a. $m\angle 1 = \frac{1}{2}(150°) = 75°$

b. $m\widehat{RSP} = 2(130°) = 260°$

EXAMPLE 2 *Finding an Angle Measure*

In the diagram below, $\overleftrightarrow{BC}$ is tangent to the circle. Find $m\angle CBD$.

SOLUTION

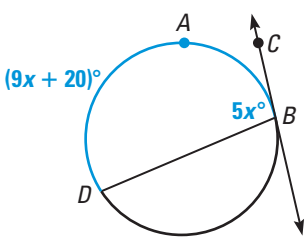
$$m\angle CBD = \frac{1}{2}m\widehat{DAB}$$

$$5x = \frac{1}{2}(9x + 20)$$

$$10x = 9x + 20$$

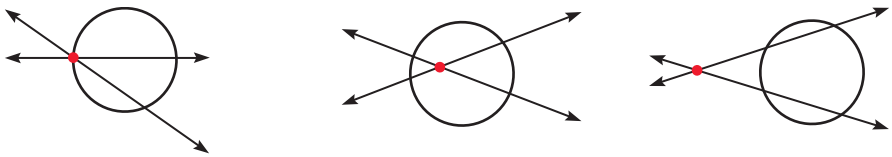
$$x = 20$$

▶ $m\angle CBD = 5(20°) = 100°$



GOAL 2 LINES INTERSECTING INSIDE OR OUTSIDE A CIRCLE

If two lines intersect a circle, there are three places where the lines can intersect.



on the circle **inside the circle** **outside the circle**

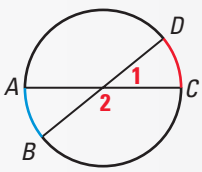
You know how to find angle and arc measures when lines intersect *on* the circle. You can use Theorems 10.13 and 10.14 to find measures when the lines intersect *inside* or *outside* the circle. You will prove these theorems in Exercises 40 and 41.

THEOREMS

THEOREM 10.13

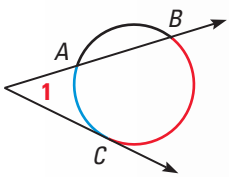
If two chords intersect in the *interior* of a circle, then the measure of each angle is one half the *sum* of the measures of the arcs intercepted by the angle and its vertical angle.

$$m\angle 1 = \frac{1}{2}(m\widehat{CD} + m\widehat{AB}),\ m\angle 2 = \frac{1}{2}(m\widehat{BC} + m\widehat{AD})$$

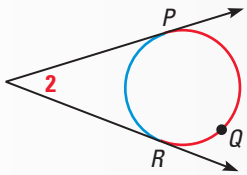


THEOREM 10.14

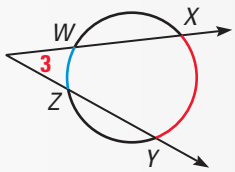
If a tangent and a secant, two tangents, or two secants intersect in the *exterior* of a circle, then the measure of the angle formed is one half the *difference* of the measures of the intercepted arcs.



$$m\angle 1 = \frac{1}{2}(m\widehat{BC} - m\widehat{AC})$$



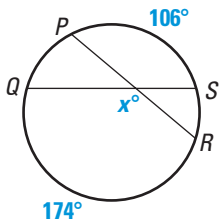
$$m\angle 2 = \frac{1}{2}(m\widehat{PQR} - m\widehat{PR})$$



$$m\angle 3 = \frac{1}{2}(m\widehat{XY} - m\widehat{WZ})$$

EXAMPLE 3 *Finding the Measure of an Angle Formed by Two Chords*

Find the value of x.



SOLUTION

$x° = \frac{1}{2}(m\widehat{PS} + m\widehat{RQ})$ **Apply Theorem 10.13.**

$x° = \frac{1}{2}(106° + 174°)$ **Substitute.**

$x = 140$ **Simplify.**

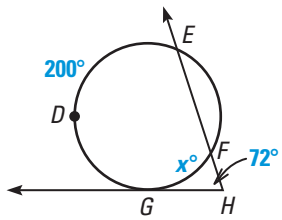
STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

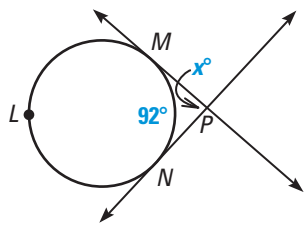
EXAMPLE 4 *Using Theorem 10.14*

Find the value of x.

a.



b.



SOLUTION

a. $m\angle GHF = \frac{1}{2}(m\widehat{EDG} - m\widehat{GF})$ **Apply Theorem 10.14.**

$72° = \frac{1}{2}(200° - x°)$ **Substitute.**

$144 = 200 - x$ **Multiply each side by 2.**

$x = 56$ **Solve for *x*.**

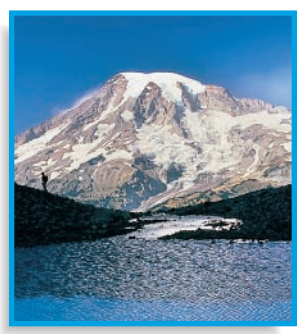
b. Because $\widehat{MN}$ and $\widehat{MLN}$ make a whole circle, $m\widehat{MLN} = 360° - 92° = 268°$.

$x = \frac{1}{2}(m\widehat{MLN} - m\widehat{MN})$ **Apply Theorem 10.14.**

$= \frac{1}{2}(268 - 92)$ **Substitute.**

$= \frac{1}{2}(176)$ **Subtract.**

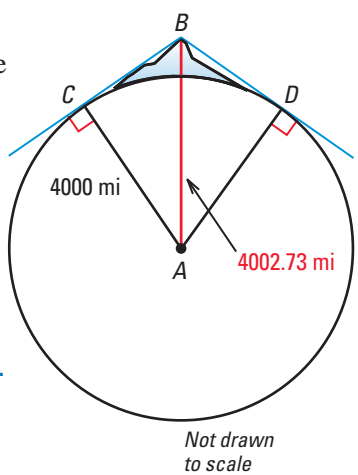
$= 88$ **Multiply.**



Mount Rainier, Washington

EXAMPLE 5 *Describing the View from Mount Rainier*

REAL LIFE **VIEWS** You are on top of Mount Rainier on a clear day. You are about 2.73 miles above sea level. Find the measure of the arc $\widehat{CD}$ that represents the part of Earth that you can see.



SOLUTION

$\overrightarrow{BC}$ and $\overrightarrow{BD}$ are tangent to Earth. You can solve right $\triangle BCA$ to see that $m\angle CBA \approx 87.9°$. So, $m\angle CBD \approx 175.8°$. Let $m\widehat{CD} = x°$.

$175.8 \approx \frac{1}{2}[(360 - x) - x]$ **Apply Theorem 10.14.**

$175.8 \approx \frac{1}{2}(360 - 2x)$ **Simplify.**

$175.8 \approx 180 - x$ **Distributive property**

$x \approx 4.2$ **Solve for *x*.**

▶ From the peak, you can see an arc of about 4°.

STUDENT HELP

Look Back
For help with solving a right triangle, see pp. 567–569.

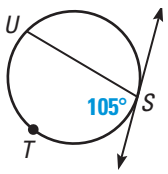
GUIDED PRACTICE

Concept Check ✓

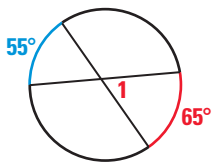
1. If a chord of a circle intersects a tangent to the circle at the point of tangency, what is the relationship between the angles formed and the intercepted arcs?

Skill Check ✓ **Find the indicated measure or value.**

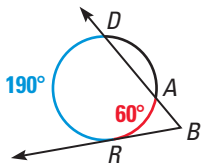
2. $m\widehat{STU}$



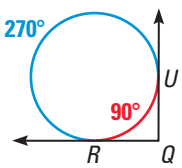
3. $m\angle 1$



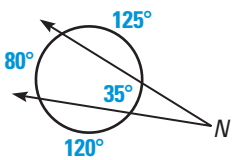
4. $m\angle DBR$



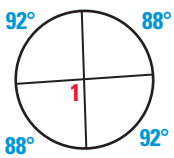
5. $m\angle RQU$



6. $m\angle N$



7. $m\angle 1$



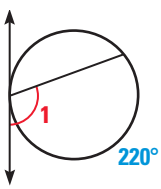
PRACTICE AND APPLICATIONS

STUDENT HELP

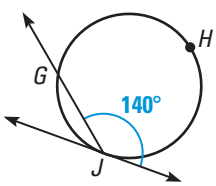
Extra Practice to help you master skills is on p. 822.

FINDING MEASURES **Find the indicated measure.**

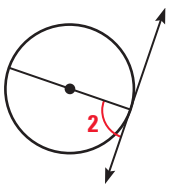
8. $m\angle 1$



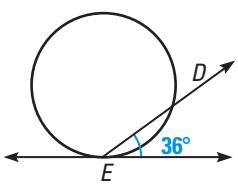
9. $m\widehat{GHJ}$



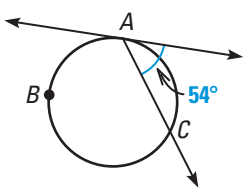
10. $m\angle 2$



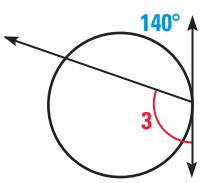
11. $m\widehat{DE}$



12. $m\widehat{ABC}$

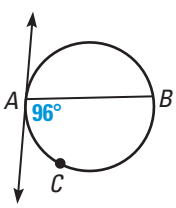


13. $m\angle 3$

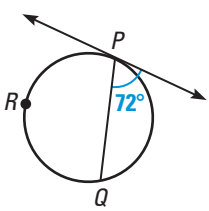


USING ALGEBRA **Find the value of *x*.**

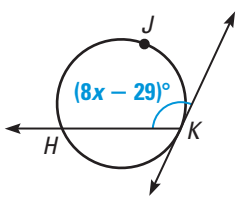
14. $m\widehat{AB} = x°$



15. $m\widehat{PQ} = (5x + 17)°$



16. $m\widehat{HJK} = (10x + 50)°$

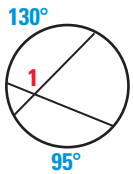


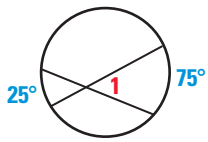
STUDENT HELP

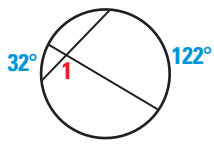
HOMEWORK HELP

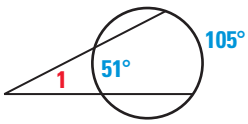
Example 1: Exs. 8–13
Example 2: Exs. 14–16
Example 3: Exs. 17–25
Example 4: Exs. 26–28
Example 5: Ex. 35

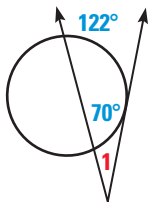
FINDING ANGLE MEASURES Find $m\angle 1$.

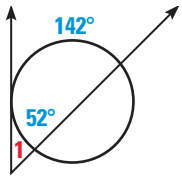
17. 

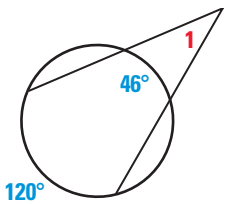
18. 

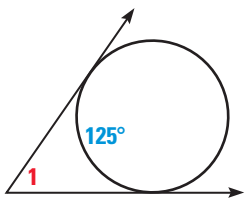
19. 

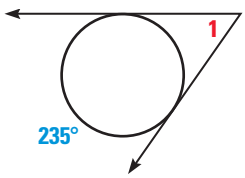
20. 

21. 

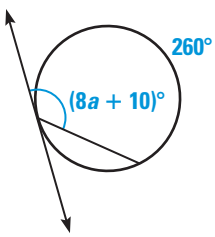
22. 

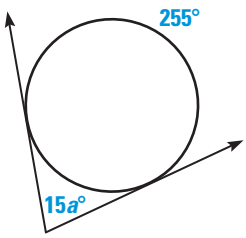
23. 

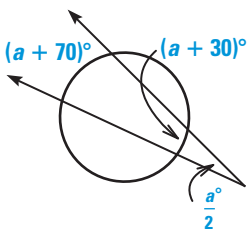
24. 

25. 

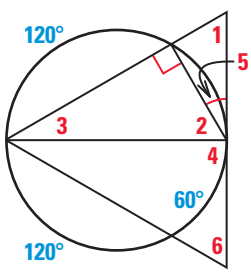
USING ALGEBRA Find the value of a.

26. 

27. 

28. 

FINDING ANGLE MEASURES Use the diagram at the right to find the measure of the angle.



29. $m\angle 1$

30. $m\angle 2$

31. $m\angle 3$

32. $m\angle 4$

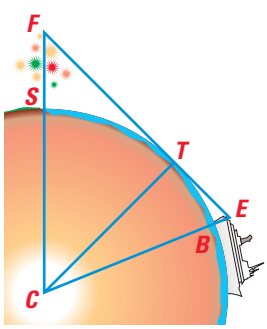
33. $m\angle 5$

34. $m\angle 6$

STUDENT HELP

INTERNET APPLICATION LINK Visit our Web site www.mcdougallittell.com.

35. **FIREWORKS** You are watching fireworks over San Diego Bay S as you sail away in a boat. The highest point the fireworks reach F is about 0.2 mile above the bay and your eyes E are about 0.01 mile above the water. At point B you can no longer see the fireworks because of the curvature of Earth. The radius of Earth is about 4000 miles and $\overline{FE}$ is tangent to Earth at T. Find $m\widehat{SB}$. Give your answer to the nearest tenth of a degree.



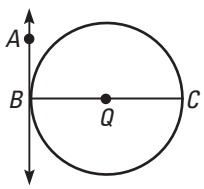
Not drawn to scale

STUDENT HELP

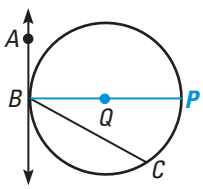
INTERNET **SOFTWARE HELP** Visit our Web site www.mcdougallittell.com to see instructions for several software applications.

36. **TECHNOLOGY** Use geometry software to construct and label circle O, $\overline{AB}$ which is tangent to $\odot O$ at point A, and any point C on $\odot O$. Then construct secant $\overline{AC}$. Measure $\angle BAC$ and $\widehat{AC}$. Compare the measures of $\angle BAC$ and its intercepted arc as you drag point C on the circle. What do you notice? What theorem from this lesson have you illustrated?

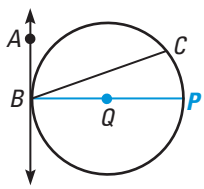
PROVING THEOREM 10.12 **The proof of Theorem 10.12 can be split into three cases, as shown in the diagrams.**



Case 1
The center of the circle is on one side of $\angle ABC$.



Case 2
The center of the circle is in the interior of $\angle ABC$.



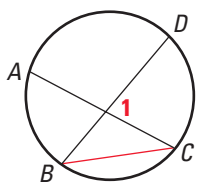
Case 3
The center of the circle is in the exterior of $\angle ABC$.

37. In Case 1, what type of chord is $\overline{BC}$? What is the measure of $\angle ABC$? What theorem earlier in this chapter supports your conclusion?

38. Write a plan for a proof of Case 2 of Theorem 10.12. (*Hint:* Use the auxiliary line and the Angle Addition Postulate.)

39. Describe how the proof of Case 3 of Theorem 10.12 is different from the proof of Case 2.

40. **PROVING THEOREM 10.13** Fill in the blanks to complete the proof of Theorem 10.13.



GIVEN ▶ Chords $\overline{AC}$ and $\overline{BD}$ intersect.

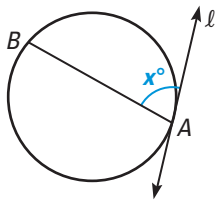
PROVE ▶ $m\angle 1 = \frac{1}{2}(m\widehat{DC} + m\widehat{AB})$

Statements	Reasons
1. Chords $\overline{AC}$ and $\overline{BD}$ intersect.	**1.** ___?___
2. Draw $\overline{BC}$.	**2.** ___?___
3. $m\angle 1 = m\angle DBC + m\angle$ ___?___	**3.** ___?___
4. $m\angle DBC = \frac{1}{2}m\widehat{DC}$	**4.** ___?___
5. $m\angle ACB = \frac{1}{2}m\widehat{AB}$	**5.** ___?___
6. $m\angle 1 = \frac{1}{2}m\widehat{DC} + \frac{1}{2}m\widehat{AB}$	**6.** ___?___
7. $m\angle 1 = \frac{1}{2}(m\widehat{DC} + m\widehat{AB})$	**7.** ___?___

41. **JUSTIFYING THEOREM 10.14** Look back at the diagrams for Theorem 10.14 on page 622. Copy the diagram for the case of a tangent and a secant and draw $\overline{BC}$. Explain how to use the Exterior Angle Theorem in the proof of this case. Then copy the diagrams for the other two cases, draw appropriate auxiliary segments, and write plans for the proofs of the cases.

Test Preparation

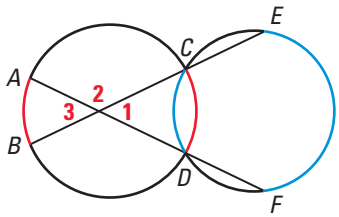
42. MULTIPLE CHOICE The diagram at the right is not drawn to scale. $\overline{AB}$ is any chord of the circle. The line is tangent to the circle at point A. Which of the following must be true?



Ⓐ $x < 90$ Ⓑ $x \leq 90$ Ⓒ $x = 90$

Ⓓ $x > 90$ Ⓔ Cannot be determined from given information

43. MULTIPLE CHOICE In the figure at the right, which relationship is *not* true?



Ⓐ $m\angle 1 = \frac{1}{2}(m\widehat{CD} + m\widehat{AB})$

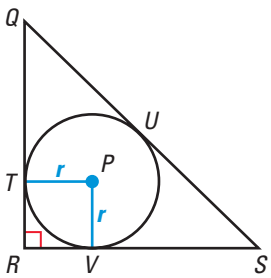
Ⓑ $m\angle 1 = \frac{1}{2}(m\widehat{EF} - m\widehat{CD})$

Ⓒ $m\angle 2 = \frac{1}{2}(m\widehat{BD} - m\widehat{AC})$

Ⓓ $m\angle 3 = \frac{1}{2}(m\widehat{EF} - m\widehat{CD})$

★ Challenge

44. PROOF Use the plan to write a paragraph proof.



GIVEN ▶ $\angle R$ is a right angle. Circle P is inscribed in $\triangle QRS$. T, U, and V are points of tangency.

PROVE ▶ $r = \frac{1}{2}(QR + RS - QS)$

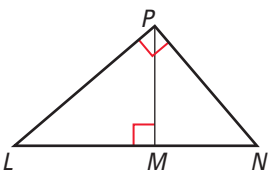
Plan for Proof Prove that $TPVR$ is a square. Then show that $\overline{QT} \cong \overline{QU}$ and $\overline{SU} \cong \overline{SV}$. Finally, use the Segment Addition Postulate and substitution.

EXTRA CHALLENGE
www.mcdougallittell.com

45. FINDING A RADIUS Use the result from Exercise 44 to find the radius of an inscribed circle of a right triangle with side lengths of 3, 4, and 5.

MIXED REVIEW

USING SIMILAR TRIANGLES Use the diagram at the right and the given information. (Review 9.1)



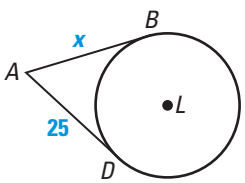
46. $MN = 9$, $PM = 12$, $LP =$?

47. $LM = 4$, $LN = 9$, $LP =$?

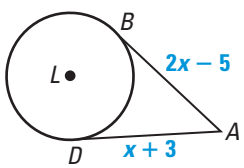
48. FINDING A RADIUS You are 10 feet from a circular storage tank. You are 22 feet from a point of tangency on the tank. Find the tank's radius. (Review 10.1)

USING ALGEBRA $\overline{AB}$ **and** $\overline{AD}$ **are tangent to** $\odot L$**. Find the value of** x**.** (Review 10.1)

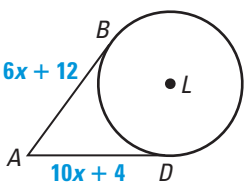
49.



50.



51.



10.5 Segment Lengths in Circles

What you should learn

GOAL 1 Find the lengths of segments of chords.

GOAL 2 Find the lengths of segments of tangents and secants.

Why you should learn it

▼ To find **real-life** measures, such as the radius of an aquarium tank in **Example 3**.

REAL LIFE

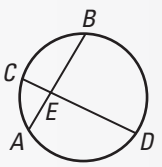
GOAL 1 FINDING LENGTHS OF SEGMENTS OF CHORDS

When two chords intersect in the interior of a circle, each chord is divided into two segments which are called *segments of a chord*. The following theorem gives a relationship between the lengths of the four segments that are formed.

THEOREM

THEOREM 10.15

If two chords intersect in the interior of a circle, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord.

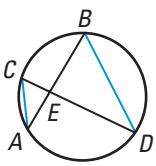


$EA \cdot EB = EC \cdot ED$

You can use similar triangles to prove Theorem 10.15.

GIVEN ▶ $\overline{AB}$, $\overline{CD}$ are chords that intersect at E.

PROVE ▶ $EA \cdot EB = EC \cdot ED$

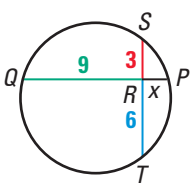


Paragraph Proof Draw $\overline{DB}$ and $\overline{AC}$. Because $\angle C$ and $\angle B$ intercept the same arc, $\angle C \cong \angle B$. Likewise, $\angle A \cong \angle D$. By the AA Similarity Postulate, $\triangle AEC \sim \triangle DEB$. So, the lengths of corresponding sides are proportional.

$\frac{EA}{ED} = \frac{EC}{EB}$	**The lengths of the sides are proportional.**
$EA \cdot EB = EC \cdot ED$	**Cross Product Property**

EXAMPLE 1 *Finding Segment Lengths*

Chords $\overline{ST}$ and $\overline{PQ}$ intersect inside the circle. Find the value of x.

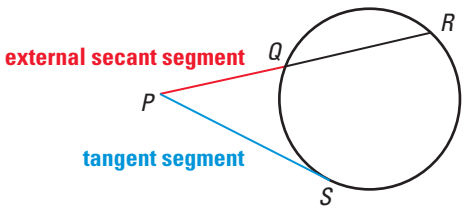


SOLUTION

$RQ \cdot RP = RS \cdot RT$	**Use Theorem 10.15.**
$9 \cdot x = 3 \cdot 6$	**Substitute.**
$9x = 18$	**Simplify.**
$x = 2$	**Divide each side by 9.**

GOAL 2 USING SEGMENTS OF TANGENTS AND SECANTS

In the figure shown below, $\overline{PS}$ is called a **tangent segment** because it is tangent to the circle at an endpoint. Similarly, $\overline{PR}$ is a **secant segment** and $\overline{PQ}$ is the **external segment** of $\overline{PR}$.

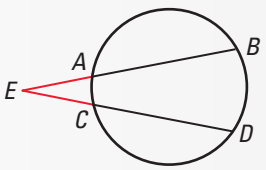


You are asked to prove the following theorems in Exercises 31 and 32.

THEOREMS

THEOREM 10.16

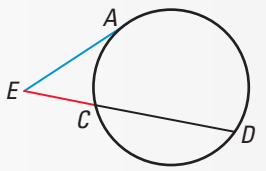
If two secant segments share the same endpoint outside a circle, then the product of the length of one secant segment and the length of its external segment equals the product of the length of the other secant segment and the length of its external segment.



$EA \cdot EB = EC \cdot ED$

THEOREM 10.17

If a secant segment and a tangent segment share an endpoint outside a circle, then the product of the length of the secant segment and the length of its external segment equals the square of the length of the tangent segment.

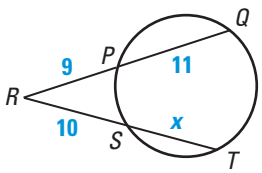


$(EA)^2 = EC \cdot ED$

EXAMPLE 2 *Finding Segment Lengths*

Using Algebra

Find the value of x.



SOLUTION

$RP \cdot RQ = RS \cdot RT$	Use Theorem 10.16.
$9 \cdot (11 + 9) = 10 \cdot (x + 10)$	Substitute.
$180 = 10x + 100$	Simplify.
$80 = 10x$	Subtract 100 from each side.
$8 = x$	Divide each side by 10.

FOCUS ON APPLICATIONS

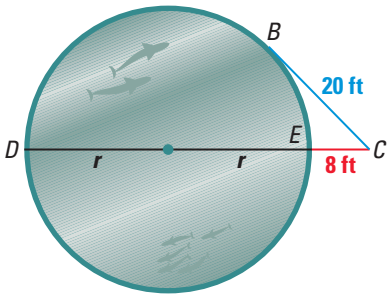
AQUARIUM TANK The Caribbean Coral Reef Tank at the New England Aquarium is a circular tank 24 feet deep. The 200,000 gallon tank contains an elaborate coral reef and many exotic fishes.

APPLICATION LINK
www.mcdougallittell.com

In Lesson 10.1, you learned how to use the Pythagorean Theorem to estimate the radius of a grain silo. Example 3 shows you another way to estimate the radius of a circular object.

EXAMPLE 3 *Estimating the Radius of a Circle*

AQUARIUM TANK You are standing at point C, about 8 feet from a circular aquarium tank. The distance from you to a point of tangency on the tank is about 20 feet. Estimate the radius of the tank.



SOLUTION

You can use Theorem 10.17 to find the radius.

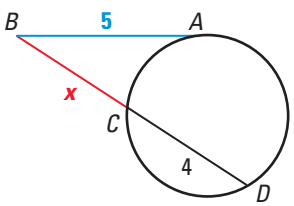
$(CB)^2 = CE \cdot CD$	**Use Theorem 10.17.**
$20^2 \approx 8 \cdot (2r + 8)$	**Substitute.**
$400 \approx 16r + 64$	**Simplify.**
$336 \approx 16r$	**Subtract 64 from each side.**
$21 \approx r$	**Divide each side by 16.**

▶ So, the radius of the tank is about 21 feet.

Using Algebra

EXAMPLE 4 *Finding Segment Lengths*

Use the figure at the right to find the value of x.



SOLUTION

$(BA)^2 = BC \cdot BD$	**Use Theorem 10.17.**
$5^2 = x \cdot (x + 4)$	**Substitute.**
$25 = x^2 + 4x$	**Simplify.**
$0 = x^2 + 4x - 25$	**Write in standard form.**
$x = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(-25)}}{2}$	**Use Quadratic Formula.**
$x = -2 \pm \sqrt{29}$	**Simplify.**

Use the positive solution, because lengths cannot be negative.

▶ So, $x = -2 + \sqrt{29} \approx 3.39$.

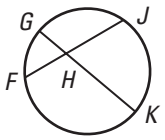
GUIDED PRACTICE

Vocabulary Check ✓

1. Sketch a circle with a secant segment. Label each endpoint and point of intersection. Then name the external segment.

Concept Check ✓

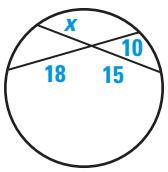
2. How are the lengths of the segments in the figure at the right related to each other?



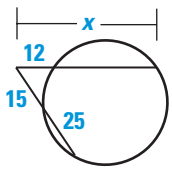
Skill Check ✓

Fill in the blanks. Then find the value of *x*.

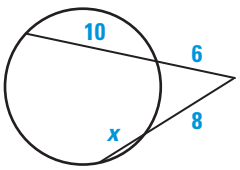
3. $x \cdot \underline{?} = 10 \cdot \underline{?}$



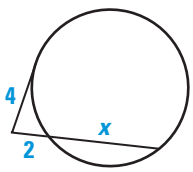
4. $\underline{?} \cdot x = \underline{?} \cdot 40$



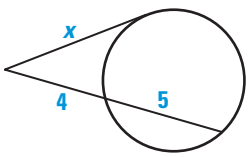
5. $6 \cdot \underline{?} = 8 \cdot \underline{?}$



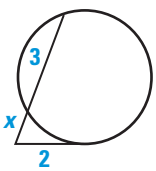
6. $4^2 = 2 \cdot (\underline{?} + x)$



7. $x^2 = 4 \cdot \underline{?}$



8. $x \cdot \underline{?} = \underline{?}$



9. **ZOO HABITAT** A zoo has a large circular aviary, a habitat for birds. You are standing about 40 feet from the aviary. The distance from you to a point of tangency on the aviary is about 60 feet. Describe how to estimate the radius of the aviary.

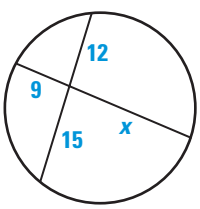
PRACTICE AND APPLICATIONS

STUDENT HELP

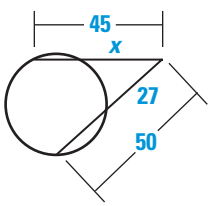
Extra Practice to help you master skills is on p. 822.

FINDING SEGMENT LENGTHS **Fill in the blanks. Then find the value of *x*.**

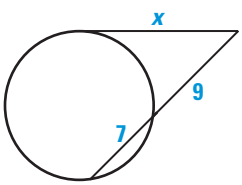
10. $x \cdot \underline{?} = 12 \cdot \underline{?}$



11. $x \cdot \underline{?} = \underline{?} \cdot 50$



12. $x^2 = 9 \cdot \underline{?}$



STUDENT HELP

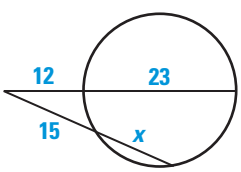
HOMEWORK HELP

Example 1: Exs. 10, 14–17, 26–29

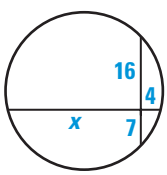
Example 2: Exs. 11, 13, 18, 19, 24, 25

FINDING SEGMENT LENGTHS **Find the value of *x*.**

13.



14.



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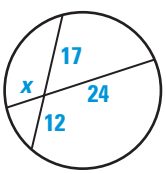


STUDENT HELP

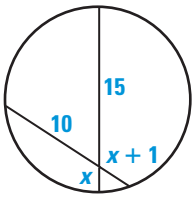
HOMEWORK HELP

Example 3: Exs. 12, 20–23, 25–27

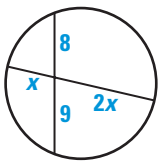
Example 4: Exs. 12, 20–23, 25–27

FINDING SEGMENT LENGTHS **Find the value of *x*.**

16.



17.



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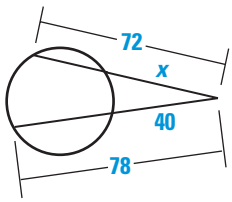

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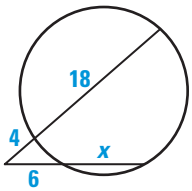

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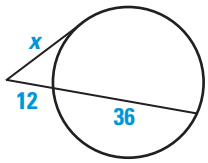

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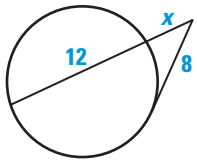


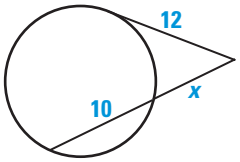
STUDENT HELP

HOMEWORK HELP

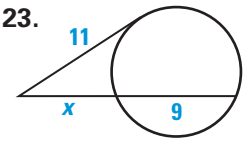
INTERNET

Visit our Web site www.mcdougallittell.com for help with using the Quadratic Formula in Exs. 21–27.

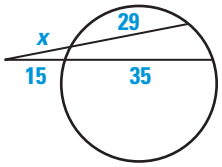
22.



23.

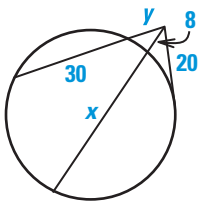


24.



USING ALGEBRA **Find the values of *x* and *y*.**

25.



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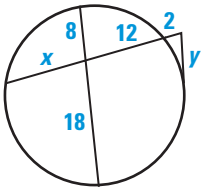
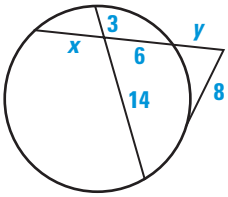
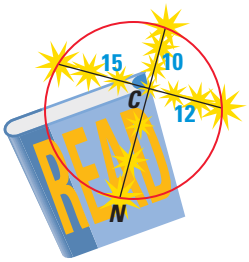


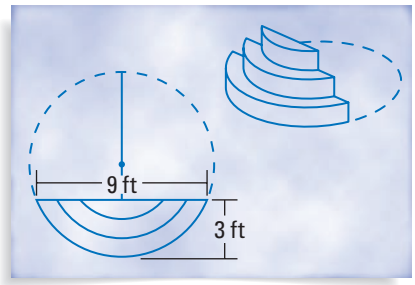
27.



28. **DESIGNING A LOGO** Suppose you are designing an animated logo for a television commercial. You want sparkles to leave point *C* and move to the circle along the segments shown. You want each of the sparkles to reach the circle at the same time. To calculate the speed for each sparkle, you need to know the distance from point *C* to the circle along each segment. What is the distance from *C* to *N*?



29. **BUILDING STAIRS** You are making curved stairs for students to stand on for photographs at a homecoming dance. The diagram shows a top view of the stairs. What is the radius of the circle shown? Explain how you can use Theorem 10.15 to find the answer.

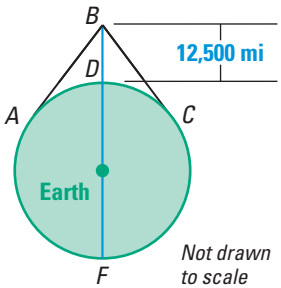


FOCUS ON APPLICATIONS

REAL LIFE **GPS** Some cars have navigation systems that use GPS to tell you where you are and how to get where you want to go.

INTERNET **APPLICATION LINK**
www.mcdougallittell.com

30. **GLOBAL POSITIONING SYSTEM** Satellites in the Global Positioning System (GPS) orbit 12,500 miles above Earth. GPS signals can't travel through Earth, so a satellite at point B can transmit signals only to points on $\widehat{AC}$. How far must the satellite be able to transmit to reach points A and C? Find BA and BC. The diameter of Earth is about 8000 miles. Give your answer to the nearest thousand miles.

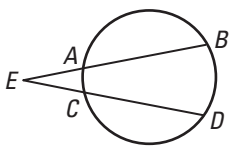


31. **PROVING THEOREM 10.16** Use the plan to write a paragraph proof.

GIVEN ▶ $\overline{EB}$ and $\overline{ED}$ are secant segments.

PROVE ▶ $EA \cdot EB = EC \cdot ED$

Plan for Proof Draw $\overline{AD}$ and $\overline{BC}$, and show that $\triangle BCE$ and $\triangle DAE$ are similar. Use the fact that corresponding sides of similar triangles are proportional.

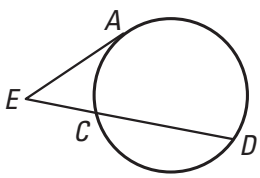


32. **PROVING THEOREM 10.17** Use the plan to write a paragraph proof.

GIVEN ▶ $\overline{EA}$ is a tangent segment and $\overline{ED}$ is a secant segment.

PROVE ▶ $(EA)^2 = EC \cdot ED$

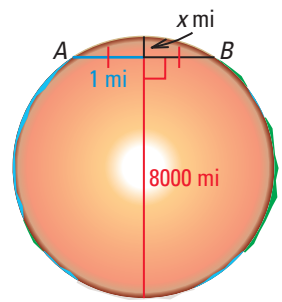
Plan for Proof Draw $\overline{AD}$ and $\overline{AC}$. Use the fact that corresponding sides of similar triangles are proportional.



Test Preparation

MULTI-STEP PROBLEM **In Exercises 33–35, use the following information.**

A person is standing at point A on a beach and looking 2 miles down the beach to point B, as shown at the right. The beach is very flat but, because of Earth's curvature, the ground between A and B is x mi higher than $\overline{AB}$.



Not drawn to scale

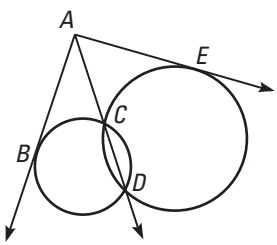
33. Find the value of x.

34. Convert your answer to inches. Round to the nearest inch.

35. *Writing* Why do you think people historically thought that Earth was flat?

★ Challenge

In the diagram at the right, $\overrightarrow{AB}$ and $\overrightarrow{AE}$ are tangents.



36. Write an equation that shows how AB is related to AC and AD.

37. Write an equation that shows how AE is related to AC and AD.

38. How is AB related to AE? Explain.

EXTRA CHALLENGE
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39. Make a conjecture about tangents to intersecting circles. Then test your conjecture by looking for a counterexample.

MIXED REVIEW

FINDING DISTANCE AND MIDPOINT **Find *AB* to the nearest hundredth. Then find the coordinates of the midpoint of $\overline{AB}$.** (Review 1.3, 1.5 for 10.6)

40. $A(2, 5), B(-3, 3)$

41. $A(6, -4), B(0, 4)$

42. $A(-8, -6), B(1, 9)$

43. $A(-1, -5), B(-10, 7)$

44. $A(0, -11), B(8, 2)$

45. $A(5, -2), B(-9, -2)$

WRITING EQUATIONS **Write an equation of a line perpendicular to the given line at the given point.** (Review 3.7 for 10.6)

46. $y = -2x - 5, (-2, -1)$

47. $y = \frac{2}{3}x + 4, (6, 8)$

48. $y = -x + 9, (0, 9)$

49. $y = 3x - 10, (2, -4)$

50. $y = \frac{1}{5}x + 1, (-10, -1)$

51. $y = -\frac{7}{3}x - 5, (-6, 9)$

DRAWING TRANSLATIONS **Quadrilateral *ABCD* has vertices *A*(−6, 8), *B*(−1, 4), *C*(−2, 2), and *D*(−7, 3). Draw its image after the translation.** (Review 7.4 for 10.6)

52. $(x, y) \rightarrow (x + 7, y)$

53. $(x, y) \rightarrow (x - 2, y + 3)$

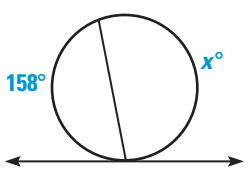
54. $(x, y) \rightarrow \left(x, y - \frac{11}{2}\right)$

QUIZ 2

Self-Test for Lessons 10.4 and 10.5

Find the value of *x*. (Lesson 10.4)

1.



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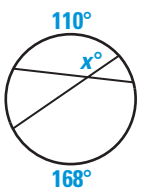

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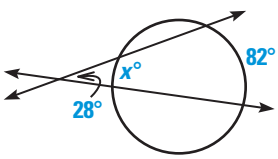
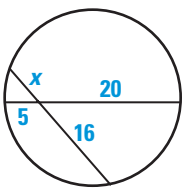
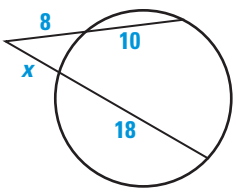


Find the value of *x*. (Lesson 10.5)

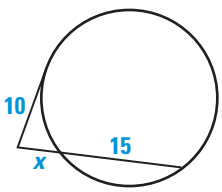
4.



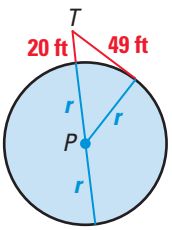
5.



6.



7. **SWIMMING POOL** You are standing 20 feet from the circular wall of an above ground swimming pool and 49 feet from a point of tangency. Describe two different methods you could use to find the radius of the pool. What is the radius? (Lesson 10.5)



10.6 Equations of Circles

What you should learn

GOAL 1 Write the equation of a circle.

GOAL 2 Use the equation of a circle and its graph to solve problems.

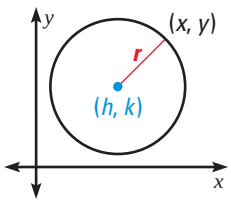
Why you should learn it

▼ To solve **real-life** problems, such as determining cellular phone coverage, as in **Exs. 41 and 42**.

REAL LIFE

GOAL 1 FINDING EQUATIONS OF CIRCLES

You can write an equation of a circle in a coordinate plane if you know its radius and the coordinates of its center. Suppose the radius of a circle is r and the center is (h, k). Let (x, y) be any point on the circle. The distance between (x, y) and (h, k) is r, so you can use the Distance Formula.



$$\sqrt{(x-h)^2+(y-k)^2} = r$$

Square both sides to find the **standard equation of a circle** with radius r and center (h, k).

Standard equation of a circle: $(x-h)^2+(y-k)^2 = r^2$

If the center is the origin, then the standard equation is $x^2 + y^2 = r^2$.

EXAMPLE 1 *Writing a Standard Equation of a Circle*

Write the standard equation of the circle with center $(-4, 0)$ and radius 7.1.

SOLUTION

$(x-h)^2+(y-k)^2 = r^2$	**Standard equation of a circle**
$[x-(-4)]^2+(y-0)^2 = 7.1^2$	**Substitute.**
$(x+4)^2+y^2 = 50.41$	**Simplify.**

EXAMPLE 2 *Writing a Standard Equation of a Circle*

The point $(1, 2)$ is on a circle whose center is $(5, -1)$. Write the standard equation of the circle.

SOLUTION

Find the radius. The radius is the distance from the point $(1, 2)$ to the center $(5, -1)$.

$r = \sqrt{(5-1)^2+(-1-2)^2}$	**Use the Distance Formula.**
$r = \sqrt{4^2+(-3)^2}$	**Simplify.**
$r = 5$	**Simplify.**

Substitute $(h, k) = (5, -1)$ and $r = 5$ into the standard equation of a circle.

$(x-5)^2+(y-(-1))^2 = 5^2$	**Standard equation of a circle**
$(x-5)^2+(y+1)^2 = 25$	**Simplify.**

GOAL 2 GRAPHING CIRCLES

If you know the equation of a circle, you can graph the circle by identifying its center and radius.

STUDENT HELP

Study Tip
You can sketch the graph of the circle in Example 3 without a compass by first plotting the four points shown in red. Then sketch a circle through the points.

EXAMPLE 3 *Graphing a Circle*

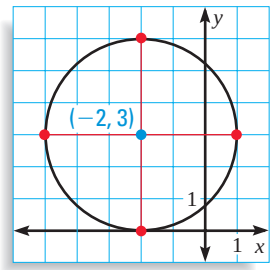
The equation of a circle is $(x + 2)^2 + (y - 3)^2 = 9$. Graph the circle.

Rewrite the equation to find the center and radius:

$$(x + 2)^2 + (y - 3)^2 = 9$$

$$[x - (-2)]^2 + (y - 3)^2 = 3^2$$

The center is $(-2, 3)$ and the radius is 3. To graph the circle, place the point of a compass at $(-2, 3)$, set the radius at 3 units, and swing the compass to draw a full circle.



EXAMPLE 4 *Applying Graphs of Circles*

THEATER LIGHTING A bank of lights is arranged over a stage. Each light illuminates a circular area on the stage. A coordinate plane is used to arrange the lights, using the corner of the stage as the origin. The equation $(x - 13)^2 + (y - 4)^2 = 16$ represents one of the disks of light.

a. Graph the disk of light.

b. Three actors are located as follows: Henry is at (11, 4), Jolene is at (8, 5), and Martin is at (15, 5). Which actors are in the disk of light?

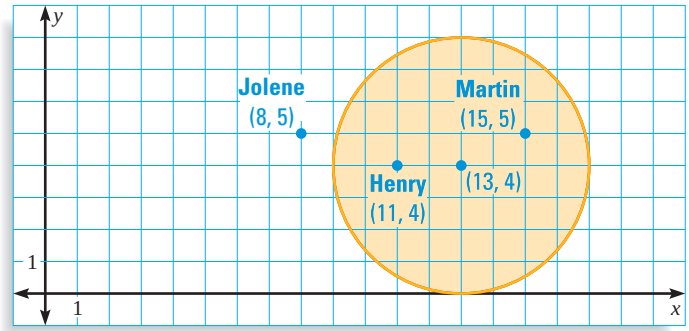
SOLUTION

a. Rewrite the equation to find the center and radius:

$$(x - 13)^2 + (y - 4)^2 = 16$$

$$(x - 13)^2 + (y - 4)^2 = 4^2$$

The center is **(13, 4)** and the radius is **4**. The circle is shown below.



b. The graph shows that Henry and Martin are both in the disk of light.

GUIDED PRACTICE

Vocabulary Check ✓

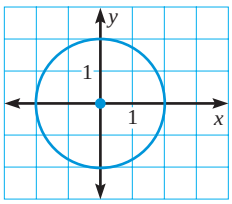
1. The standard form of an equation of a circle is ___?___.

Concept Check ✓

2. Describe how to graph the circle $(x - 3)^2 + (y - 4)^2 = 9$.

Skill Check ✓

Give the coordinates of the center and the radius. Write an equation of the circle in standard form.

3.


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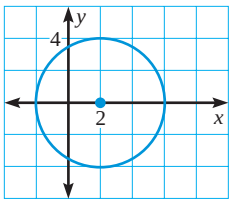

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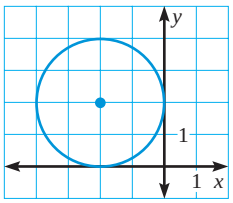


6. $P(-1, 3)$ is on a circle whose center is $C(0, 0)$. Write an equation of $\odot C$.

PRACTICE AND APPLICATIONS

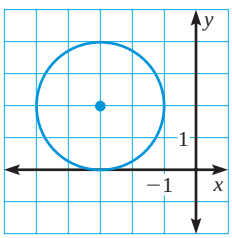
STUDENT HELP

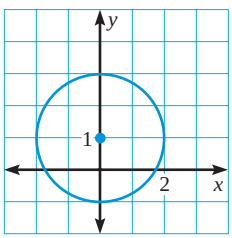
Extra Practice to help you master skills is on p. 822.

USING STANDARD EQUATIONS **Give the center and radius of the circle.**

7. $(x - 4)^2 + (y - 3)^2 = 16$
8. $(x - 5)^2 + (y - 1)^2 = 25$
9. $x^2 + y^2 = 4$
10. $(x + 2)^2 + (y - 3)^2 = 36$
11. $(x + 5)^2 + (y + 3)^2 = 1$
12. $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = \frac{1}{4}$

USING GRAPHS **Give the coordinates of the center, the radius, and the equation of the circle.**

13.


14.


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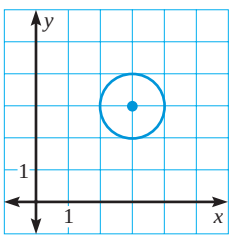
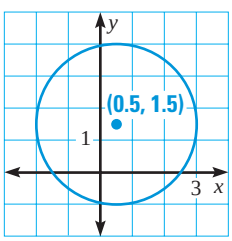
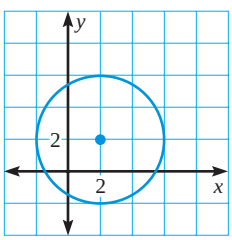
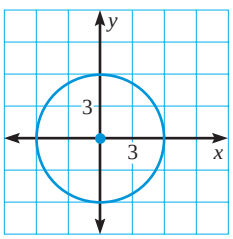


16.


17.


18.


STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 13–22
Example 2: Exs. 23–26
Example 3: Exs. 27–40
Example 4: Exs. 33–42

WRITING EQUATIONS **Write the standard equation of the circle with the given center and radius.**

19. center $(0, 0)$, radius 1
20. center $(4, 0)$, radius 4
21. center $(3, -2)$, radius 2
22. center $(-1, -3)$, radius 6

WRITING EQUATIONS **Use the given information to write the standard equation of the circle.**

23. The center is (0, 0), a point on the circle is (0, 3).

24. The center is (1, 2), a point on the circle is (4, 6).

25. The center is (3, 2), a point on the circle is (5, 2).

26. The center is (−5, 3) and the diameter is 8.

GRAPHING CIRCLES **Graph the equation.**

27. $x^2 + y^2 = 25$

28. $x^2 + (y - 4)^2 = 1$

29. $(x + 3)^2 + y^2 = 9$

30. $(x - 3)^2 + (y - 4)^2 = 16$

31. $(x + 5)^2 + (y - 1)^2 = 49$

32. $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}$

USING GRAPHS **The equation of a circle is $(x - 2)^2 + (y + 3)^2 = 4$. Tell whether each point is *on* the circle, in the *interior* of the circle, or in the *exterior* of the circle.**

33. (0, 0)

34. (2, −4)

35. (0, −3)

36. (3, −1)

37. (1, −4)

38. (2, −5)

39. (2, 0)

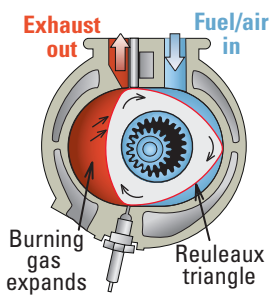
40. (2.5, −3)

CELL PHONES **In Exercises 41 and 42, use the following information.** A cellular phone network uses towers to transmit calls. Each tower transmits to a circular area. On a grid of a city, the coordinates of the location and the radius each tower covers are as follows (integers represent miles): Tower *A* is at (0, 0) and covers a 3 mile radius, Tower *B* is at (5, 3) and covers a 2.5 mile radius, and Tower *C* is at (2, 5) and covers a 2 mile radius.

41. Write the equations that represent the transmission boundaries of the towers. Graph each equation.

42. Tell which towers, if any, transmit to a phone located at $J(1, 1)$, $K(4, 2)$, $L(3.5, 4.5)$, $M(2, 2.8)$, or $N(1, 6)$.

FOCUS ON APPLICATIONS



REAL LIFE The Wankel engine is an engine with a triangular rotor that is based on a Reuleaux triangle. It has been used in sports cars, snowmobiles, and hybrid electric vehicles.

INTERNET APPLICATION LINK
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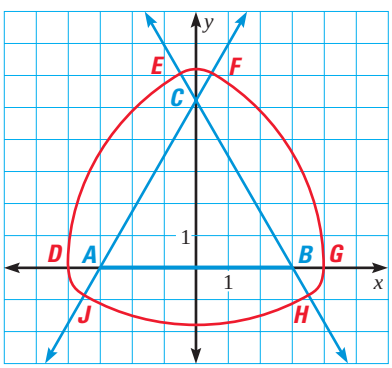
REULEAUX POLYGONS **The figure at the right is called a *Reuleaux polygon*. It is not a true polygon because its sides are not straight. $\triangle ABC$ is equilateral.**



43. $\widehat{JD}$ lies on a circle with center *A* and radius *AD*. Write an equation of this circle.

44. $\widehat{DE}$ lies on a circle with center *B* and radius *BD*. Write an equation of this circle.

45. **CONSTRUCTION** The remaining arcs of the polygon are constructed in the same way as $\widehat{JD}$ and $\widehat{DE}$ in Exercises 43 and 44. Construct a Reauleaux polygon on a piece of cardboard.

46. Cut out the Reauleaux polygon from Exercise 45. Roll it on its edge like a wheel and measure its height when it is in different orientations. Explain why a Reuleaux polygon is said to have constant width.

47. TRANSLATIONS Sketch the circle whose equation is $x^2 + y^2 = 16$. Then sketch the image of the circle after the translation $(x, y) \rightarrow (x - 2, y - 4)$. What is the equation of the image?

48. WRITING AN EQUATION A circle has a center (p, q) and is tangent to the x-axis. Write the standard equation of the circle.

Test Preparation

49. MULTIPLE CHOICE What is the standard form of the equation of a circle with center $(-3, 1)$ and radius 2?

Ⓐ $(x - 3)^2 + (y - 1)^2 = 2$ Ⓑ $(x + 3)^2 + (y - 1)^2 = 2$

Ⓒ $(x - 3)^2 + (y - 1)^2 = 4$ Ⓓ $(x + 3)^2 + (y - 1)^2 = 4$

50. MULTIPLE CHOICE The center of a circle is $(-3, 0)$ and its radius is 5. Which point does *not* lie on the circle?

Ⓐ $(2, 0)$ Ⓑ $(0, 4)$ Ⓒ $(-3, 0)$ Ⓓ $(-3, -5)$ Ⓔ $(-8, 0)$

★ Challenge

51. CRITICAL THINKING $\odot A$ and $\odot B$ are externally tangent. Suppose you know the equation of $\odot A$, the coordinates of the single point of intersection of $\odot A$ and $\odot B$, and the radius of $\odot B$. Do you know enough to find the equation of $\odot B$? Explain.

USING ALGEBRA Find the missing coordinate of the center of the circle with the given characteristics.

EXTRA CHALLENGE
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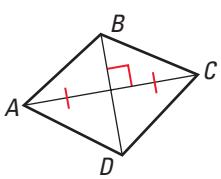
52. The center is $(1, b)$, the radius is 3, and a point on the circle is $(-2, 0)$.

53. The center is $(-3, b)$, the radius is 5, and a point on the circle is $(2, -2)$.

MIXED REVIEW

IDENTIFYING QUADRILATERALS What kind(s) of quadrilateral could *ABCD* be? *ABCD* is not drawn to scale. (Review 6.6)

54.



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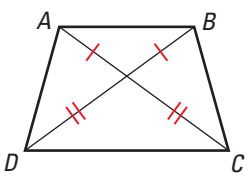

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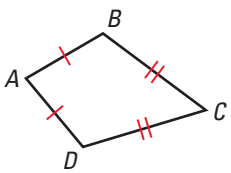


VECTORS Write the component form of vector $\overrightarrow{PQ}$. Use the component form to find the magnitude of $\overrightarrow{PQ}$ to the nearest tenth. (Review 9.7)

57. $P = (0, 0)$, $Q = (-6, 7)$

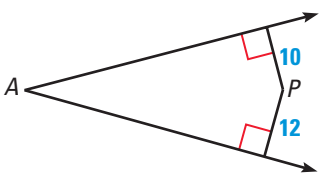
58. $P = (3, -4)$, $Q = (-11, 2)$

59. $P = (-6, -6)$, $Q = (9, -5)$

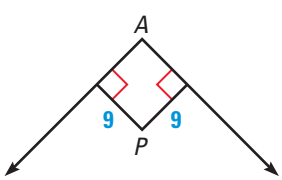
60. $P = (5, 6)$, $Q = (-3, 7)$

ANGLE BISECTORS Does *P* lie on the bisector of $\angle A$? Explain your reasoning. (Review 5.1)

61.



62.



10.7 Locus

What you should learn

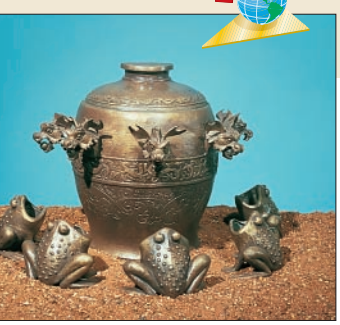
GOAL 1 Draw the locus of points that satisfy a given condition.

GOAL 2 Draw the locus of points that satisfy two or more conditions.

Why you should learn it

▼ To use **real-life** constraints, such as using seismograph readings to find an epicenter in **Example 4** and **Ex. 29**.

REAL LIFE



In ancient China, a seismometer like this one could measure the direction to an epicenter.

GOAL 1 DRAWING A LOCUS SATISFYING ONE CONDITION

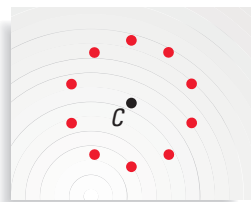
A **locus** in a plane is the set of all points in a plane that satisfy a given condition or a set of given conditions. The word *locus* is derived from the Latin word for "location." The plural of locus is *loci*, pronounced "low-sigh."

A locus is often described as the path of an object moving in a plane. For instance, the reason that many clock faces are circular is that the locus of the end of a clock's minute hand is a circle.

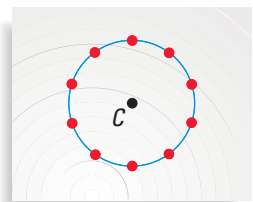
EXAMPLE 1 *Finding a Locus*

Draw point C on a piece of paper. Draw and describe the locus of all points on the paper that are 3 inches from C.

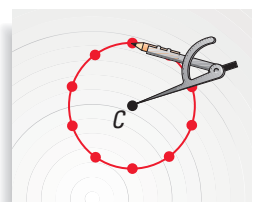
SOLUTION



1 Draw point C. Locate several points 3 inches from C.



2 Recognize a pattern: the points lie on a circle.



3 Draw the circle.

▶ The locus of points on the paper that are 3 inches from C is a circle with center C and a radius of 3 inches.

CONCEPT SUMMARY — FINDING A LOCUS

To find the locus of points that satisfy a given condition, use the following steps.

1. Draw any figures that are given in the statement of the problem.
2. Locate several points that satisfy the given condition.
3. Continue drawing points until you can recognize the pattern.
4. Draw the locus and describe it in words.

GOAL 2 LOCI SATISFYING TWO OR MORE CONDITIONS

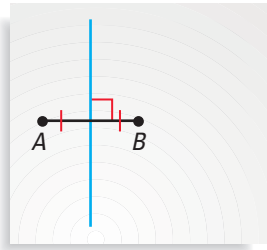
To find the locus of points that satisfy two or more conditions, first find the locus of points that satisfy each condition alone. Then find the intersection of these loci.

EXAMPLE 2 *Drawing a Locus Satisfying Two Conditions*

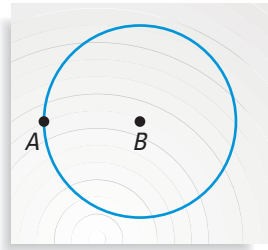
Logical Reasoning

Points A and B lie in a plane. What is the locus of points in the plane that are equidistant from points A and B and are a distance of AB from B?

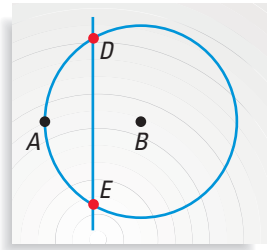
SOLUTION



The locus of all points that are equidistant from A and B is the perpendicular bisector of $\overline{AB}$.



The locus of all points that are a distance of AB from B is the circle with center B and radius AB.



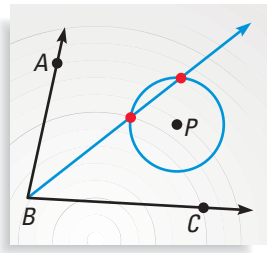
These loci intersect at D and E. So D and E are the locus of points that satisfy both conditions.

EXAMPLE 3 *Drawing a Locus Satisfying Two Conditions*

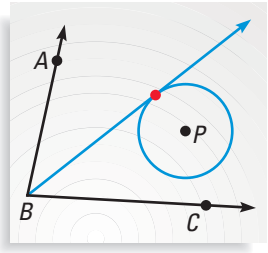
Point P is in the interior of $\angle ABC$. What is the locus of points in the interior of $\angle ABC$ that are equidistant from both sides of $\angle ABC$ and 2 inches from P? How does the location of P within $\angle ABC$ affect the locus?

SOLUTION

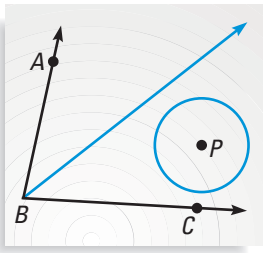
The locus of points equidistant from both sides of $\angle ABC$ is the angle bisector. The locus of points 2 inches from P is a circle. The intersection of the angle bisector and the circle depends on the location of P. The locus can be 2 points, 1 point, or 0 points.



The locus is 2 points.

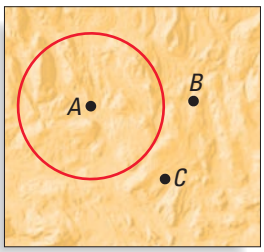


The locus is 1 point.

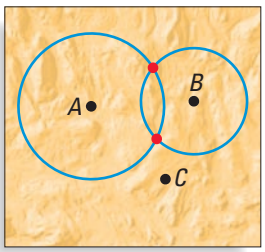


The locus is 0 points.

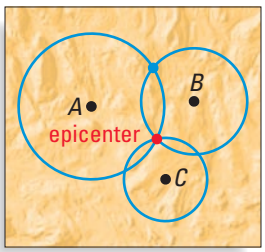
EARTHQUAKES The *epicenter* of an earthquake is the point on Earth's surface that is directly above the earthquake's origin. A seismograph can measure the distance to the epicenter, but not the direction to the epicenter. To locate the epicenter, readings from three seismographs in different locations are needed.



The reading from seismograph A tells you that the epicenter is somewhere on a circle centered at A.



The reading from B tells you that the epicenter is one of the two points of intersection of $\odot A$ and $\odot B$.



The reading from C tells you which of the two points of intersection is the epicenter.

EXAMPLE 4 *Finding a Locus Satisfying Three Conditions*

LOCATING AN EPICENTER You are given readings from three seismographs.

- At $A(-5, 5)$, the epicenter is 4 miles away.
- At $B(-4, -3.5)$, the epicenter is 5 miles away.
- At $C(1, 1.5)$, the epicenter is 7 miles away.

Where is the epicenter?

FOCUS ON CAREERS

GEOSCIENTISTS do a variety of things, including locating earthquakes, searching for oil, studying fossils, and mapping the ocean floor.

CAREER LINK
www.mcdougallittell.com

SOLUTION

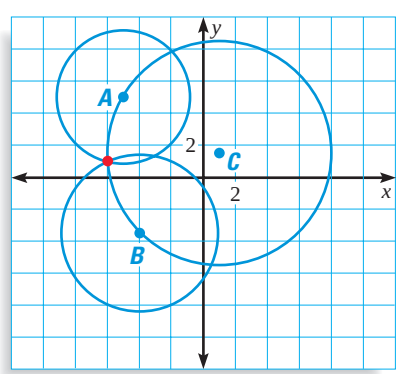
Each seismograph gives you a locus that is a circle.

Circle A has center $(-5, 5)$ and radius 4.

Circle B has center $(-4, -3.5)$ and radius 5.

Circle C has center $(1, 1.5)$ and radius 7.

Draw the three circles in a coordinate plane. The point of intersection of the three circles is the epicenter.



▶ The epicenter is at about $(-6, 1)$.

GUIDED PRACTICE

Vocabulary Check ✓

1. The radius of $\odot C$ is 3 inches. The locus of points in the plane that are more than 3 inches from C is the __?__ of $\odot C$.

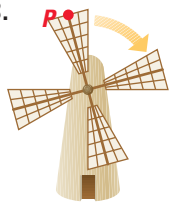
Concept Check ✓

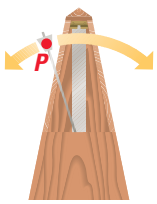
2. Draw two points A and B on a piece of paper. Draw and describe the locus of points on the paper that are equidistant from A and B.

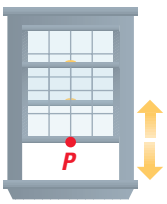
Skill Check ✓

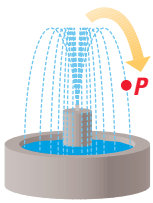
Match the object with the locus of point *P*.

A. Arc **B.** Circle **C.** Parabola **D.** Line segment

3. 

4. 

5. 

6. 

7. What is the locus of points in the coordinate plane that are equidistant from $A(0, 0)$ and $B(6, 0)$ and 5 units from A? Make a sketch.

8. Points C and D are in a plane. What is the locus of points in the plane that are 3 units from C and 5 units from D?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 822.

LOGICAL REASONING Draw the figure. Then sketch and describe the locus of points on the paper that satisfy the given condition.

9. Point P, the locus of points that are 1 inch from P

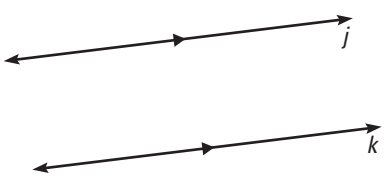
10. Line k, the locus of points that are 1 inch from k

11. Point C, the locus of points that are no more than 1 inch from C

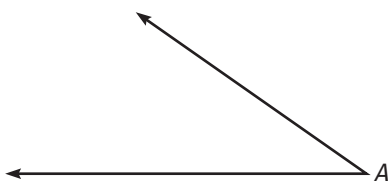
12. Line j, the locus of points that are at least 1 inch from j

LOGICAL REASONING Copy the figure. Then sketch and describe the locus of points on the paper that satisfy the given condition(s).

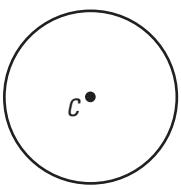
13. equidistant from j and k



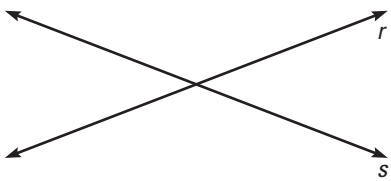
14. in the interior of $\angle A$ and equidistant from both sides of $\angle A$



15. midpoint of a radius of $\odot C$



16. equidistant from r and s



STUDENT HELP

HOMEWORK HELP

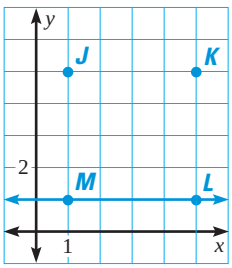
Example 1: Exs. 9–23
Example 2: Exs. 14, 24, 25
Example 3: Exs. 26, 27, 31
Example 4: Exs. 19–25, 28–30

CRITICAL THINKING **Draw $\overline{AB}$. Then sketch and describe the locus of points on the paper that satisfy the given condition.**

17. the locus of points P such that $\angle PAB$ is $30°$

18. the locus of points Q such that $\triangle QAB$ is an isosceles triangle with base $\overline{AB}$

USING ALGEBRA **Use the graph at the right to write equation(s) for the locus of points in the coordinate plane that satisfy the given condition.**



19. equidistant from J and K

20. equidistant from J and M

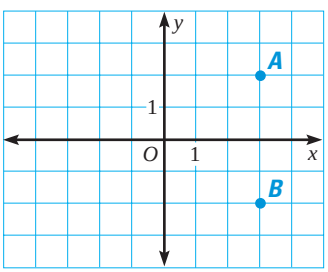
21. equidistant from M and K

22. 3 units from K

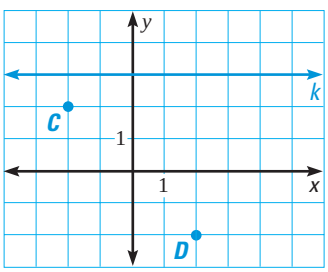
23. 3 units from $\overleftrightarrow{ML}$

COORDINATE GEOMETRY **Copy the graph. Then sketch and describe the locus of points in the plane that satisfy the given conditions. Explain your reasoning.**

24. equidistant from A and B and less than 4 units from the origin



25. equidistant from C and D and 1 unit from line k



LOGICAL REASONING **Sketch and describe the locus. How do the positions of the given points affect the locus?**

26. Point R and line k are in a plane. What is the locus of points in the plane that are 1 unit from k and 2 units from R?

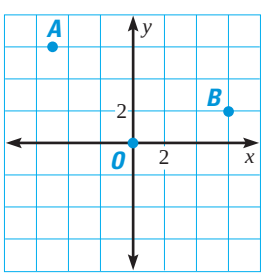
27. Noncollinear points P, Q, and R are in a plane. What is the locus of points in the plane that are equidistant from P and Q and 4 units from R?

FOCUS ON APPLICATIONS

REAL LIFE **SAN ANDREAS FAULT** In 1857, an earthquake on this fault made a river run upstream and flung the water out of a lake, stranding fish miles away.

EARTHQUAKES **In Exercises 28–30, use the following information.** You are given seismograph readings from three locations.

- At $A(-5, 6)$, the epicenter is 13 miles away.
- At $B(6, 2)$, the epicenter is 10 miles away.
- At $O(0, 0)$, the epicenter is 6 miles away.



28. For each seismograph, graph the locus of all possible locations for the epicenter.

29. Where is the epicenter?

30. People could feel the earthquake up to 14 miles away. If your friend lives at $(-3, 20)$, could your friend feel the earthquake? Explain your reasoning.

31. **TECHNOLOGY** Using geometry software, construct and label a line k and a point P not on k. Construct the locus of points that are 2 units from P. Construct the locus of points that are 2 units from k. What is the locus of points that are 2 units from P and 2 units from k? Drag P and k to determine how the location of P and k affects the locus.

32. **CRITICAL THINKING** Given points A and B, describe the locus of points P such that $\triangle APB$ is a right triangle.

Test Preparation

33. **MULTIPLE CHOICE** What is the locus of points in the coordinate plane that are 3 units from the origin?

(A) The line $x = 3$ (B) The line $y = 3$ (C) The circle $x^2 + y^2 = 3$

(D) The circle $x^2 + y^2 = 9$ (E) None of the above

34. **MULTIPLE CHOICE** Circles C and D are externally tangent. The radius of circle C is 6 centimeters and the radius of circle D is 9 centimeters. What is the locus of all points that are a distance of CD from point C?

(A) Circle with center C and a radius of 3 centimeters

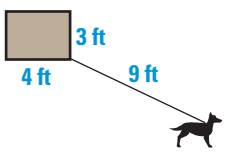
(B) Circle with center D and a radius of 3 centimeters

(C) Circle with center C and a radius of 15 centimeters

(D) Circle with center D and a radius of 15 centimeters

Challenge

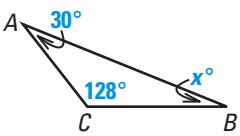
35. **DOG LEASH** A dog's leash is tied to a stake at the corner of its doghouse, as shown at the right. The leash is 9 feet long. Make a scale drawing of the doghouse and sketch the locus of points that the dog can reach.



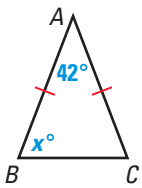
MIXED REVIEW

FINDING ANGLE MEASURES **Find the value of x.** (Review 4.1, 4.6, 6.1 for 11.1)

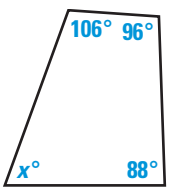
36.



37.

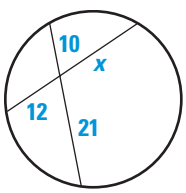


38.

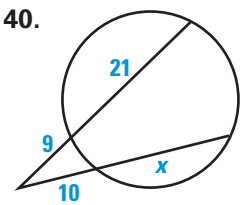


FINDING LENGTHS **Find the value of x.** (Review 10.5)

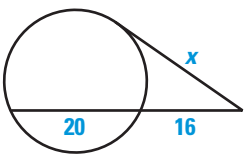
39.



40.



41.



DRAWING GRAPHS **Graph the equation.** (Review 10.6)

42. $x^2 + y^2 = 81$

43. $(x + 6)^2 + (y - 4)^2 = 9$

44. $x^2 + (y - 7)^2 = 100$

45. $(x - 4)^2 + (y - 5)^2 = 1$

QUIZ 3

Self-Test for Lessons 10.6 and 10.7

Graph the equation. (Lesson 10.6)

1. $x^2 + y^2 = 100$

2. $(x + 3)^2 + (y + 3)^2 = 49$

3. $(x - 1)^2 + y^2 = 36$

4. $(x + 4)^2 + (y - 7)^2 = 25$

5. The point $(-3, -9)$ is on a circle whose center is $(2, -2)$. What is the standard equation of the circle? (Lesson 10.6)

6. Draw point P on a piece of paper. Draw and describe the locus of points on the paper that are more than 6 units and less than 9 units from P. (Lesson 10.7)

7. Draw the locus of all points in a plane that are 4 centimeters from a ray $\overrightarrow{AB}$. (Lesson 10.7)

8. **SOCCER** In a soccer game, play begins with a kick-off. All players not involved in the kick-off must stay at least 10 yards from the ball. The ball is in the center of the field. Sketch a 50 yard by 100 yard soccer field with a ball in the center. Then draw and describe the locus of points at which the players not involved in the kick-off can stand. (Lesson 10.7)

MATH & History — History of Timekeeping

INTERNET APPLICATION LINK
www.mcdougallittell.com

THEN

SCHOLARS BELIEVE THAT the practice of dividing a circle into 360 equal parts has its origins in ancient Babylon. Around 1000 B.C., the Babylonians divided the day (one rotation of Earth) into 12 equal time units. Each unit was divided into 30 smaller units. So one of Earth's rotations was divided into $12 \times 30 = 360$ equal parts.

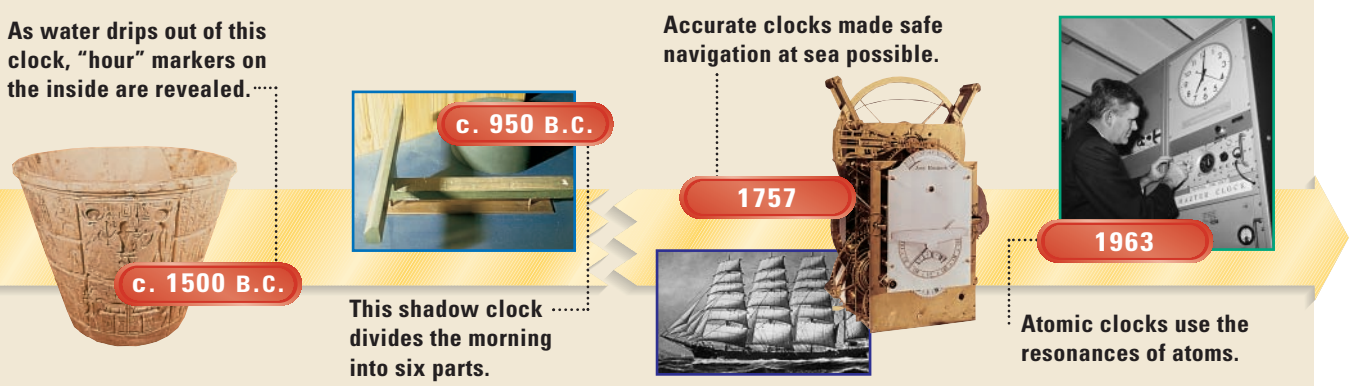
1. Before the introduction of accurate clocks, other civilizations divided the time between sunrise and sunset into 12 equal "temporary hours." These hours varied in length, depending on the time of year.

The table at the right shows the times of sunrise and sunset in New York City. To the nearest minute, find the length of a temporary hour on June 21 and the length of a temporary hour on December 21.

New York City		
Date	**Sunrise**	**Sunset**
June 21	4:25 A.M.	7:30 P.M.
Dec. 21	7:16 A.M.	4:31 P.M.

NOW

TODAY, a day is divided into 24 hours. Atomic clocks are used to give the correct time with an accuracy of better than one second in six million years.



CHAPTER 10

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Identify segments and lines related to circles. **(10.1)**	Lay the foundation for work with circles.
Use properties of tangents of circles. **(10.1)**	Find real-life distances, such as the radius of a silo. **(p. 597)**
Use properties of arcs and chords of circles. **(10.2)**	Solve real-life problems such as analyzing a procedure used to locate an avalanche rescue beacon. **(p. 609)**
Use properties of inscribed angles and inscribed polygons of circles. **(10.3)**	Reach conclusions about angles in real-life objects, such as your viewing angle at the movies. **(p. 614)**
Use angles formed by tangents, chords, and secants. **(10.4)**	Estimate distances, such as the maximum distance at which fireworks can be seen. **(p. 625)**
Find the lengths of segments of tangents, chords, and lines that intersect a circle. **(10.5)**	Find real-life distances, such as the distance a satellite transmits a signal. **(p. 634)**
Find and graph the equation of a circle. **(10.6)**	Solve real-life problems, such as determining cellular phone coverage. **(p. 639)**
Draw loci in a plane that satisfy one or more conditions. **(10.7)**	Make conclusions based on real-life constraints, such as using seismograph readings to locate the epicenter of an earthquake. **(p. 644)**

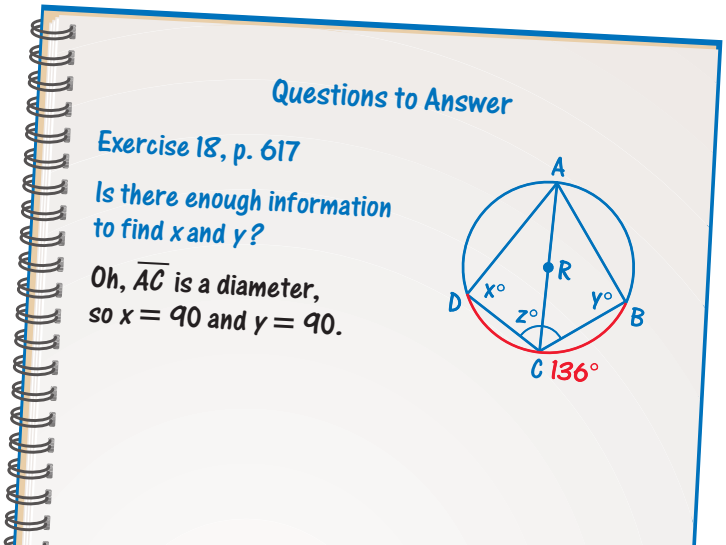
How does Chapter 10 fit into the BIGGER PICTURE of geometry?

In this chapter, you learned that circles have many connections with other geometric figures. For instance, you learned that a quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary. Circles also occur in natural settings, such as the ripples in a pond, and in manufactured structures, such as a cross section of a storage tank. The properties of circles that you studied in this chapter will help you solve problems related to mathematics and the real world.

STUDY STRATEGY

Did you answer your questions?

Your record of questions about difficult exercises, following the study strategy on page 594, may resemble this one.



CHAPTER 10

Chapter Review

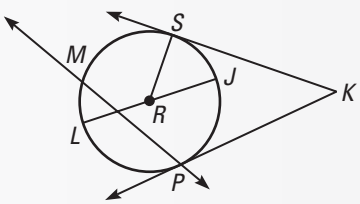
VOCABULARY

- circle, p. 595
- center of circle, p. 595
- radius of circle, p. 595
- congruent circles, p. 595
- diameter of circle, p. 595
- chord, secant, tangent, p. 595
- tangent circles, p. 596
- concentric circles, p. 596
- common tangent, p. 596
- interior of a circle, p. 596
- exterior of a circle, p. 596
- point of tangency, p. 597
- central angle, p. 603
- minor arc and its measure, p. 603
- major arc and its measure, p. 603
- semicircle, p. 603
- congruent arcs, p. 604
- inscribed angle, p. 613
- intercepted arc, p. 613
- inscribed polygon, p. 615
- circumscribed circle, p. 615
- tangent segment, p. 630
- secant segment, p. 630
- external segment, p. 630
- standard equation of a circle, p. 636
- locus, p. 642

10.1 TANGENTS TO CIRCLES

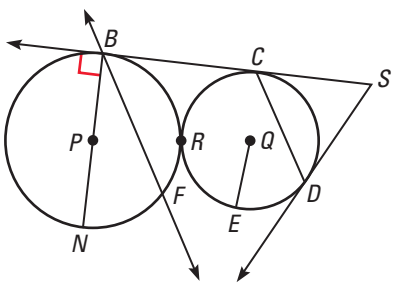
Examples on pp. 595–598

EXAMPLES In $\odot R$, R is the center. $\overline{RJ}$ is a radius, and $\overline{JL}$ is a diameter. $\overline{MP}$ is a chord, and $\overleftrightarrow{MP}$ is a secant. $\overleftrightarrow{KS}$ is a tangent and so it is perpendicular to the radius $\overline{RS}$. $\overline{KS} \cong \overline{KP}$ because they are two tangents from the same exterior point.



Name a point, segment, line, or circle that represents the phrase.

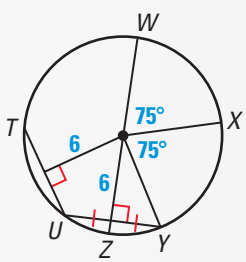
1. Diameter of $\odot P$
2. Point of tangency of $\odot Q$
3. Chord of $\odot P$
4. Center of larger circle
5. Radius of $\odot Q$
6. Common tangent
7. Secant
8. Point of tangency of $\odot P$ and $\odot Q$
9. Is $\angle PBC$ a right angle? Explain.
10. Show that $\triangle SCD$ is isosceles.



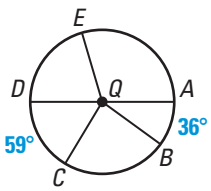
10.2 ARCS AND CHORDS

Examples on pp. 603–606

EXAMPLES $\widehat{WX}$ and $\widehat{XY}$ are congruent minor arcs with measure 75°. $\widehat{WYX}$ is a major arc, and $m\widehat{WYX} = 360° - 75° = 285°$. Chords $\overline{TU}$ and $\overline{UY}$ are congruent because they are equidistant from the center of the circle. $\widehat{TU} \cong \widehat{UY}$ because $\overline{TU} \cong \overline{UY}$. Chord $\overline{WZ}$ is a perpendicular bisector of chord $\overline{UY}$, so $\overline{WZ}$ is a diameter.



Use ⊙*Q* in the diagram to find the measure of the indicated arc. $\overline{AD}$ is a diameter, and $m\widehat{CE} = 121°$.



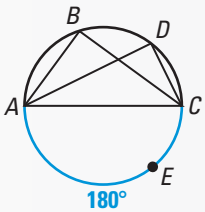
11. $\widehat{DE}$ **12.** $\widehat{AE}$ **13.** $\widehat{AEC}$

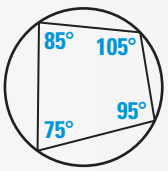
14. $\widehat{BC}$ **15.** $\widehat{BDC}$ **16.** $\widehat{BDA}$

10.3 INSCRIBED ANGLES

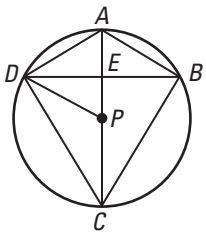
Examples on pp. 613–616

EXAMPLES $\angle ABC$ and $\angle ADC$ are congruent inscribed angles, each with measure $\frac{1}{2} \cdot m\widehat{AEC} = 90°$. Because $\triangle ADC$ is an inscribed right triangle, $\overline{AC}$ is a diameter. The quadrilateral can be inscribed in a circle because its opposite angles are supplementary.





Kite *ABCD* is inscribed in ⊙*P*. Decide whether the statement is *true* or *false*. Explain your reasoning.



17. $\angle ABC$ and $\angle ADC$ are right angles.

18. $m\angle ACD = \frac{1}{2} \cdot m\angle AED$

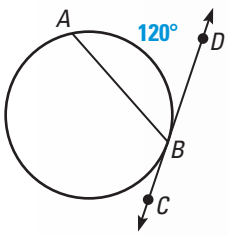
19. $m\angle DAB + m\angle BCD = 180°$

10.4 OTHER ANGLE RELATIONSHIPS IN CIRCLES

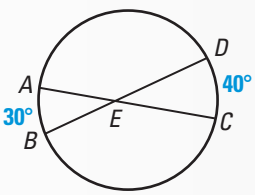
Examples on pp. 621–623

EXAMPLES

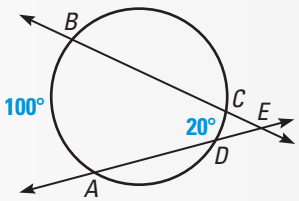
$m\angle ABD = \frac{1}{2} \cdot 120°$
$= 60°$



$m\angle CED = \frac{1}{2}(30° + 40°)$
$= 35°$

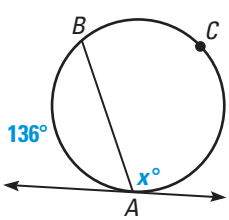


$m\angle CED = \frac{1}{2}(100° - 20°)$
$= 40°$

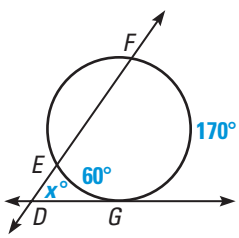


Find the value of *x*.

20.



21.



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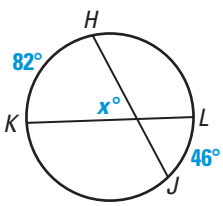

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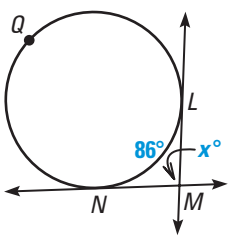


10.5 SEGMENT LENGTHS IN CIRCLES

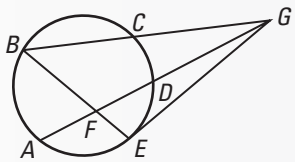
Examples on pp. 629–631

EXAMPLES $\overline{GE}$ is a tangent segment.

$BF \cdot FE = AF \cdot FD$

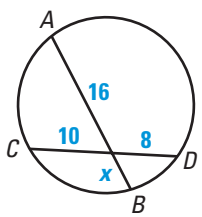
$GC \cdot GB = GD \cdot GA$

$(GE)^2 = GD \cdot GA$



Find the value of *x*.

24.



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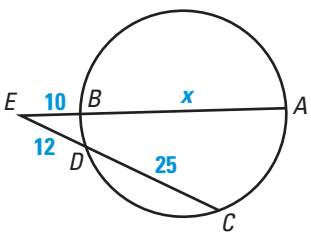

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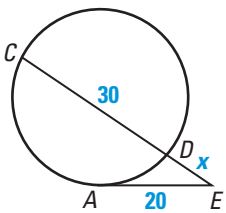
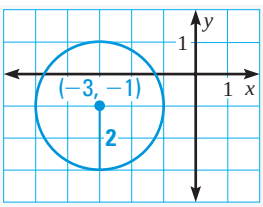


10.6 EQUATIONS OF CIRCLES

Examples on pp. 636–637

EXAMPLE $\odot C$ has center $(-3, -1)$ and radius 2. Its standard equation is

$$[x - (-3)]^2 + [y - (-1)]^2 = 2^2, \text{ or } (x + 3)^2 + (y + 1)^2 = 4.$$



Write the standard equation of the circle. Then graph the equation.

27. Center $(2, 5)$, radius 9

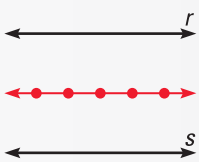
28. Center $(-4, -1)$, radius 4

29. Center $(-6, 0)$, radius $\sqrt{10}$

10.7 LOCUS

Examples on pp. 642–644

EXAMPLE To find the locus of points equidistant from two parallel lines, r and s, draw 2 parallel lines, r and s. Locate several points that are equidistant from r and s. Identify the pattern. The locus is a line parallel to r and s and halfway between them.



Draw the figure. Then sketch and describe the locus of points on the paper that satisfy the given condition(s).

30. $\triangle RST$, the locus of points that are equidistant from R and S

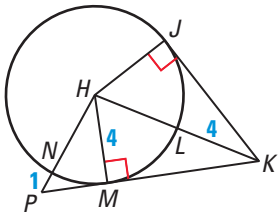
31. Line ℓ, the locus of points that are no more than 4 inches from ℓ

32. $\overline{AB}$ with length 4 cm, the locus of points 3 cm from A and 4 cm from B

CHAPTER 10

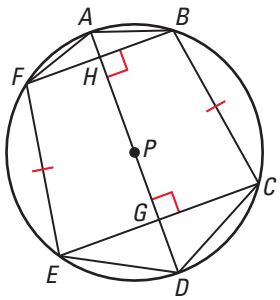
Chapter Test

Use the diagram at the right.



1. Which theorems allow you to conclude that $\overline{JK} \cong \overline{MK}$?
2. Find the lengths of $\overline{JK}$, $\overline{MP}$, and $\overline{PK}$.
3. Show that $\widehat{JL} \cong \widehat{LM}$.
4. Find the measures of $\widehat{JM}$ and $\widehat{JN}$.

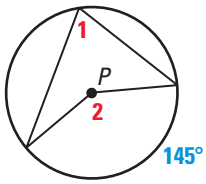
Use the diagram at the right.



5. Show that $\widehat{AF} \cong \widehat{AB}$ and $\overline{FH} \cong \overline{BH}$.
6. Show that $\widehat{FE} \cong \widehat{BC}$.
7. Suppose you were given that $PH = PG$. What could you conclude?

Find the measure of each numbered angle in ⊙*P*.

8.



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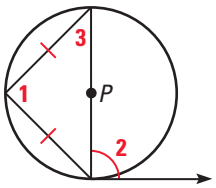

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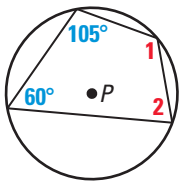

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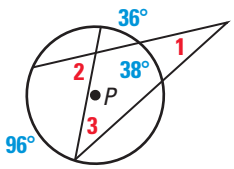
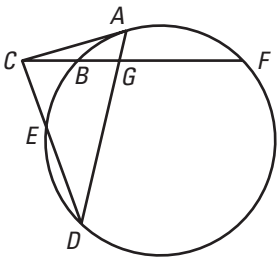


12. Sketch a pentagon *ABCDE* inscribed in a circle. Describe the relationship between (a) $\angle CDE$ and $\angle CAE$ and (b) $\angle CBE$ and $\angle CAE$.

In the diagram at the right $\overline{CA}$ is tangent to the circle at *A*.



13. If $AG = 2$, $GD = 9$, and $BG = 3$, find GF.
14. If $CF = 12$, $CB = 3$, and $CD = 9$, find CE.
15. If $BF = 9$ and $CB = 3$, find CA.
16. Graph the circle with equation $(x - 4)^2 + (y + 6)^2 = 64$.
17. Sketch and describe the locus of points in the coordinate plane that are equidistant from (0, 3) and (3, 0) and 4 units from the point (4, 0).
18. **ROCK CIRCLE** This circle of rock is in the Ténéré desert in the African country of Niger. The circle is about 60 feet in diameter. About a mile away to the north, south, east, and west, stone arrows point away from the circle. It's not known who created the circle or why. Suppose the center of the circle is at (30, 30) on a grid measured in units of feet. Write an equation for the circle.

19. **DOG RUN** A dog on a leash is able to move freely along a cable that is attached to the ground. The leash allows the dog to move anywhere within 3.5 feet from any point on the 10-foot straight cable. Draw and describe the locus of points that the dog can reach.

ACTIVITY 11.6

Using Technology

Investigating Experimental Probability

In Lesson 11.6 you found the *theoretical probability* of a dart landing in a region on a dart board. You can also find the *experimental probability* of this event using a graphing calculator simulation.

STUDENT HELP

INTERNET KEYSTROKE HELP

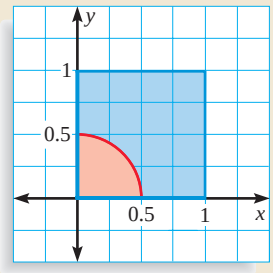
Visit our Web site www.mcdougallittell.com to see keystrokes for several models of calculators.

INVESTIGATE

1 Calculate the theoretical probability that a randomly thrown dart that lands on the dart board shown below will land in the region shaded red.

2 To find the experimental probability, you can physically throw a dart many times and record the results. You can also use a graphing calculator program that simulates throwing a dart as many times as you like. You can simulate this experiment on a TI-82 or TI-83 graphing calculator using the following program.

```
PROGRAM: DARTS
:ClrHome
:Input "HOW MANY THROWS?",N
:0 → H
:For (I, 1, N)
:rand → X
:rand → Y
:If (X² + Y²) < 0.25
:H + 1 → H
:End
:Disp "NUMBER OF HITS",H
```



3 Enter and run the program to simulate 40 throws. Determine the proportion of darts thrown that landed in the region shaded red.

CONJECTURE

1. Explain why "If $(X^2 + Y^2) < 0.25$" is in the program.
2. Compare the theoretical and experimental probabilities you found in **Steps 1**, **2**, and **3**.
3. Find the experimental probability for the entire class by combining the number of throws and the number of hits and determining the proportion of dart throws that landed in the region shaded red.
4. How does the number of trials affect the relationship between the theoretical and experimental probabilities?

EXTENSION

CRITICAL THINKING The results of a calculator simulation tend to be more reliable than those of a human-generated simulation. Explain why a calculator simulation would be easier and more accurate than a human-generated one.

11.1 Angle Measures in Polygons

What you should learn

GOAL 1 Find the measures of interior and exterior angles of polygons.

GOAL 2 Use measures of angles of polygons to solve **real-life** problems.

Why you should learn it

▼ To solve **real-life** problems, such as finding the measures of the interior angles of a home plate marker of a softball field in **Example 4**.

REAL LIFE

GOAL 1 MEASURES OF INTERIOR AND EXTERIOR ANGLES

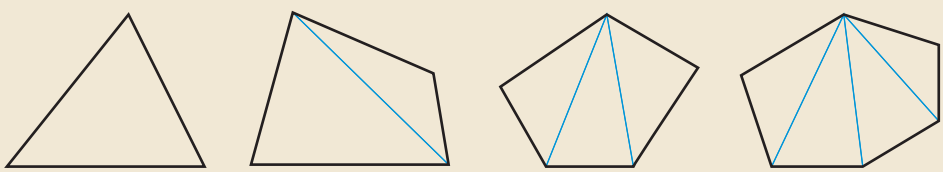
You have already learned that the name of a polygon depends on the number of sides in the polygon: triangle, quadrilateral, pentagon, hexagon, and so forth. The sum of the measures of the interior angles of a polygon also depends on the number of sides.

In Lesson 6.1, you found the sum of the measures of the interior angles of a quadrilateral by dividing the quadrilateral into two triangles. You can use this triangle method to find the sum of the measures of the interior angles of any convex polygon with n sides, called an n-gon.

ACTIVITY Developing Concepts

Investigating the Sum of Polygon Angle Measures

Draw examples of 3-sided, 4-sided, 5-sided, and 6-sided convex polygons. In each polygon, draw all the diagonals from one vertex. Notice that this divides each polygon into triangular regions.



Triangle **Quadrilateral** **Pentagon** **Hexagon**

Complete the table below. What is the pattern in the sum of the measures of the interior angles of any convex n-gon?

Polygon	Number of sides	Number of triangles	Sum of measures of interior angles
Triangle	3	1	$1 \cdot 180° = 180°$
Quadrilateral	?	?	$2 \cdot 180° = 360°$
Pentagon	?	?	?
Hexagon	?	?	?
⋮	⋮	⋮	⋮
n-gon	n	?	?

STUDENT HELP

Look Back
For help with *regular* polygons, see p. 323.

You may have found in the activity that the sum of the measures of the interior angles of a convex n-gon is $(n - 2) \cdot 180°$. This relationship can be used to find the measure of each interior angle in a *regular* n-gon, because the angles are all congruent. Exercises 43 and 44 ask you to write proofs of the following results.

THEOREMS ABOUT INTERIOR ANGLES

THEOREM 11.1 ***Polygon Interior Angles Theorem***

The sum of the measures of the interior angles of a convex n-gon is $(n - 2) \cdot 180°$.

COROLLARY TO THEOREM 11.1

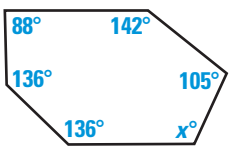
The measure of each interior angle of a regular n-gon is

$$\frac{1}{n} \cdot (n - 2) \cdot 180°, \text{ or } \frac{(n - 2) \cdot 180°}{n}.$$

EXAMPLE 1 *Finding Measures of Interior Angles of Polygons*

Using Algebra

Find the value of x in the diagram shown.



SOLUTION

The sum of the measures of the interior angles of any hexagon is $(6 - 2) \cdot 180° = 4 \cdot 180° = 720°$.

Add the measures of the interior angles of the hexagon.

$136° + 136° + 88° + 142° + 105° + x° = 720°$	**The sum is 720°.**
$607 + x = 720$	**Simplify.**
$x = 113$	**Subtract 607 from each side.**

▶ The measure of the sixth interior angle of the hexagon is 113°.

EXAMPLE 2 *Finding the Number of Sides of a Polygon*

The measure of each interior angle of a regular polygon is 140°. How many sides does the polygon have?

SOLUTION

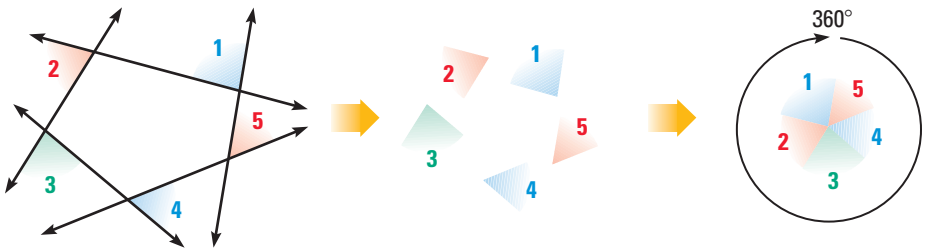
$\frac{1}{n} \cdot (n - 2) \cdot 180° = 140°$	**Corollary to Theorem 11.1**
$(n - 2) \cdot 180 = 140n$	**Multiply each side by *n*.**
$180n - 360 = 140n$	**Distributive property**
$40n = 360$	**Addition and subtraction properties of equality**
$n = 9$	**Divide each side by 40.**

▶ The polygon has 9 sides. It is a regular nonagon.

STUDENT HELP

INTERNET **HOMEWORK HELP**
Visit our Web site www.mcdougallittell.com for extra examples.

The diagrams below show that the sum of the measures of the *exterior* angles of any convex polygon is 360°. You can also find the measure of each exterior angle of a *regular* polygon. Exercises 45 and 46 ask for proofs of these results.

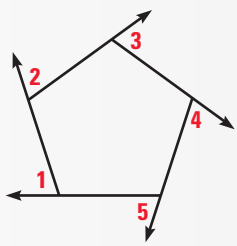


1. Shade one exterior angle at each vertex.
2. Cut out the exterior angles.
3. Arrange the exterior angles to form 360°.

THEOREMS ABOUT EXTERIOR ANGLES

THEOREM 11.2 ***Polygon Exterior Angles Theorem***

The sum of the measures of the exterior angles of a convex polygon, one angle at each vertex, is 360°.



COROLLARY TO THEOREM 11.2

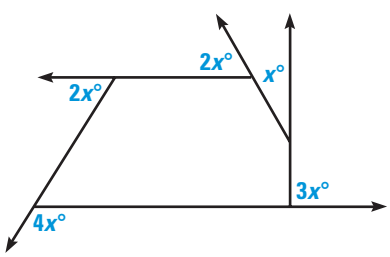
The measure of each exterior angle of a regular n-gon is $\frac{1}{n} \cdot 360°$, or $\frac{360°}{n}$.

Using Algebra

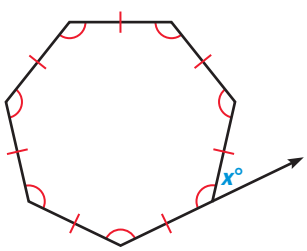
EXAMPLE 3 *Finding the Measure of an Exterior Angle*

Find the value of x in each diagram.

a.



b.



SOLUTION

a. $2x° + x° + 3x° + 4x° + 2x° = 360°$ **Use Theorem 11.2.**

$12x = 360$ **Combine like terms.**

$x = 30$ **Divide each side by 12.**

b. $x° = \frac{1}{7} \cdot 360°$ **Use $n = 7$ in the Corollary to Theorem 11.2.**

≈ 51.4 **Use a calculator.**

▶ The measure of each exterior angle of a regular heptagon is about 51.4°.

GOAL 2 USING ANGLE MEASURES IN REAL LIFE

You can use Theorems 11.1 and 11.2 and their corollaries to find angle measures.

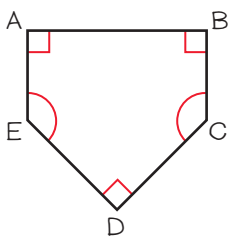
EXAMPLE 4 *Finding Angle Measures of a Polygon*

SOFTBALL A home plate marker for a softball field is a pentagon. Three of the interior angles of the pentagon are right angles. The remaining two interior angles are congruent. What is the measure of each angle?

SOLUTION

PROBLEM SOLVING STRATEGY

DRAW A SKETCH Sketch and label a diagram for the home plate marker. It is a nonregular pentagon. The right angles are $\angle A$, $\angle B$, and $\angle D$. The remaining angles are congruent. So $\angle C \cong \angle E$. The sum of the measures of the interior angles of the pentagon is 540°.



VERBAL MODEL

$$\boxed{\text{Sum of measures of interior angles}} = 3 \cdot \boxed{\text{Measure of each right angle}} + 2 \cdot \boxed{\text{Measure of } \angle C \text{ and } \angle E}$$

LABELS

Sum of measures of interior angles = **540** (degrees)

Measure of each right angle = **90** (degrees)

Measure of $\angle C$ and $\angle E$ = x (degrees)

REASONING

$540 = 3 \cdot 90 + 2x$	Write the equation.
$540 = 270 + 2x$	Simplify.
$270 = 2x$	Subtract 270 from each side.
$135 = x$	Divide each side by 2.

▶ So, the measure of each of the two congruent angles is 135°.

FOCUS ON PEOPLE

REAL LIFE **JOAN JOYCE** set a number of softball pitching records from 1956–1975. She delivered 40 pitches to slugger Ted Williams during an exhibition game in 1962. Williams only connected twice, for one foul ball and one base hit.

EXAMPLE 5 *Using Angle Measures of a Regular Polygon*

SPORTS EQUIPMENT If you were designing the home plate marker for some new type of ball game, would it be possible to make a home plate marker that is a regular polygon with each interior angle having a measure of **(a)** 135°? **(b)** 145°?

SOLUTION

a. Solve the equation $\frac{1}{n} \cdot (n - 2) \cdot 180° = 135°$ for n. You get $n = 8$.

▶ Yes, it would be possible. A polygon can have 8 sides.

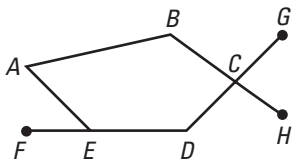
b. Solve the equation $\frac{1}{n} \cdot (n - 2) \cdot 180° = 145°$ for n. You get $n \approx 10.3$.

▶ No, it would not be possible. A polygon cannot have 10.3 sides.

GUIDED PRACTICE

Vocabulary Check ✓

1. Name an *interior angle* and an *exterior angle* of the polygon shown at the right.



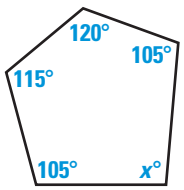
Concept Check ✓

2. How many exterior angles are there in an n-gon? Are they all considered when using the Polygon Exterior Angles Theorem? Explain.

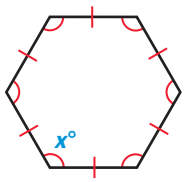
Skill Check ✓

Find the value of *x*.

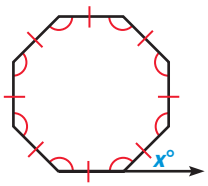
3.



4.



5.



PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 823.

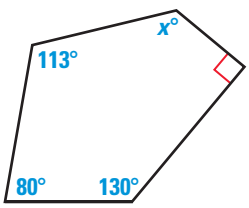
SUMS OF ANGLE MEASURES **Find the sum of the measures of the interior angles of the convex polygon.**

6. 10-gon **7.** 12-gon **8.** 15-gon **9.** 18-gon

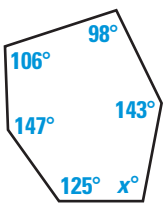
10. 20-gon **11.** 30-gon **12.** 40-gon **13.** 100-gon

ANGLE MEASURES **In Exercises 14–19, find the value of *x*.**

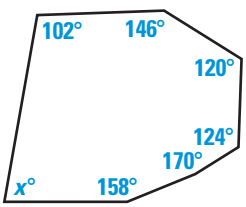
14.



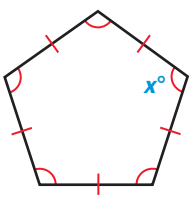
15.



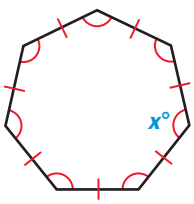
16.



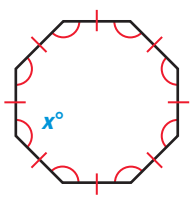
17.



18.



19.



STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 6–16, 20, 21
Example 2: Exs. 17–19, 22–28
Example 3: Exs. 29–38
Example 4: Exs. 39, 40, 49, 50
Example 5: Exs. 51–54

20. A convex quadrilateral has interior angles that measure 80°, 110°, and 80°. What is the measure of the fourth interior angle?

21. A convex pentagon has interior angles that measure 60°, 80°, 120°, and 140°. What is the measure of the fifth interior angle?

DETERMINING NUMBER OF SIDES **In Exercises 22–25, you are given the measure of each interior angle of a regular *n*-gon. Find the value of *n*.**

22. 144° **23.** 120° **24.** 140° **25.** 157.5°

CONSTRUCTION **Use a compass, protractor, and ruler to check the results of Example 2 on page 662.**

26. Draw a large angle that measures 140°. Mark congruent lengths on the sides of the angle.

27. From the end of one of the congruent lengths in Exercise 26, draw the second side of another angle that measures 140°. Mark another congruent length along this new side.

28. Continue to draw angles that measure 140° until a polygon is formed. Verify that the polygon is regular and has 9 sides.

DETERMINING ANGLE MEASURES **In Exercises 29–32, you are given the number of sides of a regular polygon. Find the measure of each exterior angle.**

29. 12 **30.** 11 **31.** 21 **32.** 15

DETERMINING NUMBER OF SIDES **In Exercises 33–36, you are given the measure of each exterior angle of a regular *n*-gon. Find the value of *n*.**

33. 60° **34.** 20° **35.** 72° **36.** 10°

37. A convex hexagon has exterior angles that measure 48°, 52°, 55°, 62°, and 68°. What is the measure of the exterior angle of the sixth vertex?

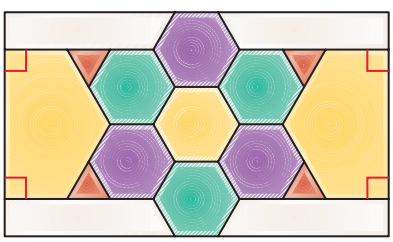
38. What is the measure of each exterior angle of a regular decagon?

FOCUS ON APPLICATIONS

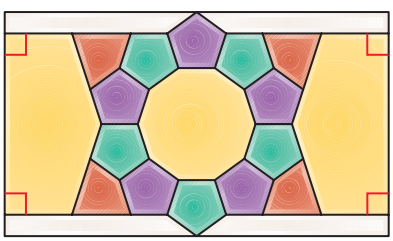
REAL LIFE **STAINED GLASS** is tinted glass that has been cut into shapes and arranged to form a picture or design. The pieces of glass are held in place by strips of lead.

STAINED GLASS WINDOWS **In Exercises 39 and 40, the purple and green pieces of glass are in the shape of regular polygons. Find the measure of each interior angle of the red and yellow pieces of glass.**

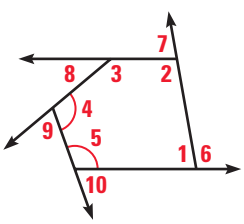
39.



40.



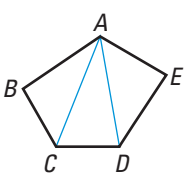
41. **FINDING MEASURES OF ANGLES** In the diagram at the right, $m\angle 2 = 100°$, $m\angle 8 = 40°$, $m\angle 4 = m\angle 5 = 110°$. Find the measures of the other labeled angles and explain your reasoning.



42. *Writing* Explain why the sum of the measures of the interior angles of any two *n*-gons with the same number of sides (two octagons, for example) is the same. Do the *n*-gons need to be regular? Do they need to be similar?

43. **PROOF** Use *ABCDE* to write a paragraph proof to prove Theorem 11.1 for pentagons.

44. **PROOF** Use a paragraph proof to prove the Corollary to Theorem 11.1.



45. **PROOF** Use this plan to write a paragraph proof of Theorem 11.2.
Plan for Proof In a convex n-gon, the sum of the measures of an interior angle and an adjacent exterior angle at any vertex is 180°. Multiply by n to get the sum of all such sums at each vertex. Then subtract the sum of the interior angles derived by using Theorem 11.1.

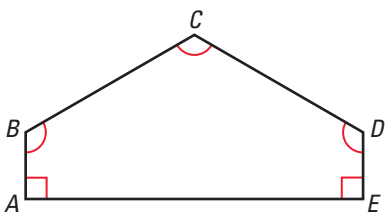
46. **PROOF** Use a paragraph proof to prove the Corollary to Theorem 11.2.

TECHNOLOGY In Exercises 47 and 48, use geometry software to construct a polygon. At each vertex, extend one of the sides of the polygon to form an exterior angle.

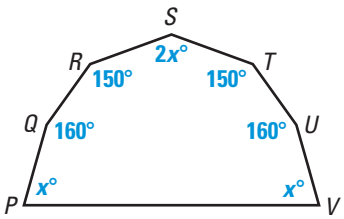
47. Measure each exterior angle and verify that the sum of the measures is 360°.

48. Move any vertex to change the shape of your polygon. What happens to the measures of the exterior angles? What happens to their sum?

49. **HOUSES** Pentagon $ABCDE$ is an outline of the front of a house. Find the measure of each angle.



50. **TENTS** Heptagon $PQRSTUV$ is an outline of a camping tent. Find the unknown angle measures.

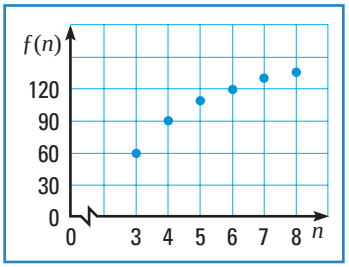


POSSIBLE POLYGONS Would it be possible for a regular polygon to have interior angles with the angle measure described? Explain.

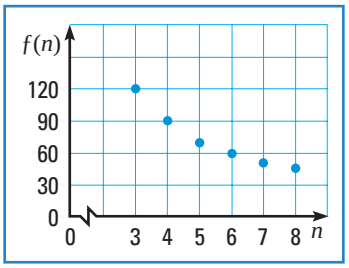
51. 150° **52.** 90° **53.** 72° **54.** 18°

USING ALGEBRA In Exercises 55 and 56, you are given a function and its graph. In each function, n is the number of sides of a polygon and $f(n)$ is measured in degrees. How does the function relate to polygons? What happens to the value of $f(n)$ as n gets larger and larger?

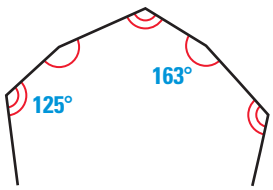
55. $f(n) = \dfrac{180n - 360}{n}$



56. $f(n) = \dfrac{360}{n}$



57. **LOGICAL REASONING** You are shown part of a convex n-gon. The pattern of congruent angles continues around the polygon. Use the Polygon Exterior Angles Theorem to find the value of n.



Test Preparation

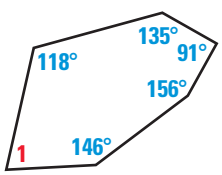
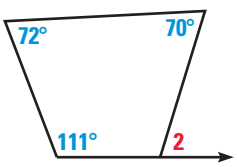
QUANTITATIVE COMPARISON In Exercises 58–61, choose the statement that is true about the given quantities.

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
58.	The sum of the interior angle measures of a decagon	The sum of the interior angle measures of a 15-gon
59.	The sum of the exterior angle measures of an octagon	$8(45°)$
60.	$m\angle 1$ 	$m\angle 2$ 
61.	Number of sides of a polygon with an exterior angle measuring 72°	Number of sides of a polygon with an exterior angle measuring 144°

★ Challenge

62. Polygon $STUVWXYZ$ is a regular octagon. Suppose sides $\overline{ST}$ and $\overline{UV}$ are extended to meet at a point R. Find the measure of $\angle TRU$.

MIXED REVIEW

FINDING AREA Find the area of the triangle described. (Review 1.7 for 11.2)

63. base: 11 inches; height: 5 inches

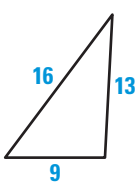
64. base: 43 meters; height: 11 meters

65. vertices: $A(2, 0)$, $B(7, 0)$, $C(5, 15)$

66. vertices: $D(-3, 3)$, $E(3, 3)$, $F(-7, 11)$

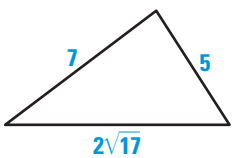
VERIFYING RIGHT TRIANGLES Tell whether the triangle is a right triangle. (Review 9.3)

67.

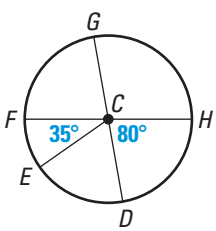


68.

69.



FINDING MEASUREMENTS $\overline{GD}$ and $\overline{FH}$ are diameters of circle C. Find the indicated arc measure. (Review 10.2)



70. $m\widehat{DH}$

71. $m\widehat{ED}$

72. $m\widehat{EH}$

73. $m\widehat{EHG}$

11.2 Areas of Regular Polygons

What you should learn

GOAL 1 Find the area of an equilateral triangle.

GOAL 2 Find the area of a regular polygon, such as the area of a dodecagon in **Example 4**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the area of a hexagonal mirror on the Hobby-Eberly Telescope in **Exs. 45 and 46**.

REAL LIFE

GOAL 1 FINDING THE AREA OF AN EQUILATERAL TRIANGLE

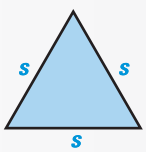
The area of *any* triangle with base length b and height h is given by $A = \frac{1}{2}bh$. The following formula for equilateral triangles, however, uses only the side length.

THEOREM

THEOREM 11.3 ***Area of an Equilateral Triangle***

The area of an equilateral triangle is one fourth the square of the length of the side times $\sqrt{3}$.

$$A = \frac{1}{4}\sqrt{3}s^2$$



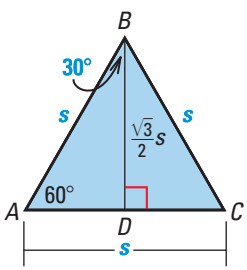
EXAMPLE 1 *Proof of Theorem 11.3*

Prove Theorem 11.3. Refer to the figure below.

SOLUTION

GIVEN ▶ $\triangle ABC$ is equilateral.

PROVE ▶ Area of $\triangle ABC$ is $A = \frac{1}{4}\sqrt{3}s^2$.



Paragraph Proof Draw the altitude from B to side $\overline{AC}$. Then $\triangle ABD$ is a 30°-60°-90° triangle. From Lesson 9.4, the length of $\overline{BD}$, the side opposite the 60° angle in $\triangle ABD$, is $\frac{\sqrt{3}}{2}s$. Using the formula for the area of a triangle,

$$A = \frac{1}{2}bh = \frac{1}{2}(s)\left(\frac{\sqrt{3}}{2}s\right) = \frac{1}{4}\sqrt{3}s^2.$$

EXAMPLE 2 *Finding the Area of an Equilateral Triangle*

Find the area of an equilateral triangle with 8 inch sides.

SOLUTION

Use $s = 8$ in the formula from Theorem 11.3.

$$A = \frac{1}{4}\sqrt{3}s^2 = \frac{1}{4}\sqrt{3}(8^2) = \frac{1}{4}\sqrt{3}(64) = \frac{1}{4}(64)\sqrt{3} = 16\sqrt{3} \text{ square inches}$$

▶ Using a calculator, the area is about 27.7 square inches.

STUDENT HELP

Study Tip

Be careful with radical signs. Notice in Example 1 that $\sqrt{3}s^2$ and $\sqrt{3s^2}$ do not mean the same thing.

GOAL 2 FINDING THE AREA OF A REGULAR POLYGON

You can use equilateral triangles to find the area of a regular hexagon.

ACTIVITY Developing Concepts

Investigating the Area of a Regular Hexagon

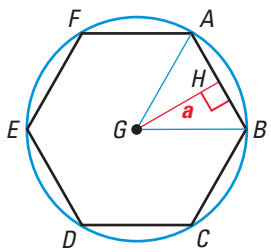
Use a protractor and ruler to draw a regular hexagon. Cut out your hexagon. Fold and draw the three lines through opposite vertices. The point where these lines intersect is the *center* of the hexagon.

1. How many triangles are formed? What kind of triangles are they?
2. Measure a side of the hexagon. Find the area of one of the triangles. What is the area of the entire hexagon? Explain your reasoning.

Think of the hexagon in the activity above, or another regular polygon, as inscribed in a circle.

The **center of the polygon** and **radius of the polygon** are the center and radius of its circumscribed circle, respectively.

The distance from the center to any side of the polygon is called the **apothem of the polygon**. The apothem is the height of a triangle between the center and two consecutive vertices of the polygon.



Hexagon *ABCDEF* with center *G*, radius *GA*, and apothem *GH*

STUDENT HELP

Study Tip
In a regular polygon, the length of each side is the same. If this length is s and there are n sides, then the perimeter P of the polygon is $n \cdot s$, or $P = ns$.

As in the activity, you can find the area of any regular n-gon by dividing the polygon into congruent triangles.

$$\begin{aligned} A &= \boxed{\text{area of one triangle}} \cdot \boxed{\text{number of triangles}} \\ &= \left(\frac{1}{2} \cdot \text{apothem} \cdot \text{side length } s\right) \cdot \text{number of sides} \\ &= \frac{1}{2} \cdot \text{apothem} \cdot \text{number of sides} \cdot \text{side length } s \\ &= \frac{1}{2} \cdot \text{apothem} \cdot \text{perimeter of polygon} \end{aligned}$$

The number of congruent triangles formed will be the same as the number of sides of the polygon.

This approach can be used to find the area of any regular polygon.

THEOREM

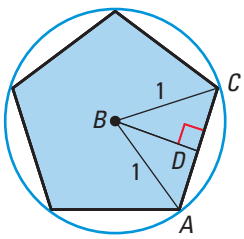
THEOREM 11.4 ***Area of a Regular Polygon***

The area of a regular n-gon with side length s is half the product of the apothem a and the perimeter P, so $A = \frac{1}{2}aP$, or $A = \frac{1}{2}a \cdot ns$.

A **central angle of a regular polygon** is an angle whose vertex is the center and whose sides contain two consecutive vertices of the polygon. You can divide 360° by the number of sides to find the measure of each central angle of the polygon.

EXAMPLE 3 *Finding the Area of a Regular Polygon*

A regular pentagon is inscribed in a circle with radius 1 unit. Find the area of the pentagon.



SOLUTION

To apply the formula for the area of a regular pentagon, you must find its apothem and perimeter.

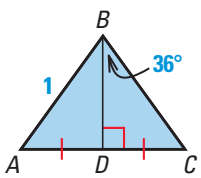
The measure of central $\angle ABC$ is $\frac{1}{5} \cdot 360°$, or 72°.

In isosceles triangle $\triangle ABC$, the altitude to base $\overline{AC}$ also bisects $\angle ABC$ and side $\overline{AC}$. The measure of $\angle DBC$, then, is 36°. In right triangle $\triangle BDC$, you can use trigonometric ratios to find the lengths of the legs.

STUDENT HELP

Look Back
For help with trigonometric ratios, see p. 558.

$$\cos 36° = \frac{BD}{BC} = \frac{BD}{1} = BD \qquad \sin 36° = \frac{DC}{BC} = \frac{DC}{1} = DC$$



So, the pentagon has an apothem of $a = BD = \cos 36°$ and a perimeter of $P = 5(AC) = 5(2 \cdot DC) = 10 \sin 36°$. The area of the pentagon is

$$A = \frac{1}{2}aP = \frac{1}{2}(\cos 36°)(10 \sin 36°) \approx 2.38 \text{ square units.}$$

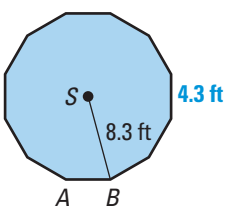
EXAMPLE 4 *Finding the Area of a Regular Dodecagon*

FOCUS ON APPLICATIONS

REAL LIFE FOUCAULT PENDULUMS swing continuously in a straight line. Watching the pendulum, though, you may think its path shifts. Instead, it is Earth and you that are turning. The floor under this pendulum in Texas rotates fully about every 48 hours.

INTERNET APPLICATION LINK
www.mcdougallittell.com

PENDULUMS The enclosure on the floor underneath the Foucault Pendulum at the Houston Museum of Natural Sciences in Houston, Texas, is a regular dodecagon with a side length of about 4.3 feet and a radius of about 8.3 feet. What is the floor area of the enclosure?



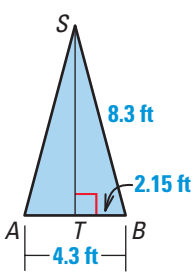
SOLUTION

A dodecagon has 12 sides. So, the perimeter of the enclosure is

$$P \approx 12(4.3) = 51.6 \text{ feet.}$$

In $\triangle SBT$, $BT = \frac{1}{2}(BA) = \frac{1}{2}(4.3) = 2.15$ feet. Use the Pythagorean Theorem to find the apothem ST.

$$a = \sqrt{8.3^2 - 2.15^2} \approx 8 \text{ feet}$$

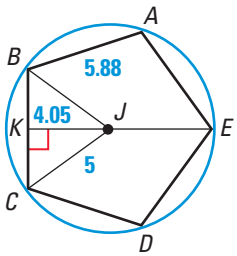


So, the floor area of the enclosure is

$$A = \frac{1}{2}aP \approx \frac{1}{2}(8)(51.6) = 206.4 \text{ square feet.}$$

GUIDED PRACTICE

Vocabulary Check ✓ **In Exercises 1–4, use the diagram shown.**



1. Identify the *center* of polygon *ABCDE*.
2. Identify the *radius* of the polygon.
3. Identify a *central angle* of the polygon.
4. Identify a segment whose length is the *apothem*.

Concept Check ✓

5. In a regular polygon, how do you find the measure of each central angle?

Skill Check ✓

6. What is the area of an equilateral triangle with 3 inch sides?

STOP SIGN The stop sign shown is a regular octagon. Its perimeter is about 80 inches and its height is about 24 inches.

7. What is the measure of each central angle?
8. Find the apothem, radius, and area of the stop sign.

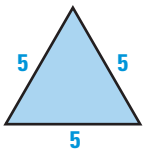
PRACTICE AND APPLICATIONS

STUDENT HELP

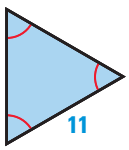
Extra Practice to help you master skills is on p. 823.

FINDING AREA Find the area of the triangle.

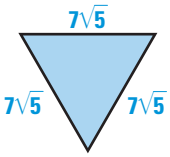
9.



10.



11.

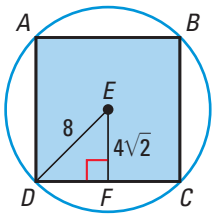


MEASURES OF CENTRAL ANGLES Find the measure of a central angle of a regular polygon with the given number of sides.

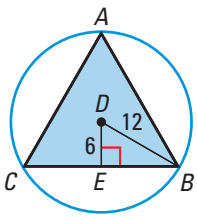
12. 9 sides **13.** 12 sides **14.** 15 sides **15.** 180 sides

FINDING AREA Find the area of the inscribed regular polygon shown.

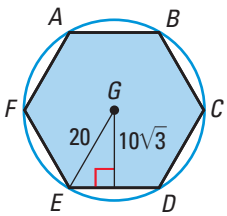
16.



17.



18.



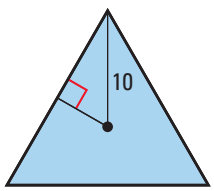
STUDENT HELP

HOMEWORK HELP

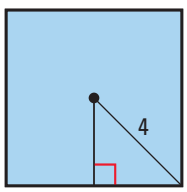
Example 1: Exs. 9–11, 17, 19, 25, 33
Example 2: Exs. 9–11, 17, 19, 25, 33
Example 3: Exs. 12–24, 26, 34
Example 4: Exs. 34, 45–49

PERIMETER AND AREA Find the perimeter and area of the regular polygon.

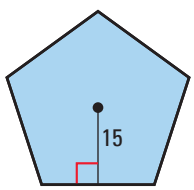
19.



20.

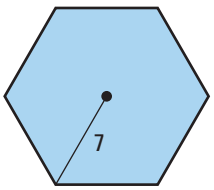


21.

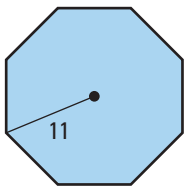


PERIMETER AND AREA **In Exercises 22–24, find the perimeter and area of the regular polygon.**

22.



23.



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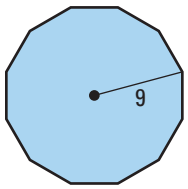


25. **AREA** Find the area of an equilateral triangle that has a height of 15 inches.

26. **AREA** Find the area of a regular dodecagon (or 12-gon) that has 4 inch sides.

LOGICAL REASONING **Decide whether the statement is *true* or *false*. Explain your choice.**

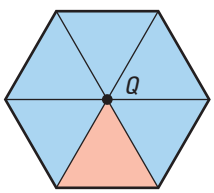
27. The area of a regular polygon of fixed radius r increases as the number of sides increases.

28. The apothem of a regular polygon is always less than the radius.

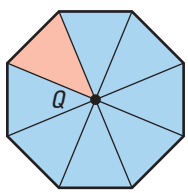
29. The radius of a regular polygon is always less than the side length.

AREA **In Exercises 30–32, find the area of the regular polygon. The area of the portion shaded in red is given. Round answers to the nearest tenth.**

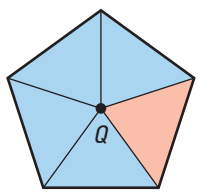
30. Area $= 16\sqrt{3}$



31. Area $= 4 \tan 67.5°$



32. Area $= \tan 54°$



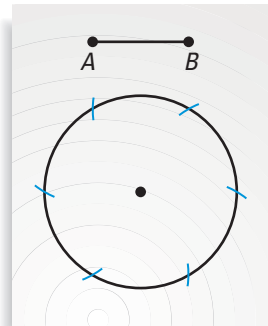
33. **USING THE AREA FORMULAS** Show that the area of a regular hexagon is six times the area of an equilateral triangle with the same side length.

$\left(\textit{Hint}\text{: Show that for a hexagon with side lengths } s, \frac{1}{2}aP = 6 \cdot \left(\frac{1}{4}\sqrt{3}s^2\right).\right)$

34. **BASALTIC COLUMNS** Suppose the top of one of the columns along the Giant's Causeway (see p. 659) is in the shape of a regular hexagon with a diameter of 18 inches. What is its apothem?

CONSTRUCTION **In Exercises 35–39, use a straightedge and a compass to construct a regular hexagon and an equilateral triangle.**

35. Draw $\overline{AB}$ with a length of 1 inch. Open the compass to 1 inch and draw a circle with that radius.



36. Using the same compass setting, mark off equal parts along the circle.

37. Connect the six points where the compass marks and circle intersect to draw a regular hexagon.

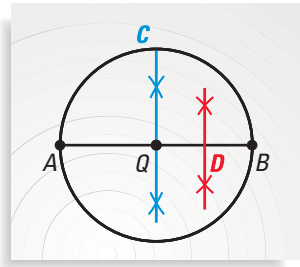
38. What is the area of the hexagon?

39. *Writing* Explain how you could use this construction to construct an equilateral triangle.

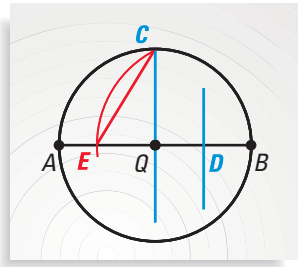
STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell for help with construction in Exs. 40–44.

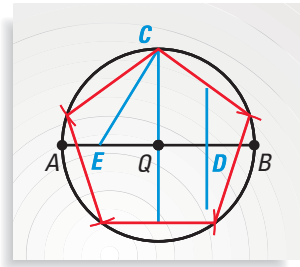
CONSTRUCTION **In Exercises 40–44, use a straightedge and a compass to construct a regular pentagon as shown in the diagrams below.**



Exs. 40, 41



Ex. 42



Exs. 43, 44

40. Draw a circle with center Q. Draw a diameter $\overline{AB}$. Construct the perpendicular bisector of $\overline{AB}$ and label its intersection with the circle as point C.

41. Construct point D, the midpoint of $\overline{QB}$.

42. Place the compass point at D. Open the compass to the length DC and draw an arc from C so it intersects $\overline{AB}$ at a point, E. Draw $\overline{CE}$.

43. Open the compass to the length CE. Starting at C, mark off equal parts along the circle.

44. Connect the five points where the compass marks and circle intersect to draw a regular pentagon. What is the area of your pentagon?

FOCUS ON CAREERS

REAL LIFE **ASTRONOMERS** use physics and mathematics to study the universe, including the sun, moon, planets, stars, and galaxies.

CAREER LINK www.mcdougallittell.com

TELESCOPES **In Exercises 45 and 46, use the following information.**

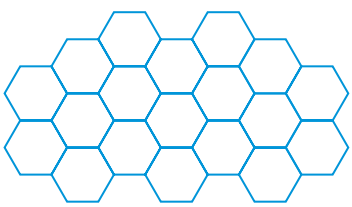
The Hobby-Eberly Telescope in Fort Davis, Texas, is the largest optical telescope in North America. The primary mirror for the telescope consists of 91 smaller mirrors forming a hexagon shape. Each of the smaller mirror parts is itself a hexagon with side length 0.5 meter.

45. What is the apothem of one of the smaller mirrors?

46. Find the perimeter and area of one of the smaller mirrors.

TILING **In Exercises 47–49, use the following information.**

You are tiling a bathroom floor with tiles that are regular hexagons, as shown. Each tile has 6 inch sides. You want to choose different colors so that no two adjacent tiles are the same color.



47. What is the minimum number of colors that you can use?

48. What is the area of each tile?

49. The floor that you are tiling is rectangular. Its width is 6 feet and its length is 8 feet. At least how many tiles of each color will you need?

Test Preparation

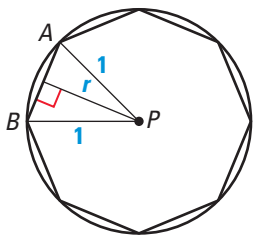
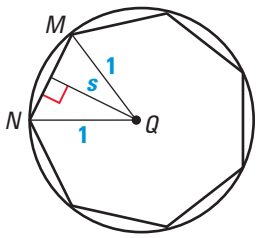
QUANTITATIVE COMPARISON In Exercises 50–52, choose the statement that is true about the given quantities.

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

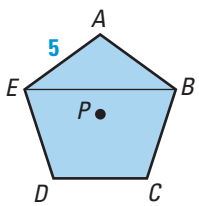
Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
		
50.	$m\angle APB$	$m\angle MQN$
51.	Apothem r	Apothem s
52.	Perimeter of octagon with center P	Perimeter of heptagon with center Q

★ Challenge

53. USING DIFFERENT METHODS Find the area of $ABCDE$ by using two methods. First, use the formula $A = \frac{1}{2}aP$, or $A = \frac{1}{2}a \cdot ns$. Second, add the areas of the smaller polygons. Check that both methods yield the same area.



EXTRA CHALLENGE
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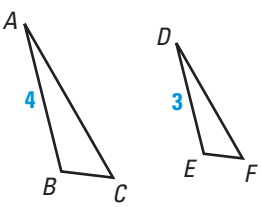
MIXED REVIEW

SOLVING PROPORTIONS Solve the proportion. (Review 8.1 for 11.3)

54. $\frac{x}{6} = \frac{11}{12}$ **55.** $\frac{20}{4} = \frac{15}{x}$ **56.** $\frac{12}{x+7} = \frac{13}{x}$ **57.** $\frac{x+6}{9} = \frac{x}{11}$

USING SIMILAR POLYGONS In the diagram shown, $\triangle ABC \sim \triangle DEF$. Use the figures to determine whether the statement is true. (Review 8.3 for 11.3)

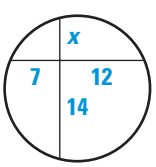


58. $\frac{AC}{BC} = \frac{DF}{EF}$ **59.** $\frac{DF}{AC} = \frac{EF + DE + DF}{BC + AB + AC}$

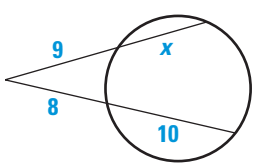
60. $\angle B \cong \angle E$ **61.** $\overline{BC} \cong \overline{EF}$

FINDING SEGMENT LENGTHS Find the value of x. (Review 10.5)

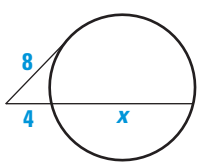
62.



63.



64.



11.3 Perimeters and Areas of Similar Figures

What you should learn

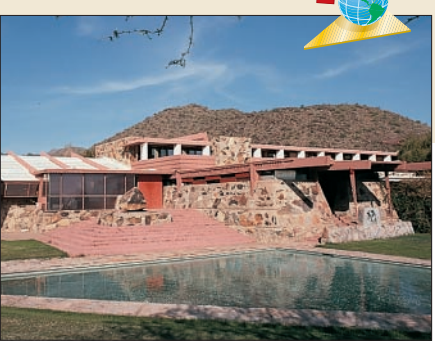
GOAL 1 Compare perimeters and areas of similar figures.

GOAL 2 Use perimeters and areas of similar figures to solve **real-life** problems, as applied in **Example 2**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the area of the walkway around a polygonal pool in **Exs. 25–27**.

REAL LIFE



Frank Lloyd Wright included this triangular pool and walkway in his design of *Taliesin West* in Scottsdale, Arizona.

GOAL 1 COMPARING PERIMETER AND AREA

For any polygon, the *perimeter of the polygon* is the sum of the lengths of its sides and the *area of the polygon* is the number of square units contained in its interior.

In Lesson 8.3, you learned that if two polygons are *similar*, then the ratio of their perimeters is the same as the ratio of the lengths of their corresponding sides. In Activity 11.3 on page 676, you may have discovered that the ratio of the areas of two similar polygons is *not* this same ratio, as shown in Theorem 11.5. Exercise 22 asks you to write a proof of this theorem for rectangles.

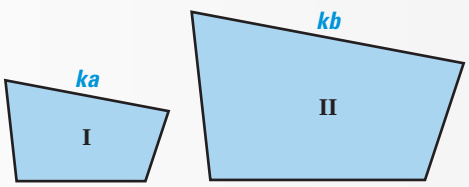
THEOREM

THEOREM 11.5 ***Areas of Similar Polygons***

If two polygons are similar with the lengths of corresponding sides in the ratio of $a:b$, then the ratio of their areas is $a^2:b^2$.

$$\frac{\text{Side length of Quad. I}}{\text{Side length of Quad. II}} = \frac{a}{b}$$

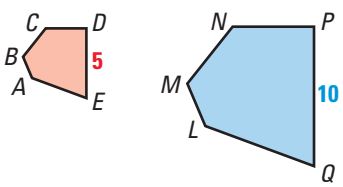
$$\frac{\text{Area of Quad. I}}{\text{Area of Quad. II}} = \frac{a^2}{b^2}$$



Quad. I ~ **Quad.** II

EXAMPLE 1 *Finding Ratios of Similar Polygons*

Pentagons *ABCDE* and *LMNPQ* are similar.



a. Find the ratio (red to blue) of the perimeters of the pentagons.

b. Find the ratio (red to blue) of the areas of the pentagons.

SOLUTION

The ratio of the lengths of corresponding sides in the pentagons is $\frac{5}{10} = \frac{1}{2}$, or $1:2$.

a. The ratio of the perimeters is also $1:2$. So, the perimeter of pentagon *ABCDE* is half the perimeter of pentagon *LMNPQ*.

b. Using Theorem 11.5, the ratio of the areas is $1^2:2^2$, or $1:4$. So, the area of pentagon *ABCDE* is one fourth the area of pentagon *LMNPQ*.

STUDENT HELP

Study Tip

The ratio "*a* to *b*," for example, can be written using a fraction bar $\left(\frac{a}{b}\right)$ or a colon ($a:b$).

GOAL 2 USING PERIMETER AND AREA IN REAL LIFE

EXAMPLE 2 *Using Areas of Similar Figures*

COMPARING COSTS You are buying photographic paper to print a photo in different sizes. An 8 inch by 10 inch sheet of the paper costs \$.42. What is a reasonable cost for a 16 inch by 20 inch sheet?

SOLUTION

Because the ratio of the lengths of the sides of the two rectangular pieces of paper is $1:2$, the ratio of the areas of the pieces of paper is $1^2:2^2$, or $1:4$. Because the cost of the paper should be a function of its area, the larger piece of paper should cost about four times as much, or \$1.68.

EXAMPLE 3 *Finding Perimeters and Areas of Similar Polygons*

OCTAGONAL FLOORS A trading pit at the Chicago Board of Trade is in the shape of a series of regular octagons. One octagon has a side length of about 14.25 feet and an area of about 980.4 square feet. Find the area of a smaller octagon that has a perimeter of about 76 feet.

FOCUS ON APPLICATIONS

CHICAGO BOARD OF TRADE
Commodities such as grains, coffee, and financial securities are exchanged at this marketplace. Associated traders stand on the descending steps in the same "pie-slice" section of an octagonal pit. The different levels allow buyers and sellers to see each other as orders are yelled out.

APPLICATION LINK
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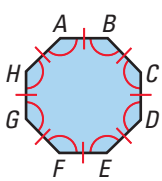
SOLUTION

All regular octagons are similar because all corresponding angles are congruent and the corresponding side lengths are proportional.

Draw and label a sketch.

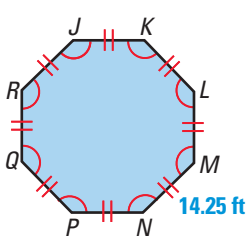


Find the ratio of the side lengths of the two octagons, which is the same as the ratio of their perimeters.

$$\frac{\text{perimeter of } ABCDEFGH}{\text{perimeter of } JKLMNPQR} = \frac{a}{b} \approx \frac{76}{8(14.25)} = \frac{76}{114} = \frac{2}{3}$$

Calculate the area of the smaller octagon. Let A represent the area of the smaller octagon. The ratio of the areas of the smaller octagon to the larger is $a^2:b^2 = 2^2:3^2$, or $4:9$.

$\frac{A}{980.4} = \frac{4}{9}$	**Write proportion.**
$9A = 980.4 \cdot 4$	**Cross product property**
$A = \frac{3921.6}{9}$	**Divide each side by 9.**
$A \approx 435.7$	**Use a calculator.**



▶ The area of the smaller octagon is about 435.7 square feet.

GUIDED PRACTICE

Vocabulary Check ✓

1. If two polygons are *similar* with the lengths of corresponding sides in the ratio of $a:b$, then the ratio of their perimeters is __?__ and the ratio of their areas is __?__.

Concept Check ✓

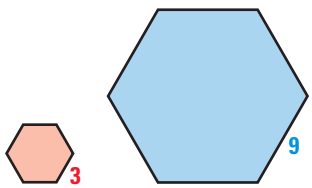
Tell whether the statement is *true* or *false*. Explain.

2. Any two regular polygons with the same number of sides are similar.
3. Doubling the side length of a square doubles the area.

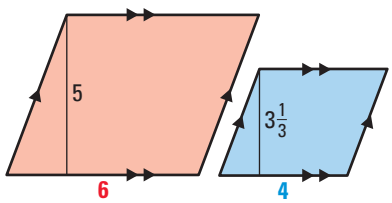
Skill Check ✓

In Exercises 4 and 5, the red and blue figures are similar. Find the ratio (red to blue) of their perimeters and of their areas.

4.



5.



6. **PHOTOGRAPHY** Use the information from Example 2 on page 678 to find a reasonable cost for a sheet of 4 inch by 5 inch photographic paper.

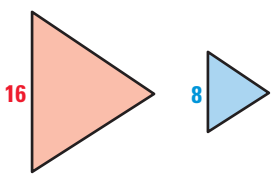
PRACTICE AND APPLICATIONS

STUDENT HELP

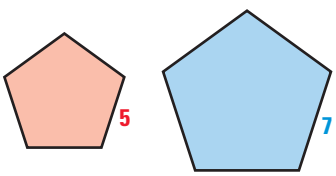
Extra Practice to help you master skills is on p. 823.

FINDING RATIOS **In Exercises 7–10, the polygons are similar. Find the ratio (red to blue) of their perimeters and of their areas.**

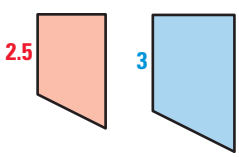
7.



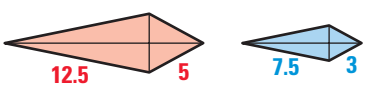
8.



9.



10.



LOGICAL REASONING **In Exercises 11–13, complete the statement using *always, sometimes,* or *never.***

11. Two similar hexagons __?__ have the same perimeter.
12. Two rectangles with the same area are __?__ similar.
13. Two regular pentagons are __?__ similar.
14. **HEXAGONS** The ratio of the lengths of corresponding sides of two similar hexagons is $2:5$. What is the ratio of their areas?
15. **OCTAGONS** A regular octagon has an area of 49 m^2. Find the scale factor of this octagon to a similar octagon that has an area of 100 m^2.

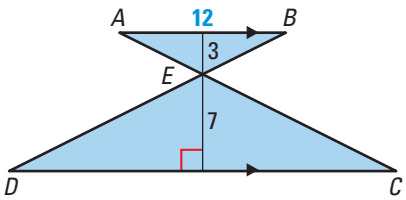
STUDENT HELP

HOMEWORK HELP

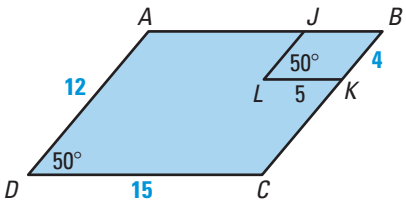
Example 1: Exs. 7–10, 14–18
Example 2: Exs. 23, 24
Example 3: Exs. 25–28

16. RIGHT TRIANGLES $\triangle ABC$ is a right triangle whose hypotenuse $\overline{AC}$ is 8 inches long. Given that the area of $\triangle ABC$ is 13.9 square inches, find the area of similar triangle $\triangle DEF$ whose hypotenuse $\overline{DF}$ is 20 inches long.

17. FINDING AREA Explain why $\triangle CDE$ is similar to $\triangle ABE$. Find the area of $\triangle CDE$.



18. FINDING AREA Explain why $\square JBKL \sim \square ABCD$. The area of $\square JBKL$ is 15.3 square inches. Find the area of $\square ABCD$.



STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with scale factors in Exs. 19–21.

19. SCALE FACTOR Regular pentagon $ABCDE$ has a side length of $6\sqrt{5}$ centimeters. Regular pentagon $QRSTU$ has a perimeter of 40 centimeters. Find the ratio of the perimeters of $ABCDE$ to $QRSTU$.

20. SCALE FACTOR A square has a perimeter of 36 centimeters. A smaller square has a side length of 4 centimeters. What is the ratio of the areas of the larger square to the smaller one?

21. SCALE FACTOR A regular nonagon has an area of 90 square feet. A similar nonagon has an area of 25 square feet. What is the ratio of the perimeters of the first nonagon to the second?

22. PROOF Prove Theorem 11.5 for rectangles.

RUG COSTS Suppose you want to be sure that a large rug is priced fairly. The price of a small rug (29 inches by 47 inches) is $79 and the price of the large rug (4 feet 10 inches by 7 feet 10 inches) is $299.

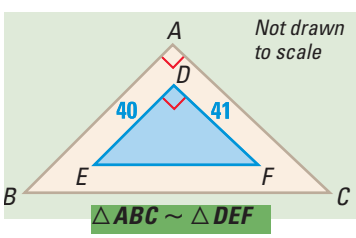
23. What are the areas of the two rugs? What is the ratio of the areas?

24. Compare the rug costs. Do you think the large rug is a good buy? Explain.

FOCUS ON APPLICATIONS

REAL LIFE FORT JEFFERSON is in the Dry Tortugas National Park 70 miles west of Key West, Florida. The fort has been used as a prison, a navy base, a seaplane port, and an observation post.

TRIANGULAR POOL In Exercises 25–27, use the following information. The pool at *Taliesin West* (see page 677) is a right triangle with legs of length 40 feet and 41 feet.



25. Find the area of the triangular pool, $\triangle DEF$.

26. The walkway bordering the pool is 40 inches wide, so the scale factor of the similar triangles is about 1.3 : 1. Find AB.

27. Find the area of $\triangle ABC$. What is the area of the walkway?

28. FORT JEFFERSON The outer wall of Fort Jefferson, which was originally constructed in the mid-1800s, is in the shape of a hexagon with an area of about 466,170 square feet. The length of one side is about 477 feet. The inner courtyard is a similar hexagon with an area of about 446,400 square feet. Calculate the length of a corresponding side in the inner courtyard to the nearest foot.

Test Preparation

29. MULTI-STEP PROBLEM Use the following information about similar triangles $\triangle ABC$ and $\triangle DEF$.

The scale factor of $\triangle ABC$ to $\triangle DEF$ is $15:2$.

The area of $\triangle ABC$ is $25x$. The area of $\triangle DEF$ is $x - 5$.

The perimeter of $\triangle ABC$ is $8 + y$. The perimeter of $\triangle DEF$ is $3y - 19$.

a. Use the scale factor to find the ratio of the area of $\triangle ABC$ to the area of $\triangle DEF$.

b. Write and solve a proportion to find the value of x.

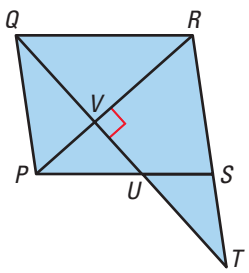
c. Use the scale factor to find the ratio of the perimeter of $\triangle ABC$ to the perimeter of $\triangle DEF$.

d. Write and solve a proportion to find the value of y.

e. *Writing* Explain how you could find the value of z if $AB = 22.5$ and the length of the corresponding side $\overline{DE}$ is $13z - 10$.

★ Challenge

Use the figure shown at the right. *PQRS* is a parallelogram.



30. Name three pairs of similar triangles and explain how you know that they are similar.

31. The ratio of the area of $\triangle PVQ$ to the area of $\triangle RVT$ is $9:25$, and the length RV is 10. Find PV.

32. If VT is 15, find VQ, VU, and UT.

33. Find the ratio of the areas of each pair of similar triangles that you found in Exercise 30.

EXTRA CHALLENGE
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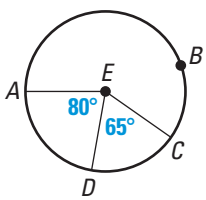
MIXED REVIEW

FINDING MEASURES In Exercises 34–37, use the diagram shown at the right. (Review 10.2 for 11.4)



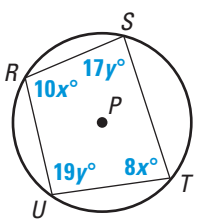
34. Find $m\widehat{AD}$.

35. Find $m\angle AEC$.

36. Find $m\widehat{AC}$.

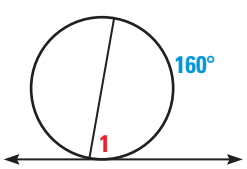
37. Find $m\widehat{ABC}$.

38. USING AN INSCRIBED QUADRILATERAL In the diagram shown at the right, quadrilateral $RSTU$ is inscribed in circle P. Find the values of x and y, and use them to find the measures of the angles of $RSTU$. (Review 10.3)

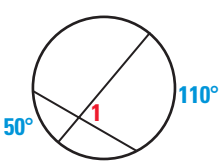


FINDING ANGLE MEASURES Find the measure of $\angle 1$. (Review 10.4 for 11.4)

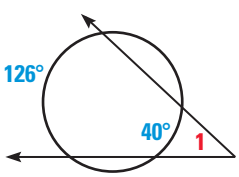
39.



40.



41.



QUIZ 1

Self-Test for Lessons 11.1–11.3

1. Find the sum of the measures of the interior angles of a convex 20-gon. **(Lesson 11.1)**

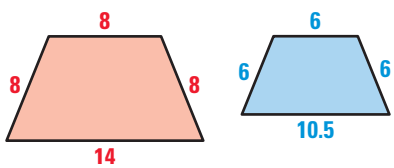
2. What is the measure of each exterior angle of a regular 25-gon? **(Lesson 11.1)**

3. Find the area of an equilateral triangle with a side length of 17 inches. **(Lesson 11.2)**

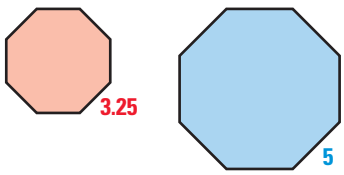
4. Find the area of a regular nonagon with an apothem of 9 centimeters. **(Lesson 11.2)**

In Exercises 5 and 6, the polygons are similar. Find the ratio (red to blue) of their perimeters and of their areas. **(Lesson 11.3)**

5.



6.



7. **CARPET** You just carpeted a 9 foot by 12 foot room for $480. The carpet is priced by the square foot. About how much would you expect to pay for the same carpet in another room that is 21 feet by 28 feet? **(Lesson 11.3)**

MATH & History — History of Approximating Pi

APPLICATION LINK
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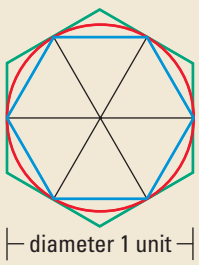
THEN

THOUSANDS OF YEARS AGO, people first noticed that the circumference of a circle is the product of its diameter and a value that is a little more than three. Over time, various methods have been used to find better approximations of this value, called π (*pi*).

1. In the third century B.C., Archimedes approximated the value of π by calculating the perimeters of inscribed and circumscribed regular polygons of a circle with diameter 1 unit. Copy the diagram and follow the steps below to use his method.
   - Find the perimeter of the **inscribed** hexagon in terms of the length of the diameter of the circle.
   - Draw a radius of the **circumscribed** hexagon. Find the length of one side of the hexagon. Then find its perimeter.
   - Write an inequality that approximates the value of π:

$$\text{perimeter of inscribed hexagon} < \pi < \text{perimeter of circumscribed hexagon}$$



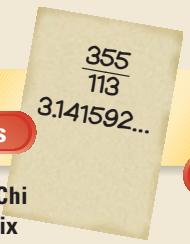
NOW

MATHEMATICIANS use computers to calculate the value of π to billions of decimal places.

200s B.C.

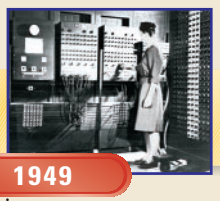
Archimedes uses perimeters of polygons.

A.D. 400s



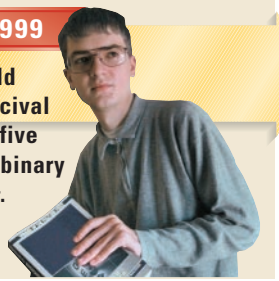
Tsu Chung Chi finds π to six decimal places.

1949



ENIAC computer finds π to 2037 decimal places.

1999



17 year old Colin Percival finds the five trillionth binary digit of π.

11.4 Circumference and Arc Length

What you should learn

GOAL 1 Find the circumference of a circle and the length of a circular arc.

GOAL 2 Use circumference and arc length to solve **real-life** problems such as finding the distance around a track in **Example 5**.

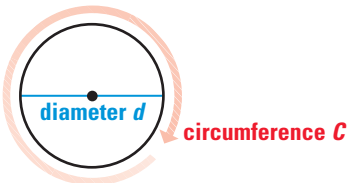
Why you should learn it

▼ To solve **real-life** problems, such as finding the number of revolutions a tire needs to make to travel a given distance in **Example 4** and **Exs. 39–41**.

REAL LIFE

GOAL 1 FINDING CIRCUMFERENCE AND ARC LENGTH

The **circumference** of a circle is the distance around the circle. For all circles, the ratio of the circumference to the diameter is the same. This ratio is known as π, or *pi*.



THEOREM

THEOREM 11.6 ***Circumference of a Circle***

The circumference C of a circle is $C = \pi d$ or $C = 2\pi r$, where d is the diameter of the circle and r is the radius of the circle.

EXAMPLE 1 ***Using Circumference***

Find (**a**) the circumference of a circle with radius 6 centimeters and (**b**) the radius of a circle with circumference 31 meters. Round decimal answers to two decimal places.

SOLUTION

a. $C = 2\pi r$

$= 2 \cdot \pi \cdot 6$

$= 12\pi$ **Use a calculator.**

≈ 37.70

▶ So, the circumference is about 37.70 centimeters.

b. $C = 2\pi r$

$31 = 2\pi r$

$\frac{31}{2\pi} = r$ **Use a calculator.**

$4.93 \approx r$

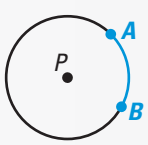
▶ So, the radius is about 4.93 meters.

An **arc length** is a portion of the circumference of a circle. You can use the measure of the arc (in degrees) to find its length (in linear units).

COROLLARY

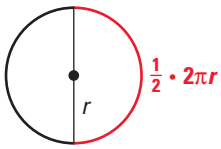
ARC LENGTH COROLLARY

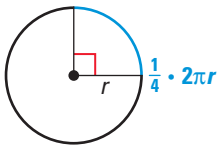
In a circle, the ratio of the length of a given arc to the circumference is equal to the ratio of the measure of the arc to 360°.



$$\frac{\text{Arc length of } \widehat{AB}}{2\pi r} = \frac{m\widehat{AB}}{360°}, \text{ or Arc length of } \widehat{AB} = \frac{m\widehat{AB}}{360°} \cdot 2\pi r$$

The length of a **semicircle** is one half the circumference, and the length of a **90° arc** is one quarter of the circumference.

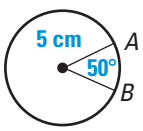




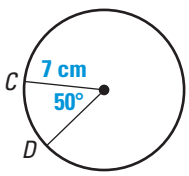
EXAMPLE 2 *Finding Arc Lengths*

Find the length of each arc.

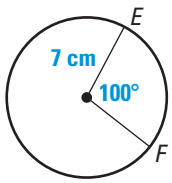
a.



b.



c.



STUDENT HELP

Study Tip

Throughout this chapter, you should use the π key on a calculator, then round decimal answers to two decimal places unless instructed otherwise.

SOLUTION

a. Arc length of $\widehat{AB} = \frac{50°}{360°} \cdot 2\pi(5) \approx 4.36$ centimeters

b. Arc length of $\widehat{CD} = \frac{50°}{360°} \cdot 2\pi(7) \approx 6.11$ centimeters

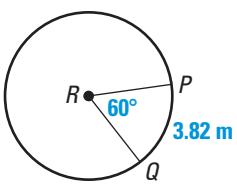
c. Arc length of $\widehat{EF} = \frac{100°}{360°} \cdot 2\pi(7) \approx 12.22$ centimeters

In parts (a) and (b) in Example 2, note that the arcs have the same measure, but different lengths because the circumferences of the circles are not equal.

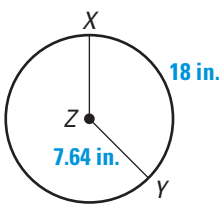
EXAMPLE 3 *Using Arc Lengths*

Find the indicated measure.

a. Circumference



b. $m\widehat{XY}$



SOLUTION

a.
$$\frac{\text{Arc length of } \widehat{PQ}}{2\pi r} = \frac{m\widehat{PQ}}{360°}$$
$$\frac{3.82}{2\pi r} = \frac{60°}{360°}$$
$$\frac{3.82}{2\pi r} = \frac{1}{6}$$
$$3.82(6) = 2\pi r$$
$$22.92 = 2\pi r$$

▶ So, $C = 2\pi r \approx 22.92$ meters.

b.
$$\frac{\text{Arc length of } \widehat{XY}}{2\pi r} = \frac{m\widehat{XY}}{360°}$$
$$\frac{18}{2\pi(7.64)} = \frac{m\widehat{XY}}{360°}$$
$$360° \cdot \frac{18}{2\pi(7.64)} = m\widehat{XY}$$
$$135° \approx m\widehat{XY}$$

▶ So, $m\widehat{XY} \approx 135°$.

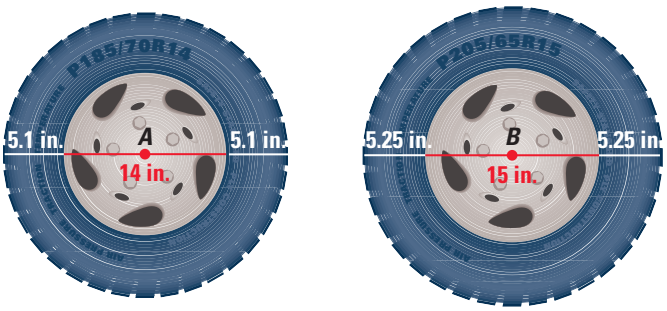
STUDENT HELP

INTERNET **HOMEWORK HELP**
Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 CIRCUMFERENCE CIRCUMFERENCES

EXAMPLE 4 *Comparing Circumferences*

TIRE REVOLUTIONS Tires from two different automobiles are shown below. How many revolutions does each tire make while traveling 100 feet? Round decimal answers to one decimal place.



SOLUTION

Tire A has a diameter of $14 + 2(5.1)$, or 24.2 inches. Its circumference is $\pi(24.2)$, or about 76.03 inches.

Tire B has a diameter of $15 + 2(5.25)$, or 25.5 inches. Its circumference is $\pi(25.5)$, or about 80.11 inches.

Divide the distance traveled by the tire circumference to find the number of revolutions made. First convert 100 feet to 1200 inches.

Tire A: $\dfrac{100 \text{ ft}}{76.03 \text{ in.}} = \dfrac{1200 \text{ in.}}{76.03 \text{ in.}}$

≈ 15.8 revolutions

Tire B: $\dfrac{100 \text{ ft}}{80.11 \text{ in.}} = \dfrac{1200 \text{ in.}}{80.11 \text{ in.}}$

≈ 15.0 revolutions

FOCUS ON PEOPLE

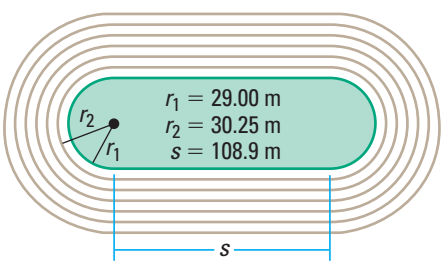
JACOB HEILVEIL was born in Korea and now lives in the United States. He was the top American finisher in the 10,000 meter race at the 1996 Paralympics held in Atlanta, Georgia.

EXAMPLE 5 *Finding Arc Length*

TRACK The track shown has six lanes. Each lane is 1.25 meters wide. There is a 180° arc at each end of the track. The radii for the arcs in the first two lanes are given.

a. Find the distance around Lane 1.

b. Find the distance around Lane 2.



SOLUTION

The track is made up of two semicircles and two straight sections with length s. To find the total distance around each lane, find the sum of the lengths of each part. Round decimal answers to one decimal place.

a. Distance $= 2s + 2\pi r_1$

$= 2(108.9) + 2\pi(29.00)$

≈ 400.0 meters

b. Distance $= 2s + 2\pi r_2$

$= 2(108.9) + 2\pi(30.25)$

≈ 407.9 meters

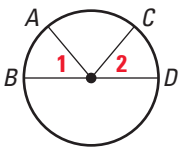
GUIDED PRACTICE

Vocabulary Check ✓

1. What is the difference between *arc measure* and *arc length*?

Concept Check ✓

2. In the diagram, $\overline{BD}$ is a diameter and $\angle 1 \cong \angle 2$. Explain why $\widehat{AB}$ and $\widehat{CD}$ have the same length.

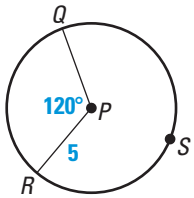


Skill Check ✓

In Exercises 3–8, match the measure with its value.

A. $\frac{10}{3}\pi$ **B.** 10π **C.** $\frac{20}{3}\pi$

D. 10 **E.** 5π **F.** $120°$



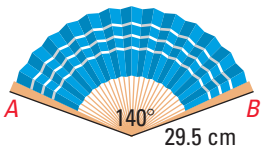
3. $m\widehat{QR}$
4. Diameter of $\odot P$
5. Length of $\widehat{QSR}$
6. Circumference of $\odot P$
7. Length of $\widehat{QR}$
8. Length of semicircle of $\odot P$

Is the statement *true* or *false*? If it is false, provide a counterexample.

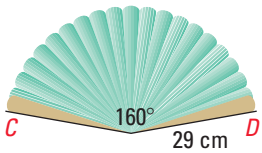
9. Two arcs with the same measure have the same length.
10. If the radius of a circle is doubled, its circumference is multiplied by 4.
11. Two arcs with the same length have the same measure.

FANS Find the indicated measure.

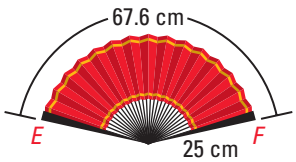
12. Length of $\widehat{AB}$



13. Length of $\widehat{CD}$



14. $m\widehat{EF}$



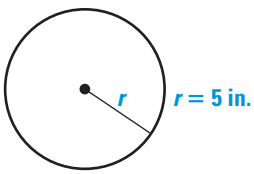
PRACTICE AND APPLICATIONS

STUDENT HELP

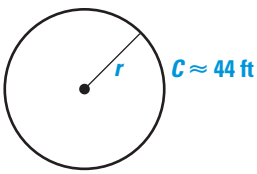
Extra Practice to help you master skills is on p. 824.

USING CIRCUMFERENCE In Exercises 15 and 16, find the indicated measure.

15. Circumference



16. Radius



17. Find the circumference of a circle with diameter 8 meters.
18. Find the circumference of a circle with radius 15 inches. (Leave your answer in terms of π.)
19. Find the radius of a circle with circumference 32 yards.

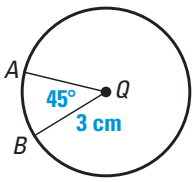
STUDENT HELP

HOMEWORK HELP

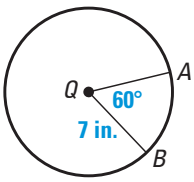
Example 1: Exs. 15–19
Example 2: Exs. 20–23
Example 3: Exs. 24–29
Example 4: Exs. 39–41
Example 5: Exs. 42–46

FINDING ARC LENGTHS **In Exercises 20–22, find the length of $\widehat{AB}$.**

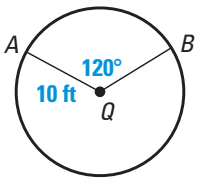
20.



21.



22.

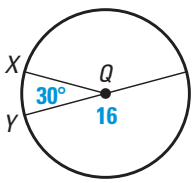


23. FINDING VALUES Complete the table.

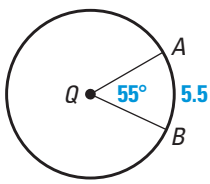
Radius	?	3	0.6	3.5	?	$3\sqrt{3}$
$m\widehat{AB}$	45°	30°	?	192°	90°	?
Length of $\widehat{AB}$	3π	?	0.4π	?	2.55π	3.09π

FINDING MEASURES **Find the indicated measure.**

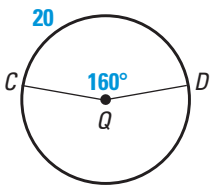
24. Length of $\widehat{XY}$



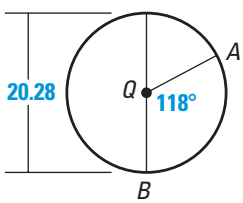
25. Circumference



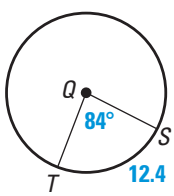
26. Radius



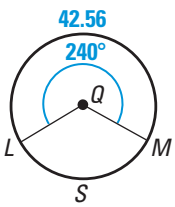
27. Length of $\widehat{AB}$



28. Circumference

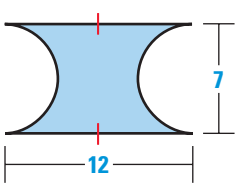


29. Radius

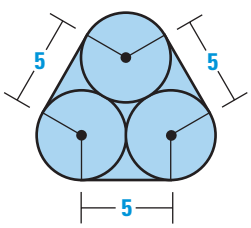


CALCULATING PERIMETERS **In Exercises 30–32, the region is bounded by circular arcs and line segments. Find the perimeter of the region.**

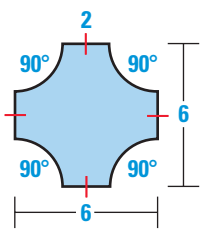
30.



31.

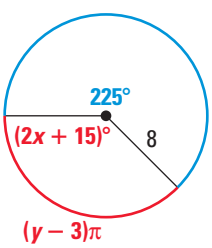


32.

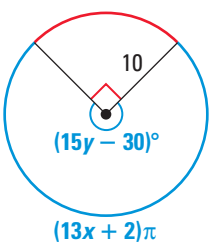


USING ALGEBRA **Find the values of *x* and *y*.**

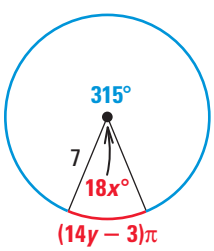
33.



34.



35.



USING ALGEBRA **Find the circumference of the circle whose equation is given. (Leave your answer in terms of π.)**

36. $x^2 + y^2 = 9$ **37.** $x^2 + y^2 = 28$ **38.** $(x + 1)^2 + (y - 5)^2 = 4$

AUTOMOBILE TIRES **In Exercises 39–41, use the table below. The table gives the rim diameters and sidewall widths of three automobile tires.**

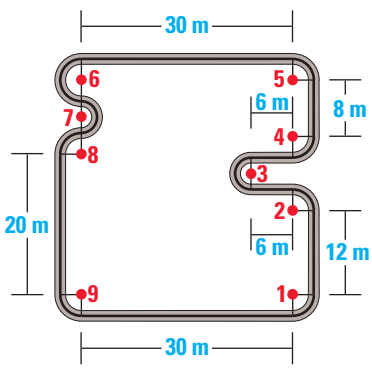
	Rim diameter	Sidewall width
Tire A	15 in.	4.60 in.
Tire B	16 in.	4.43 in.
Tire C	17 in.	4.33 in.

39. Find the diameter of each automobile tire.

40. How many revolutions does each tire make while traveling 500 feet?

41. A student determines that the circumference of a tire with a rim diameter of 15 inches and a sidewall width of 5.5 inches is 64.40 inches. Explain the error.

GO-CART TRACK **Use the diagram of the go-cart track for Exercises 42 and 43. Turns 1, 2, 4, 5, 6, 8, and 9 all have a radius of 3 meters. Turns 3 and 7 each have a radius of 2.25 meters.**



42. Calculate the length of the track.

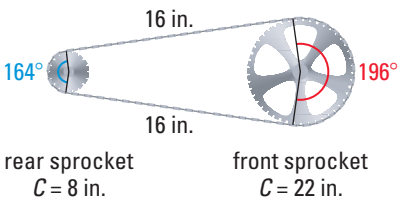
43. How many laps do you need to make to travel 1609 meters (about 1 mile)?

44. **MOUNT RAINIER** In Example 5 on page 623 of Lesson 10.4, you calculated the measure of the arc of Earth's surface seen from the top of Mount Rainier. Use that information to calculate the distance in miles that can be seen looking in one direction from the top of Mount Rainier.

FOCUS ON APPLICATIONS

REAL LIFE **MT. RAINIER**, at 14,410 ft high, is the tallest mountain in Washington State.

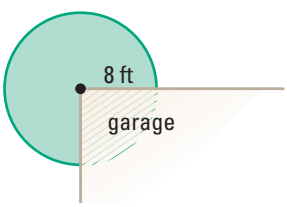
BICYCLES **Use the diagram of a bicycle chain for a fixed gear bicycle in Exercises 45 and 46.**



45. The chain travels along the front and rear sprockets. The circumference of each sprocket is given. About how long is the chain?

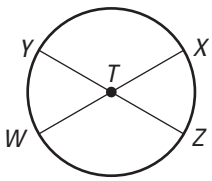
46. On a chain, the teeth are spaced in $\frac{1}{2}$ inch intervals. How many teeth are there on this chain?

47. **ENCLOSING A GARDEN** Suppose you have planted a circular garden adjacent to one of the corners of your garage, as shown at the right. If you want to fence in your garden, how much fencing do you need?



Test Preparation

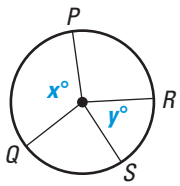
48. MULTIPLE CHOICE In the diagram shown, $\overline{YZ}$ and $\overline{WX}$ each measure 8 units and are diameters of $\odot T$. If $\widehat{YX}$ measures 120°, what is the length of $\widehat{XZ}$?



Ⓐ $\frac{2}{3}\pi$ Ⓑ $\frac{4}{3}\pi$ Ⓒ $\frac{8}{3}\pi$

Ⓓ 4π Ⓔ 8π

49. MULTIPLE CHOICE In the diagram shown, the ratio of the length of $\widehat{PQ}$ to the length of $\widehat{RS}$ is 2 to 1. What is the ratio of x to y?



Ⓐ 4 to 1 Ⓑ 2 to 1 Ⓒ 1 to 1

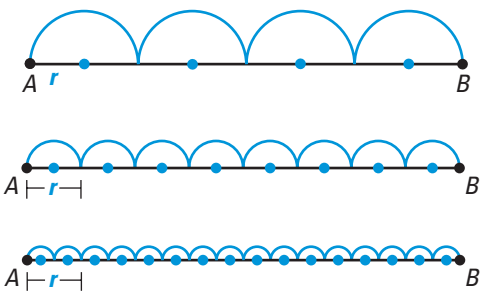
Ⓓ 1 to 2 Ⓔ 1 to 4

★ Challenge

CALCULATING ARC LENGTHS Suppose $\overline{AB}$ is divided into four congruent segments and semicircles with radius r are drawn.

50. What is the sum of the four arc lengths if the radius of each arc is r?

51. Imagine that $\overline{AB}$ is divided into n congruent segments and that semicircles are drawn. What would the sum of the arc lengths be for 8 segments? 16 segments? n segments? Does the number of segments matter?



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

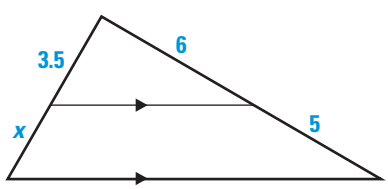
FINDING AREA In Exercises 52–55, the radius of a circle is given. Use the formula $A = \pi r^2$ to calculate the area of the circle. (Review 1.7 for 11.5)

52. $r = 9$ ft **53.** $r = 3.3$ in. **54.** $r = \frac{27}{5}$ cm **55.** $r = 4\sqrt{11}$ m

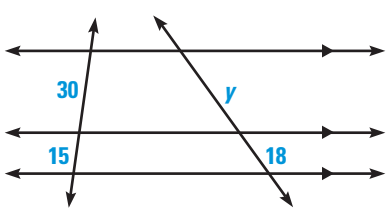
56. USING ALGEBRA Line n_1 has the equation $y = \frac{2}{3}x + 8$. Line n_2 is parallel to n_1 and passes through the point $(9, -2)$. Write an equation for n_2. (Review 3.6)

USING PROPORTIONALITY THEOREMS In Exercises 57 and 58, find the value of the variable. (Review 8.6)

57.



58.



CALCULATING ARC MEASURES You are given the measure of an inscribed angle of a circle. Find the measure of its intercepted arc. (Review 10.3)

59. 48° **60.** 88° **61.** 129° **62.** 15.5°

11.5 Areas of Circles and Sectors

What you should learn

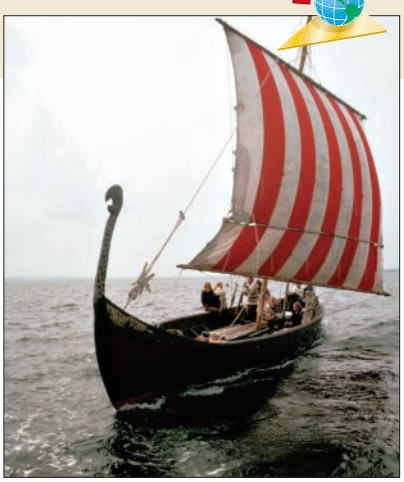
GOAL 1 Find the area of a circle and a sector of a circle.

GOAL 2 Use areas of circles and sectors to solve **real-life** problems, such as finding the area of a boomerang in **Example 6**.

Why you should learn it

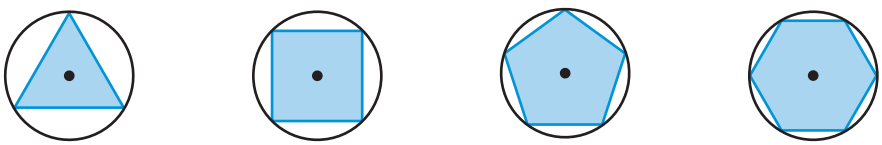
▼ To solve **real-life** problems, such as finding the area of portions of tree trunks that are used to build Viking ships in **Exs. 38 and 39**.

REAL LIFE



GOAL 1 AREAS OF CIRCLES AND SECTORS

The diagrams below show regular polygons inscribed in circles with radius r. Exercise 42 on page 697 demonstrates that as the number of sides increases, the area of the polygon approaches the value πr^2.

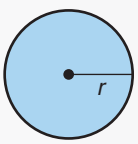


3-gon 4-gon 5-gon 6-gon

THEOREM

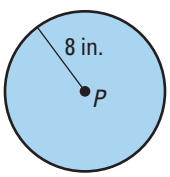
THEOREM 11.7 ***Area of a Circle***

The area of a circle is π times the square of the radius, or $A = \pi r^2$.

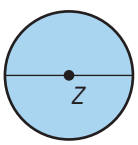


EXAMPLE 1 ***Using the Area of a Circle***

a. Find the area of $\odot P$.



b. Find the diameter of $\odot Z$.



Area of $\odot Z = 96$ cm^2

SOLUTION

a. Use $r = 8$ in the area formula.

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \cdot 8^2 \\ &= 64\pi \\ &\approx 201.06 \end{aligned}$$

▶ So, the area is 64π, or about 201.06, square inches.

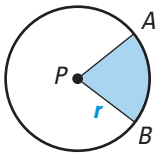
b. The diameter is twice the radius.

$$\begin{aligned} A &= \pi r^2 \\ 96 &= \pi r^2 \\ \frac{96}{\pi} &= r^2 \\ 30.56 &\approx r^2 \\ 5.53 &\approx r \end{aligned}$$

Find the square roots.

▶ The diameter of the circle is about 2(5.53), or about 11.06, centimeters.

A **sector of a circle** is the region bounded by two radii of the circle and their intercepted arc. In the diagram, sector APB is bounded by $\overline{AP}$, $\overline{BP}$, and $\widehat{AB}$. The following theorem gives a method for finding the area of a sector.



THEOREM

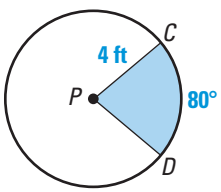
THEOREM 11.8 ***Area of a Sector***

The ratio of the area A of a sector of a circle to the area of the circle is equal to the ratio of the measure of the intercepted arc to 360°.

$$\frac{A}{\pi r^2} = \frac{m\widehat{AB}}{360°}, \text{ or } A = \frac{m\widehat{AB}}{360°} \cdot \pi r^2$$

EXAMPLE 2 *Finding the Area of a Sector*

Find the area of the sector shown at the right.



SOLUTION

Sector CPD intercepts an arc whose measure is 80°. The radius is 4 feet.

$$A = \frac{m\widehat{CD}}{360°} \cdot \pi r^2 \qquad \textbf{Write the formula for the area of a sector.}$$

$$= \frac{80°}{360°} \cdot \pi \cdot 4^2 \qquad \textbf{Substitute known values.}$$

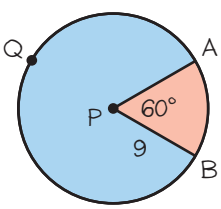
$$\approx 11.17 \qquad \textbf{Use a calculator.}$$

▶ So, the area of the sector is about 11.17 square feet.

EXAMPLE 3 *Finding the Area of a Sector*

A and B are two points on a $\odot P$ with radius 9 inches and $m\angle APB = 60°$. Find the areas of the sectors formed by $\angle APB$.

SOLUTION



Draw a diagram of $\odot P$ and $\angle APB$. Shade the sectors.

Label a point Q on the major arc.

Find the measures of the minor and major arcs.

Because $m\angle APB = 60°$, $m\widehat{AB} = 60°$ and $m\widehat{AQB} = 360° - 60° = 300°$.

Use the formula for the area of a sector.

$$\text{Area of small sector} = \frac{60°}{360°} \cdot \pi \cdot 9^2 = \frac{1}{6} \cdot \pi \cdot 81 \approx 42.41 \text{ square inches}$$

$$\text{Area of larger sector} = \frac{300°}{360°} \cdot \pi \cdot 9^2 = \frac{5}{6} \cdot \pi \cdot 81 \approx 212.06 \text{ square inches}$$

STUDENT HELP

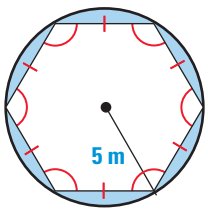
INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 USING AREAS OF CIRCLES AND REGIONS

You may need to divide a figure into different regions to find its area. The regions may be polygons, circles, or sectors. To find the area of the entire figure, add or subtract the areas of the separate regions as appropriate.

EXAMPLE 4 *Finding the Area of a Region*

Find the area of the shaded region shown at the right.



SOLUTION

The diagram shows a regular hexagon inscribed in a circle with radius 5 meters. The shaded region is the part of the circle that is outside of the hexagon.

$$\boxed{\text{Area of shaded region}} = \boxed{\text{Area of circle}} - \boxed{\text{Area of hexagon}}$$

$$= \pi r^2 - \frac{1}{2}aP$$

$$= \pi \cdot 5^2 - \frac{1}{2} \cdot \left(\frac{5}{2}\sqrt{3}\right) \cdot (6 \cdot 5)$$

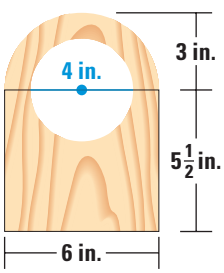
The apothem of a hexagon is $\frac{1}{2} \cdot$ side length $\cdot \sqrt{3}$.

$$= 25\pi - \frac{75}{2}\sqrt{3}$$

▶ So, the area of the shaded region is $25\pi - \frac{75}{2}\sqrt{3}$, or about 13.59 square meters.

EXAMPLE 5 *Finding the Area of a Region*

REAL LIFE **WOODWORKING** You are cutting the front face of a clock out of wood, as shown in the diagram. What is the area of the front of the case?



SOLUTION

The front of the case is formed by a rectangle and a sector, with a circle removed. Note that the intercepted arc of the sector is a semicircle.

$$\boxed{\text{Area}} = \boxed{\text{Area of rectangle}} + \boxed{\text{Area of sector}} - \boxed{\text{Area of circle}}$$

$$= 6 \cdot \frac{11}{2} + \frac{180°}{360°} \cdot \pi \cdot 3^2 - \pi \cdot \left(\frac{1}{2} \cdot 4\right)^2$$

$$= 33 + \frac{1}{2} \cdot \pi \cdot 9 - \pi \cdot (2)^2$$

$$= 33 + \frac{9}{2}\pi - 4\pi$$

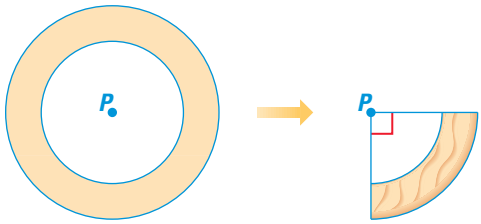
$$\approx 34.57$$

▶ The area of the front of the case is about 34.57 square inches.

FOCUS ON APPLICATIONS

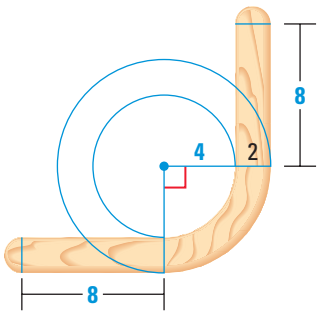
BOOMERANGS are slightly curved at the ends and travel in an arc when thrown. Small boomerangs used for sport make a full circle and return to the thrower.

Complicated shapes may involve a number of regions. In Example 6, the curved region is a portion of a ring whose edges are formed by concentric circles. Notice that the area of a portion of the ring is the difference of the areas of two sectors.



EXAMPLE 6 *Finding the Area of a Boomerang*

BOOMERANGS Find the area of the boomerang shown. The dimensions are given in inches. Give your answer in terms of π and to two decimal places.

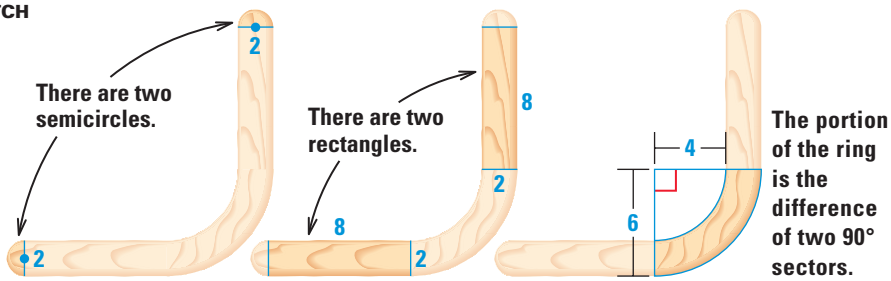


SOLUTION

Separate the boomerang into different regions. The regions are two semicircles (at the ends), two rectangles, and a portion of a ring. Find the area of each region and add these areas together.

PROBLEM SOLVING STRATEGY

DRAW AND LABEL A SKETCH Draw and label a sketch of each region in the boomerang.



VERBAL MODEL

$$\boxed{\text{Area of boomerang}} = 2 \cdot \boxed{\text{Area of semicircle}} + 2 \cdot \boxed{\text{Area of rectangle}} + \boxed{\text{Area of portion of ring}}$$

LABELS

Area of semicircle $= \frac{1}{2} \cdot \pi \cdot 1^2$ (square inches)

Area of rectangle $= 8 \cdot 2$ (square inches)

Area of portion of ring $= \frac{1}{4} \cdot \pi \cdot 6^2 - \frac{1}{4} \cdot \pi \cdot 4^2$ (square inches)

REASONING

$$\begin{aligned}\boxed{\text{Area of boomerang}} &= 2\left(\frac{1}{2} \cdot \pi \cdot 1^2\right) + 2(8 \cdot 2) + \left(\frac{1}{4} \cdot \pi \cdot 6^2 - \frac{1}{4} \cdot \pi \cdot 4^2\right)\\ &= 2\left(\frac{1}{2} \cdot \pi \cdot 1\right) + 2 \cdot 16 + \left(\frac{1}{4} \cdot \pi \cdot 36 - \frac{1}{4} \cdot \pi \cdot 16\right)\\ &= \pi + 32 + (9\pi - 4\pi)\\ &= 6\pi + 32\end{aligned}$$

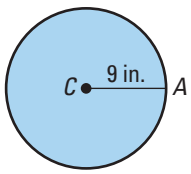
▶ So, the area of the boomerang is $(6\pi + 32)$, or about 50.85 square inches.

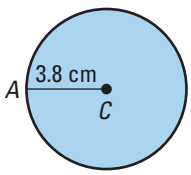
GUIDED PRACTICE

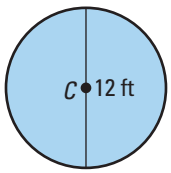
Vocabulary Check ✓ 1. Describe the boundaries of a *sector of a circle*.

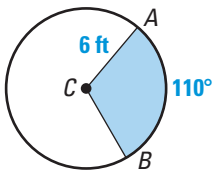
Concept Check ✓ 2. In Example 5 on page 693, explain why the expression $\pi \cdot \left(\frac{1}{2} \cdot 4\right)^2$ represents the area of the circle cut from the wood.

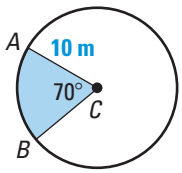
Skill Check ✓ **In Exercises 3–8, find the area of the shaded region.**

3.


4.


5.


6.


7.


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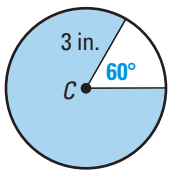
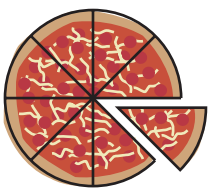


9. **PIECES OF PIZZA** Suppose the pizza shown is divided into 8 equal pieces. The diameter of the pizza is 16 inches. What is the area of one piece of pizza?

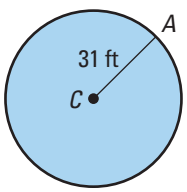


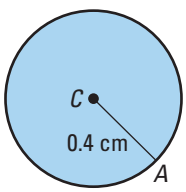
PRACTICE AND APPLICATIONS

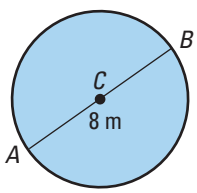
STUDENT HELP

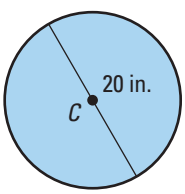
Extra Practice to help you master skills is on p. 824.

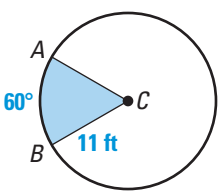
FINDING AREA **In Exercises 10–18, find the area of the shaded region.**

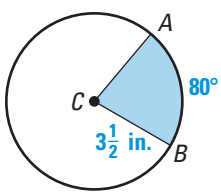
10.


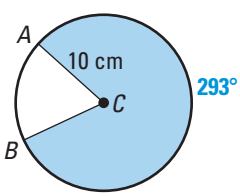
11.


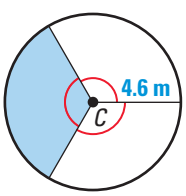
12.


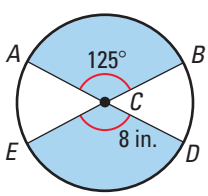
13.


14.


15.


16.


17.


18.


STUDENT HELP

HOMEWORK HELP

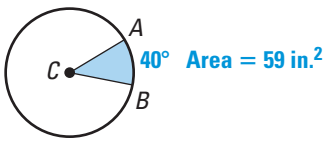
Example 1: Exs. 10–13, 19, 20
Example 2: Exs. 14–18, 21, 22, 29
Example 3: Exs. 14–18, 21, 22, 29
Example 4: Exs. 23–28, 35–37
Example 5: Exs. 23–28, 35–37
Example 6: Exs. 38–40

19. **USING AREA** What is the area of a circle with diameter 20 feet?

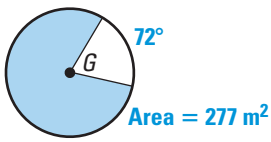
20. **USING AREA** What is the radius of a circle with area 50 square meters?

USING AREA **Find the indicated measure. The area given next to the diagram refers to the shaded region only.**

21. Find the radius of $\odot C$.

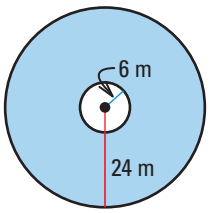


22. Find the diameter of $\odot G$.

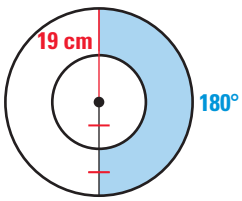


FINDING AREA **Find the area of the shaded region.**

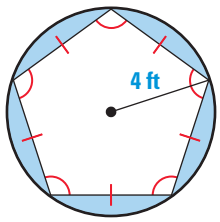
23.



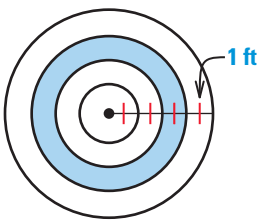
24.



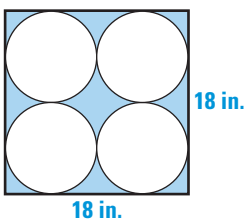
25.



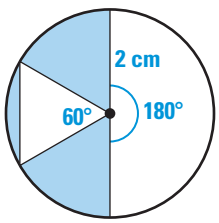
26.



27.



28.



FINDING A PATTERN **In Exercises 29–32, consider an arc of a circle with a radius of 3 inches.**

29. Copy and complete the table. Round your answers to the nearest tenth.

Measure of arc, x	30°	60°	90°	120°	150°	180°
Area of corresponding sector, y	?	?	?	?	?	?

30. **USING ALGEBRA** Graph the data in the table.

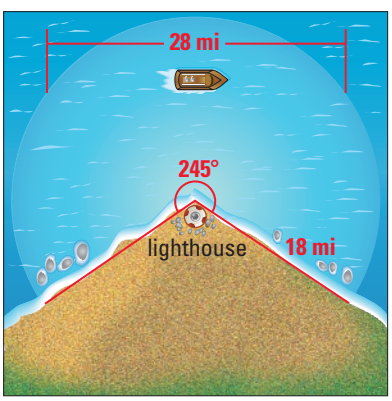
31. **USING ALGEBRA** Is the relationship between x and y linear? Explain.

32. **LOGICAL REASONING** If Exercises 29–31 were repeated using a circle with a 5 inch radius, would the areas in the table change? Would your answer to Exercise 31 change? Explain your reasoning.

FOCUS ON APPLICATIONS

REAL LIFE **LIGHTHOUSES** use special lenses that increase the intensity of the light projected. Some lenses are 8 feet high and 6 feet in diameter.

LIGHTHOUSES **The diagram shows a projected beam of light from a lighthouse.**

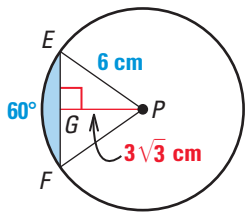


33. What is the area of water that can be covered by the light from the lighthouse?

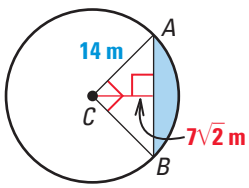
34. Suppose a boat traveling along a straight line is illuminated by the lighthouse for approximately 28 miles of its route. What is the closest distance between the lighthouse and the boat?

USING AREA **In Exercises 35–37, find the area of the shaded region in the circle formed by a chord and its intercepted arc. (*Hint:* Find the difference between the areas of a sector and a triangle.)**

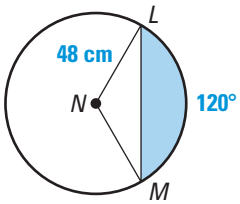
35.



36.



37.



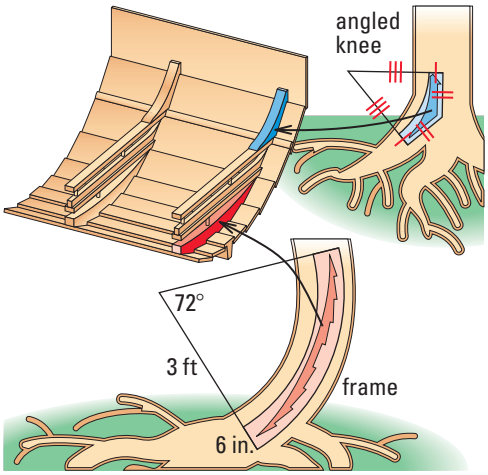
FOCUS ON APPLICATIONS

VIKING LONGSHIPS
The planks in the hull of a longship were cut in a radial pattern from a single green log, providing uniform resiliency and strength.

APPLICATION LINK
www.mcdougallittell.com

VIKING LONGSHIPS **Use the information below for Exercises 38 and 39.**

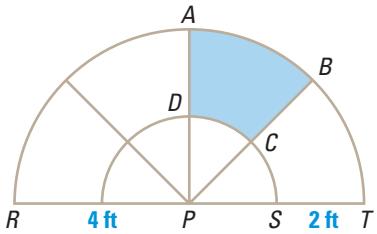
When Vikings constructed *longships*, they cut hull-hugging frames from curved trees. Straight trees provided angled knees, which were used to brace the frames.



38. Find the area of a cross-section of the frame piece shown in red.

39. *Writing* The angled knee piece shown in blue has a cross section whose shape results from subtracting a sector from a kite. What measurements would you need to know to find its area?

40. **WINDOW DESIGN** The window shown is in the shape of a semicircle with radius 4 feet. The distance from S to T is 2 feet, and the measure of $\widehat{AB}$ is 45°. Find the area of the glass in the region $ABCD$.

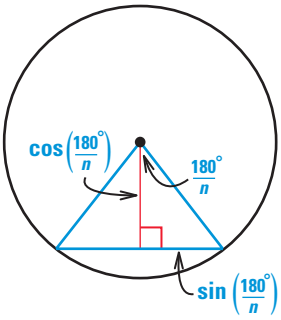


41. **LOGICAL REASONING** Suppose a circle has a radius of 4.5 inches. If you double the radius of the circle, does the area of the circle double as well? What happens to the circle's circumference? Explain.

42. **TECHNOLOGY** The area of a regular n-gon inscribed in a circle with radius 1 unit can be written as

$$A = \frac{1}{2}\left(\cos\left(\frac{180°}{n}\right)\right)\left(2n \cdot \sin\left(\frac{180°}{n}\right)\right).$$

Use a spreadsheet to make a table. The first column is for the number of sides n and the second column is for the area of the n-gon. Fill in your table up to a 16-gon. What do you notice as n gets larger and larger?

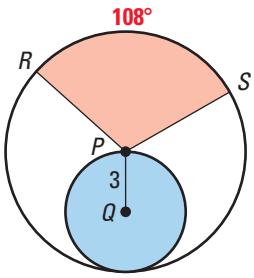


STUDENT HELP

Look Back
to Activity 11.4 on p. 690 for help with spreadsheets.

Test Preparation

⊙*Q* and ⊙*P* are tangent. Use the diagram for Exercises 43 and 44.



43. MULTIPLE CHOICE If ⊙Q is cut away, what is the remaining area of ⊙P?

Ⓐ 6π Ⓑ 9π Ⓒ 27π

Ⓓ 60π Ⓔ 180π

44. MULTIPLE CHOICE What is the area of the region shaded in red?

Ⓐ 0.3 Ⓑ 1.8π Ⓒ 6π

Ⓓ 10.8π Ⓔ 108π

★ Challenge

EXTRA CHALLENGE

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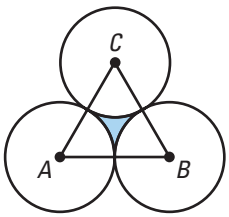
45. FINDING AREA Find the area between the three congruent tangent circles. The radius of each circle is 6 centimeters. (*Hint:* $\triangle ABC$ is equilateral.)



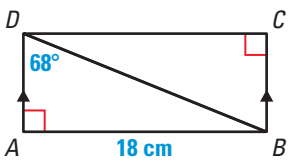
MIXED REVIEW

SIMPLIFYING RATIOS In Exercises 46–49, simplify the ratio. (Review 8.1 for 11.6)

46. $\frac{8 \text{ cats}}{20 \text{ cats}}$ **47.** $\frac{6 \text{ teachers}}{32 \text{ teachers}}$ **48.** $\frac{12 \text{ inches}}{63 \text{ inches}}$ **49.** $\frac{52 \text{ weeks}}{143 \text{ weeks}}$

50. The length of the diagonal of a square is 30. What is the length of each side? (Review 9.4)

FINDING MEASURES Use the diagram to find the indicated measure. Round decimals to the nearest tenth. (Review 9.6)



51. BD **52.** DC

53. $m\angle DBC$ **54.** BC

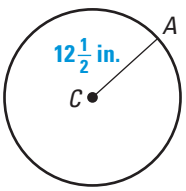
WRITING EQUATIONS Write the standard equation of the circle with the given center and radius. (Review 10.6)

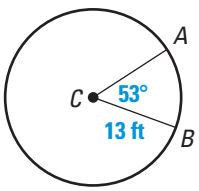
55. center $(-2, -7)$, radius 6 **56.** center $(0, -9)$, radius 10

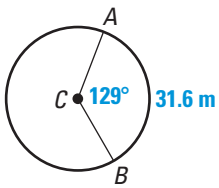
57. center $(-4, 5)$, radius 3.2 **58.** center $(8, 2)$, radius $\sqrt{11}$

FINDING MEASURES Find the indicated measure. (Review 11.4)

59. Circumference **60.** Length of $\widehat{AB}$ **61.** Radius







11.6 Geometric Probability

What you should learn

GOAL 1 Find a geometric probability.

GOAL 2 Use geometric probability to solve **real-life** problems, as applied in **Example 2**.

Why you should learn it

▼ Geometric probability is one model for calculating **real-life** probabilities, such as the probability that a bus will be waiting outside a hotel in **Ex. 28**.

REAL LIFE

GOAL 1 FINDING A GEOMETRIC PROBABILITY

A **probability** is a number from 0 to 1 that represents the chance that an event will occur. Assuming that all outcomes are equally likely, an event with a probability of 0 *cannot* occur. An event with a probability of 1 is *certain* to occur, and an event with a probability of 0.5 is just as likely to occur as not.

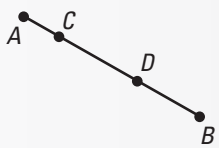
In an earlier course, you may have evaluated probabilities by counting the number of favorable outcomes and dividing that number by the total number of possible outcomes. In this lesson, you will use a related process in which the division involves geometric measures such as length or area. This process is called **geometric probability**.

GEOMETRIC PROBABILITY

PROBABILITY AND LENGTH

Let $\overline{AB}$ be a segment that contains the segment $\overline{CD}$. If a point K on $\overline{AB}$ is chosen at random, then the probability that it is on $\overline{CD}$ is as follows:

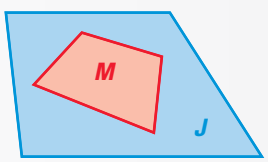
$$P(\text{Point } K \text{ is on } \overline{CD}) = \frac{\text{Length of } \overline{CD}}{\text{Length of } \overline{AB}}$$



PROBABILITY AND AREA

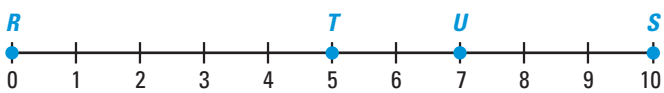
Let J be a region that contains region M. If a point K in J is chosen at random, then the probability that it is in region M is as follows:

$$P(\text{Point } K \text{ is in region } M) = \frac{\text{Area of } M}{\text{Area of } J}$$



EXAMPLE 1 *Finding a Geometric Probability*

Find the probability that a point chosen at random on $\overline{RS}$ is on $\overline{TU}$.



SOLUTION

$$P(\text{Point is on } \overline{TU}) = \frac{\text{Length of } \overline{TU}}{\text{Length of } \overline{RS}} = \frac{2}{10} = \frac{1}{5}$$

▶ The probability can be written as $\frac{1}{5}$, 0.2, or 20%.

STUDENT HELP

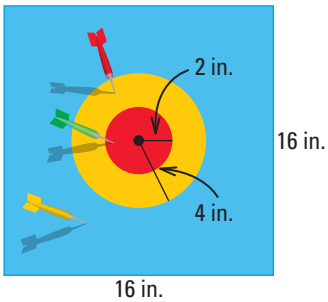
Study Tip

When applying a formula for geometric probability, it is important that every point on the segment or in the region is equally likely to be chosen.

GOAL 2 USING GEOMETRIC PROBABILITY IN REAL LIFE

EXAMPLE 2 *Using Areas to Find a Geometric Probability*

REAL LIFE **DART BOARD** A dart is tossed and hits the dart board shown. The dart is equally likely to land on any point on the dart board. Find the probability that the dart lands in the red region.



SOLUTION

Find the ratio of the area of the red region to the area of the dart board.

$$P(\text{Dart lands in red region}) = \frac{\text{Area of red region}}{\text{Area of dart board}}$$

$$= \frac{\pi(2^2)}{16^2}$$

$$= \frac{4\pi}{256}$$

$$\approx 0.05$$

▶ The probability that the dart lands in the red region is about 0.05, or 5%.

EXAMPLE 3 *Using a Segment to Find a Geometric Probability*

REAL LIFE **TRANSPORTATION** You are visiting San Francisco and are taking a trolley ride to a store on Market Street. You are supposed to meet a friend at the store at 3:00 P.M. The trolleys run every 10 minutes and the trip to the store is 8 minutes. You arrive at the trolley stop at 2:48 P.M. What is the probability that you will arrive at the store by 3:00 P.M.?

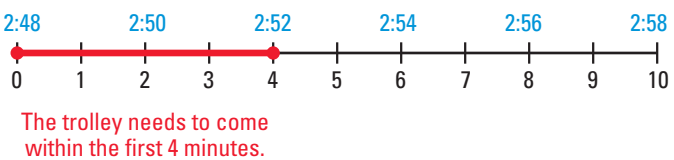
Logical Reasoning

SOLUTION

To begin, find the greatest amount of time you can afford to wait for the trolley and still get to the store by 3:00 P.M.

Because the ride takes 8 minutes, you need to catch the trolley no later than 8 minutes before 3:00 P.M., or in other words by 2:52 P.M.

So, you can afford to wait 4 minutes (2:52 − 2:48 = 4 min). You can use a line segment to model the probability that the trolley will come within 4 minutes.



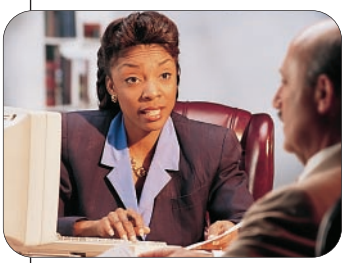
$$P(\text{Get to store by 3:00}) = \frac{\text{Favorable waiting time}}{\text{Maximum waiting time}} = \frac{4}{10} = \frac{2}{5}$$

▶ The probability is $\frac{2}{5}$, or 40%.

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

FOCUS ON CAREERS

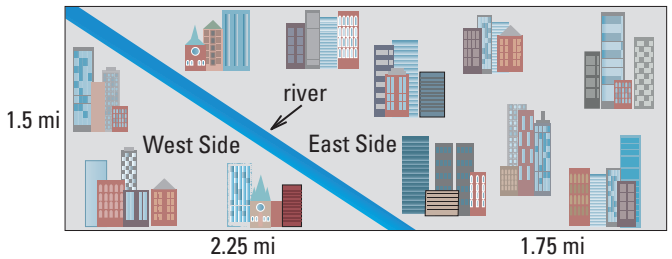


EMPLOYMENT COUNSELORS help people make decisions about career choices. A counselor evaluates a client's interests and skills and works with the client to locate and apply for appropriate jobs.

CAREER LINK
www.mcdougallittell.com

EXAMPLE 4 *Finding a Geometric Probability*

JOB LOCATION You work for a temporary employment agency. You live on the west side of town and prefer to work there. The work assignments are spread evenly throughout the rectangular region shown. Find the probability that an assignment chosen at random for you is on the west side of town.



SOLUTION

The west side of town is approximately triangular. Its area is $\frac{1}{2} \cdot 2.25 \cdot 1.5$, or about 1.69 square miles. The area of the rectangular region is $1.5 \cdot 4$, or 6 square miles. So, the probability that the assignment is on the west side of town is

$$P(\text{Assignment is on west side}) = \frac{\text{Area of west side}}{\text{Area of rectangular region}} \approx \frac{1.69}{6} \approx 0.28.$$

▶ So, the probability that the work assignment is on the west side is about 28%.

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain how a *geometric probability* is different from a *probability* found by dividing the number of favorable outcomes by the total number of possible outcomes.

Concept Check ✓

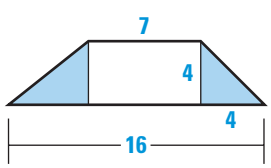
Determine whether you would use the *length method* or *area method* to find the geometric probability. Explain your reasoning.

2. The probability that an outcome lies in a triangular region
3. The probability that an outcome occurs within a certain time period

Skill Check ✓

In Exercises 4–7, *K* is chosen at random on $\overline{AF}$. Find the probability that *K* is on the indicated segment.

4. $\overline{AB}$
5. $\overline{BD}$
6. $\overline{BF}$
7. Explain the relationship between your answers to Exercises 4 and 6.
8. Find the probability that a point chosen at random in the trapezoid shown lies in either of the shaded regions.



PRACTICE AND APPLICATIONS

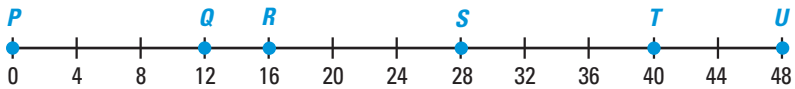
STUDENT HELP

Extra Practice to help you master skills is on p. 824.

PROBABILITY ON A SEGMENT **In Exercises 9–12, find the probability that a point *A*, selected randomly on $\overline{GN}$, is on the given segment.**

9. $\overline{GH}$ **10.** $\overline{JL}$ **11.** $\overline{JN}$ **12.** $\overline{GJ}$

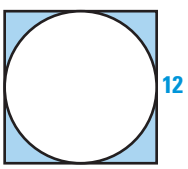
PROBABILITY ON A SEGMENT **In Exercises 13–16, find the probability that a point *K*, selected randomly on $\overline{PU}$, is on the given segment.**



13. $\overline{PQ}$ **14.** $\overline{PS}$ **15.** $\overline{SU}$ **16.** $\overline{PU}$

FINDING A GEOMETRIC PROBABILITY **Find the probability that a randomly chosen point in the figure lies in the shaded region.**

17.



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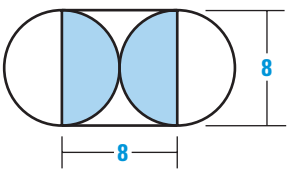

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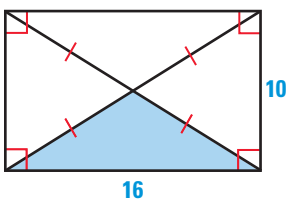

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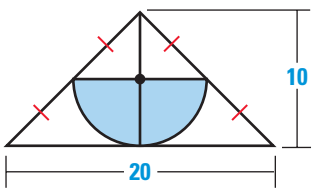
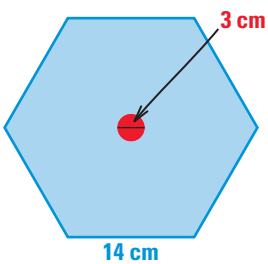


TARGETS **A regular hexagonal shaped target with sides of length 14 centimeters has a circular bull's eye with a diameter of 3 centimeters. In Exercises 21–23, darts are thrown and hit the target at random.**



21. What is the probability that a dart that hits the target will hit the bull's eye?

22. Estimate how many times a dart will hit the bull's eye if 100 darts hit the target.

23. Find the probability that a dart will hit the bull's eye if the bull's eye's radius is doubled.

24. **LOGICAL REASONING** The midpoint of $\overline{JK}$ is M. What is the probability that a randomly selected point on $\overline{JK}$ is closer to M than to J or to K?

25. **LOGICAL REASONING** A circle with radius $\sqrt{2}$ units is circumscribed about a square with side length 2 units. Find the probability that a randomly chosen point will be inside the circle but outside the square.

STUDENT HELP

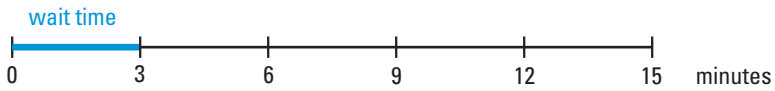
HOMEWORK HELP

Example 1: Exs. 9–16
Example 2: Exs. 17–23, 29–34
Example 3: Exs. 26–28
Example 4: Exs. 40–42

26. **FIRE ALARM** Suppose that your school day begins at 7:30 A.M. and ends at 3:00 P.M. You eat lunch at 11:00 A.M. If there is a fire drill at a random time during the day, what is the probability that it begins before lunch?

27. **PHONE CALL** You are expecting a call from a friend anytime between 6:00 P.M. and 7:00 P.M. Unexpectedly, you have to run an errand for a relative and are gone from 5:45 P.M. until 6:10 P.M. What is the probability that you missed your friend's call?

28. **TRANSPORTATION** Buses arrive at a resort hotel every 15 minutes. They wait for three minutes while passengers get on and get off, and then the buses depart. What is the probability that there is a bus waiting when a hotel guest walks out of the door at a randomly chosen time?

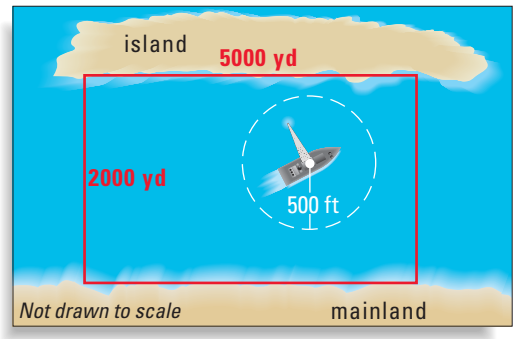


FOCUS ON APPLICATIONS

REAL LIFE SHIP SALVAGE Searchers for sunken items such as ships, planes, or even a space capsule, use charts, sonar, and video cameras in their search and recovery expeditions.

INTERNET APPLICATION LINK
www.mcdougallittell.com

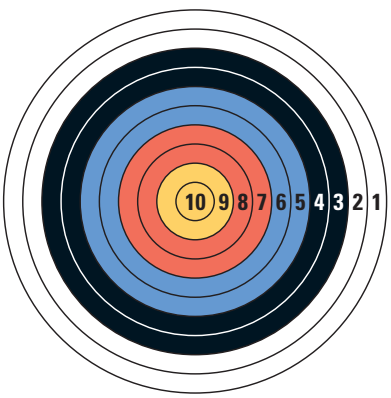
SHIP SALVAGE **In Exercises 29 and 30, use the following information.** A ship is known to have sunk off the coast, between an island and the mainland as shown. A salvage vessel anchors at a random spot in this rectangular region for divers to search for the ship.



29. Find the approximate area of the rectangular region where the ship sank.

30. The divers search 500 feet in all directions from a point on the ocean floor directly below the salvage vessel. Estimate the probability that the divers will find the sunken ship on the first try.

ARCHERY **In Exercises 31–35, use the following information.** Imagine that an arrow hitting the target shown is equally likely to hit any point on the target. The 10-point circle has a 4.8 inch diameter and each of the other rings is 2.4 inches wide. Find the probability that the arrow hits the region described.



31. The 10-point region

32. The yellow region

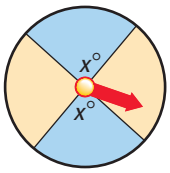
33. The white region

34. The 5-point region

35. **CRITICAL THINKING** Does the geometric probability model hold true when an expert archer shoots an arrow? Explain your reasoning.

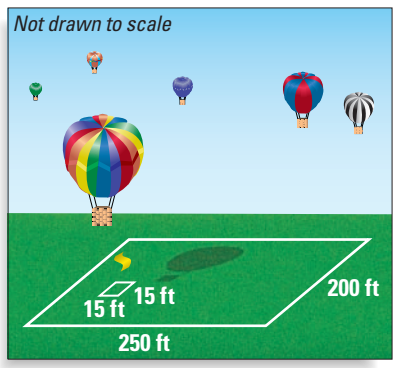
36. **USING ALGEBRA** If $0 < y < 1$ and $0 < x < 1$, find the probability that $y < x$. Begin by sketching the graph, and then use the *area method* to find the probability.

USING ALGEBRA **Find the value of x so that the probability of the spinner landing on a blue sector is the value given.**



37. $\frac{1}{3}$ **38.** $\frac{1}{4}$ **39.** $\frac{1}{6}$

BALLOON RACE **In Exercises 40–42, use the following information.** In a "Hare and Hounds" balloon race, one balloon (the hare) leaves the ground first. About ten minutes later, the other balloons (the hounds) leave. The hare then lands and marks a square region as the target. The hounds each try to drop a marker in the target zone.



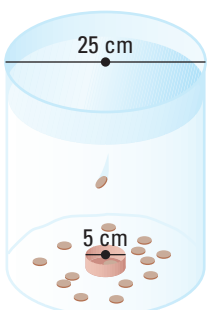
40. Suppose that a hound's marker dropped onto a rectangular field that is 200 feet by 250 feet is equally likely to land anywhere in the field. The target region is a 15 foot square that lies in the field. What is the probability that the marker lands in the target region?

41. If the area of the target region is doubled, how does the probability change?

42. If each side of the target region is doubled, how does the probability change?

Test Preparation

43. **MULTI-STEP PROBLEM** Use the following information.
You organize a fund-raiser at your school. You fill a large glass jar that has a 25 centimeter diameter with water. You place a dish that has a 5 centimeter diameter at the bottom of the jar. A person donates a coin by dropping it in the jar. If the coin lands in the dish, the person wins a small prize.



a. Calculate the probability that a coin dropped, with an equally likely chance of landing anywhere at the bottom of the jar, lands in the dish.

b. Use the probability in part (a) to estimate the average number of coins needed to win a prize.

c. From past experience, you expect about 250 people to donate 5 coins each. How many prizes should you buy?

d. *Writing* Suppose that instead of the dish, a circle with a diameter of 5 centimeters is painted on the bottom of the jar, and any coin touching the circle wins a prize. Will the probability change? Explain.

★ Challenge

44. **USING ALGEBRA** Graph the lines $y = x$ and $y = 3$ in a coordinate plane. A point is chosen randomly from within the boundaries $0 < y < 4$ and $0 < x < 4$. Find the probability that the coordinates of the point are a solution of this system of inequalities:

$$y < 3$$
$$y > x$$

EXTRA CHALLENGE
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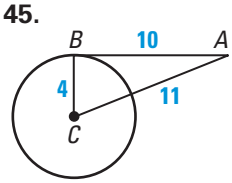
Mixed Review

Determining Tangency **Tell whether $\overleftrightarrow{AB}$ is tangent to $\odot C$. Explain your reasoning.** (Review 10.1)

45.



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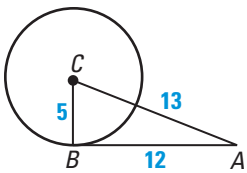

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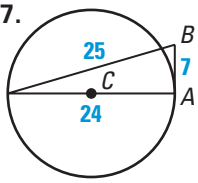


Describing Lines **In Exercises 48–51, graph the line with the circle $(x - 2)^2 + (y + 4)^2 = 16$. Is the line a *tangent* or a *secant*?** (Review 10.6)

48. $x = -y$

49. $y = 0$

50. $x = 6$

51. $y = x - 1$

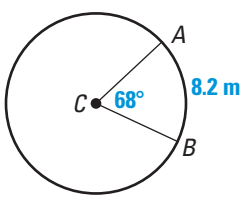
52. **Locus** Find the locus of all points in the coordinate plane that are equidistant from points (3, 2) and (1, 2) and within $\sqrt{2}$ units of the point (1, −1). (Review 10.7)

Quiz 2

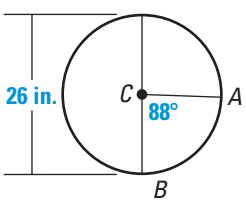
Self-Test for Lessons 11.4–11.6

Find the indicated measure. (Lesson 11.4)

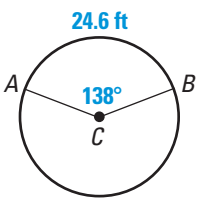
1. Circumference



2. Length of $\widehat{AB}$

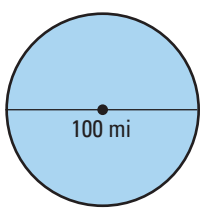


3. Radius

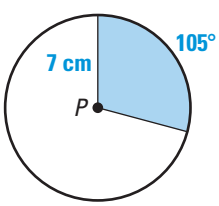


In Exercises 4–6, find the area of the shaded region. (Lesson 11.5)

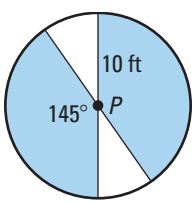
4.



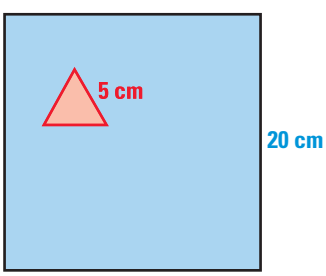
5.



6.



7. **Targets** A square target with 20 cm sides includes a triangular region with equal side lengths of 5 cm. A dart is thrown and hits the target at random. Find the probability that the dart hits the triangle. (Lesson 11.6)



CHAPTER 12

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Use properties of polyhedra. **(12.1)**	Classify crystals by their shape. **(p. 725)**
Find the surface area of prisms and cylinders. **(12.2)**	Determine the surface area of a wax cylinder record. **(p. 733)**
Find the surface area of pyramids and cones. **(12.3)**	Find the area of each lateral face of a pyramid, such as the Pyramid Arena in Tennessee. **(p. 735)**
Find the volume of prisms and cylinders. **(12.4)**	Find the volume of a fish tank, such as the tank at the New England Aquarium. **(p. 748)**
Find the volume of pyramids and cones. **(12.5)**	Find the volume of a volcano, such as Mount St. Helens. **(p. 757)**
Find the surface area and volume of a sphere. **(12.6)**	Find the surface area of a planet, such as Earth. **(p. 763)**
Find the surface area and volume of similar solids. **(12.7)**	Use the scale factor of a model car to determine dimensions on the actual car. **(p. 770)**

How does Chapter 12 fit into the BIGGER PICTURE of geometry?

Solids can be assigned three types of measure. For instance, the height and radius of a cylinder are one-dimensional measures. The surface area of a cylinder is a two-dimensional measure, and the volume of a cylinder is a three-dimensional measure. Assigning measures to plane regions and to solids is one of the primary goals of geometry. In fact, the word geometry means "Earth measure."

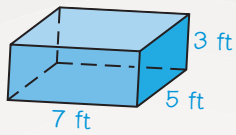
STUDY STRATEGY

How did generalizing formulas help you?

The list of similar concepts you made, following the **Study Strategy** on p. 718, may resemble this one.

Generalizing Formulas

The same concept is used to find the surface area of a prism and the surface area of a cylinder. For example, the surface areas can be found by adding twice the area of the base, $2B$, to the lateral area L.



$$\begin{aligned} S &= 2B + L \\ &= 2(l \cdot w) + Ph \\ &= 2(7 \cdot 5) + 24 \cdot 3 \\ &= 142 \text{ ft}^2 \end{aligned}$$

$$\begin{aligned} S &= 2B + L \\ &= 2(\pi r^2) + Ch \\ &= 2(\pi(6)^2) + (\pi \cdot 12)7 \\ &= 156\pi \text{ m}^2 \end{aligned}$$

CHAPTER 12

Chapter Review

VOCABULARY

- polyhedron, p. 719
- face, p. 719
- edge, p. 719
- vertex, p. 719
- regular polyhedron, p. 720
- convex, p. 720
- cross section, p. 720
- Platonic solids, p. 721
- tetrahedron, p. 721
- octahedron, p. 721
- dodecahedron, p. 721
- icosahedron, p. 721
- prism, p. 728
- bases, p. 728
- lateral faces, p. 728
- right prism, p. 728
- oblique prism, p. 728
- surface area of a polyhedron, p. 728
- lateral area of a polyhedron, p. 728
- net, p. 729
- cylinder, p. 730
- right cylinder, p. 730
- lateral area of a cylinder, p. 730
- surface area of a cylinder, p. 730
- pyramid, p. 735
- regular pyramid, p. 735
- circular cone, p. 737
- lateral surface of a cone, p. 737
- right cone, p. 737
- volume of a solid, p. 743
- sphere, p. 759
- center of a sphere, p. 759
- radius of a sphere, p. 759
- chord of a sphere, p. 759
- diameter of a sphere, p. 759
- great circle, p. 760
- hemisphere, p. 760
- similar solids, p. 766

12.1 EXPLORING SOLIDS

Examples on pp. 719–722

EXAMPLE The solid at the right has 6 faces and 10 edges. The number of vertices can be found using Euler's Theorem.

$$F + V = E + 2$$
$$6 + V = 10 + 2$$
$$V = 6$$

Use Euler's Theorem to find the unknown number.

1. Faces: 32
Vertices: ___?___
Edges: 90

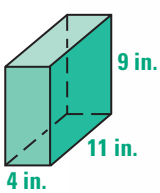
2. Faces: ___?___
Vertices: 6
Edges: 10

3. Faces: 5
Vertices: 5
Edges: ___?___

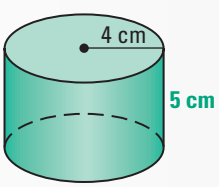
12.2 SURFACE AREA OF PRISMS AND CYLINDERS

Examples on pp. 728–731

EXAMPLES The surface area of a right prism and a right cylinder are shown.



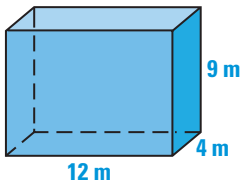
$$S = 2B + Ph$$
$$= 2(44) + 30(9)$$
$$= 358 \text{ in.}^2$$



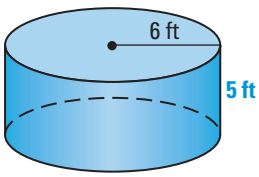
$$S = 2\pi r^2 + 2\pi rh$$
$$= 2\pi(4^2) + 2\pi(4)(5)$$
$$\approx 226.2 \text{ cm}^2$$

Find the surface area of the right prism or right cylinder. Round your result to two decimal places.

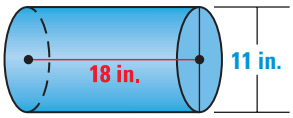
4.



5.



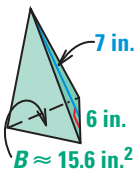
6.



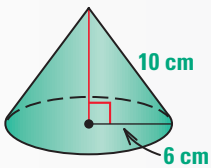
12.3 Surface Area of Pyramids and Cones

Examples on pp. 735–737

EXAMPLES The surface area of a regular pyramid and a right cone are shown.



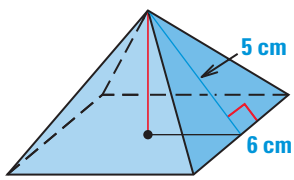
$$S = B + \frac{1}{2}P\ell$$
$$\approx 15.6 + \frac{1}{2}(18)(7)$$
$$\approx 78.6 \text{ in.}^2$$



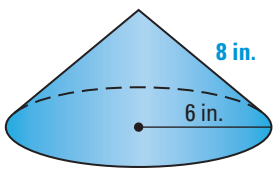
$$S = \pi r^2 + \pi r\ell$$
$$= \pi(6)^2 + \pi(6)(10)$$
$$\approx 301.6 \text{ cm}^2$$

Find the surface area of the regular pyramid or right cone. Round your result to two decimal places.

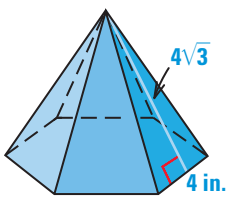
7.



8.



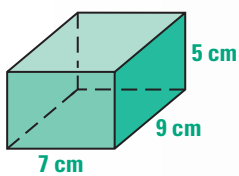
9.



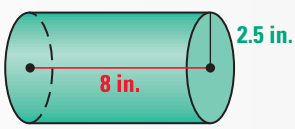
12.4 Volume of Prisms and Cylinders

Examples on pp. 743–745

EXAMPLES The volume of a rectangular prism and a right cylinder are shown.



$$V = Bh = (7 \cdot 9)(5) = 315 \text{ cm}^3$$



$$V = \pi r^2 h = \pi(2.5^2)(8) \approx 157.1 \text{ in.}^3$$

Find the volume of the described solid.

10. A side of a cube measures 8 centimeters.

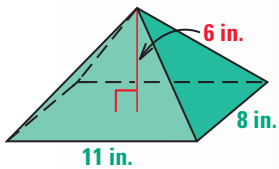
11. A right prism has a height of 37.2 meters and regular hexagonal bases, each with a base edge of 21 meters.

12. A right cylinder has a radius of 3.5 inches and a height of 8 inches.

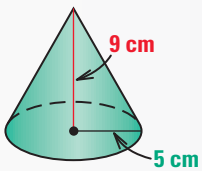
12.5 Volume of Pyramids and Cones

Examples on pp. 752–754

EXAMPLES The volume of a right pyramid and a right cone are shown.



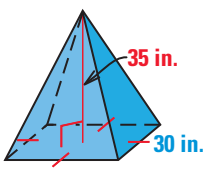
$$V = \frac{1}{3}Bh$$
$$= \frac{1}{3}(11 \cdot 8)(6)$$
$$= 176 \text{ in.}^3$$



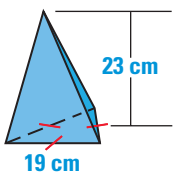
$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(5^2)(9)$$
$$\approx 235.6 \text{ cm}^3$$

Find the volume of the pyramid or cone.

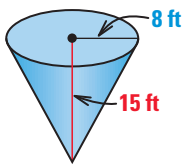
13.



14.



15.



12.6 Surface Area and Volume of Spheres

Examples on pp. 759–761

EXAMPLES The surface area and volume of the sphere are shown.

$$S = 4\pi r^2 = 4\pi(7^2) \approx 615.8 \text{ in.}^2$$
$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(7^3) \approx 1436.8 \text{ in.}^3$$

16. Find the surface area and volume of a sphere with a radius of 14 meters.

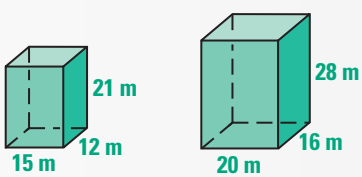
17. Find the surface area and volume of a sphere with a radius of 0.5 inch.

12.7 Similar Solids

Examples on pp. 766–768

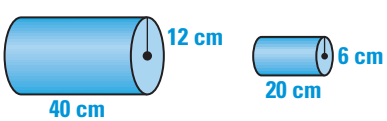
EXAMPLES The ratios of the corresponding linear measurements of the two right prisms are equal, so the solids are similar with a scale factor of 3:4.

lengths: $\frac{15}{20} = \frac{3}{4}$ **widths:** $\frac{12}{16} = \frac{3}{4}$ **heights:** $\frac{21}{28} = \frac{3}{4}$

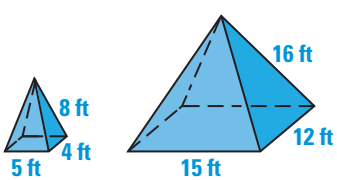


Decide whether the solids are similar. If so, find their scale factor.

18.



19.

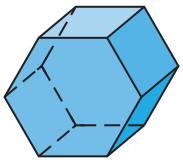


CHAPTER 12

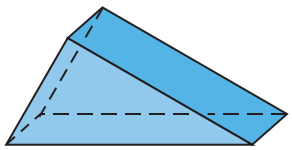
Chapter Test

Determine the number of faces, vertices, and edges of the solids.

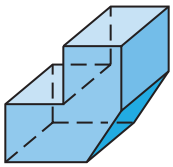
1.



2.



3.



USING ALGEBRA Sketch the solid described and find its missing measurement. (*B* is the base area, *P* is the base perimeter, *h* is the height, *S* is the surface area, *r* is the radius, and ℓ is the slant height.)

4. Right rectangular prism: $B = 44$ m^2, $P = 30$ m, $h = 7$ m, $S =$ __?__

5. Right cylinder: $r = 8.6$ in., $h =$ __?__, $S = 784\pi$ in.^{2}

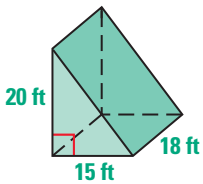
6. Regular pyramid: $B = 100$ ft^2, $P = 40$ ft, $\ell =$ __?__, $S = 340$ ft^2

7. Right cone: $r = 12$ yd, $\ell = 17$ yd, $S =$ __?__

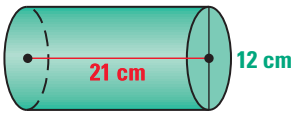
8. Sphere: $r = 34$ cm, $S =$ __?__

In Exercises 9–11, find the volume of the right solid.

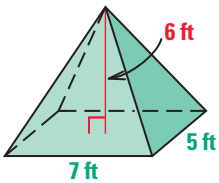
9.



10.



11.

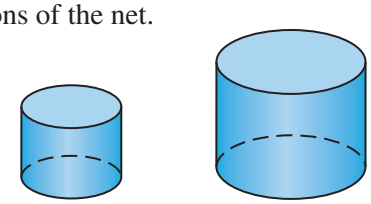


12. Draw a net for each solid in Exercises 9–11. Label the dimensions of the net.

13. The scale factor of two spheres is 1 : 5. The radius of the smaller sphere is 3 centimeters. What is the volume of the larger sphere?

14. Describe the possible intersections of a plane and a sphere.

15. What is the scale factor of the two cylinders at the right?



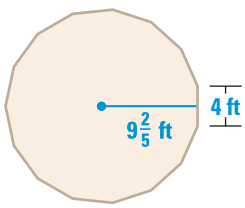
16. CANNED GOODS Find the volume and surface area of a prism with a height of 6 inches and a 4 inch by 4 inch square base. Compare the results with the volume and surface area of a cylinder with a height of 7.64 inches and a diameter of 4 inches.

SILOS Suppose you are building a silo. The shape of your silo is a right prism with a regular 15-gon for a base, as shown. The height of your silo is 59 feet.

17. What is the area of the floor of your silo?

18. Find the lateral area and volume of your silo.

19. What are the lateral area and volume of a larger silo that is in a 1 : 1.25 ratio with yours?



12.1 Exploring Solids

What you should learn

GOAL 1 Use properties of polyhedra.

GOAL 2 Use Euler's Theorem in **real-life** situations, such as analyzing the molecular structure of salt in **Example 5**.

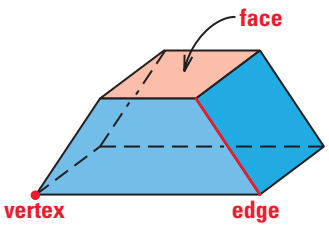
Why you should learn it

▼ You can use properties of polyhedra to classify various crystals, as in **Exs. 39–41**.

REAL LIFE

GOAL 1 USING PROPERTIES OF POLYHEDRA

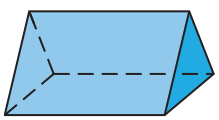
A **polyhedron** is a solid that is bounded by polygons, called **faces**, that enclose a single region of space. An **edge** of a polyhedron is a line segment formed by the intersection of two faces. A **vertex** of a polyhedron is a point where three or more edges meet. The plural of polyhedron is *polyhedra*, or polyhedrons.



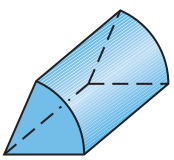
EXAMPLE 1 *Identifying Polyhedra*

Decide whether the solid is a polyhedron. If so, count the number of faces, vertices, and edges of the polyhedron.

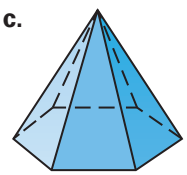
a.



b.



c.



SOLUTION

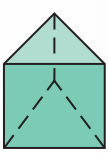
a. This is a polyhedron. It has 5 faces, 6 vertices, and 9 edges.

b. This is not a polyhedron. Some of its faces are not polygons.

c. This is a polyhedron. It has 7 faces, 7 vertices, and 12 edges.

CONCEPT SUMMARY TYPES OF SOLIDS

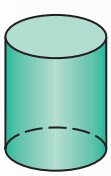
Of the five solids below, the prism and pyramid are polyhedra. The cone, cylinder, and sphere are not polyhedra.



Prism

Pyramid

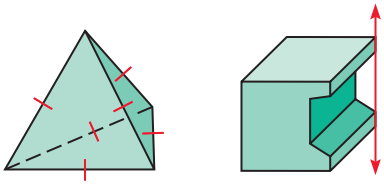
Cone



Cylinder

Sphere

A polyhedron is **regular** if all of its faces are congruent regular polygons. A polyhedron is **convex** if any two points on its surface can be connected by a segment that lies entirely inside or on the polyhedron. If this segment goes outside the polyhedron, then the polyhedron is *nonconvex*, or *concave*.



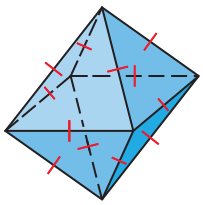
regular, convex

nonregular, nonconvex

EXAMPLE 2 *Classifying Polyhedra*

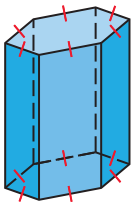
Is the octahedron convex? Is it regular?

a.



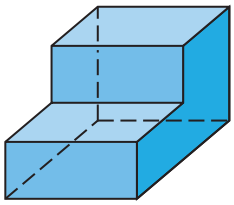
convex, regular

b.



convex, nonregular

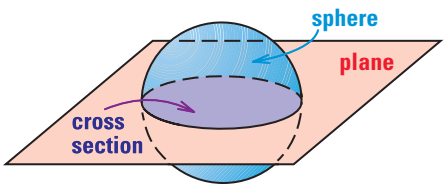
c.



nonconvex, nonregular

. . . . . .

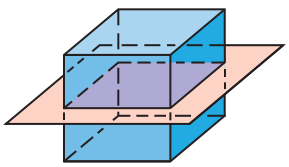
Imagine a plane slicing through a solid. The intersection of the plane and the solid is called a **cross section**. For instance, the diagram shows that the intersection of a plane and a sphere is a circle.



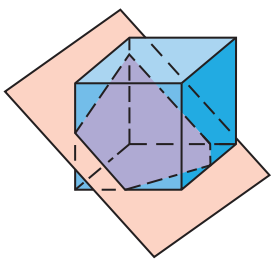
EXAMPLE 3 *Describing Cross Sections*

Describe the shape formed by the intersection of the plane and the cube.

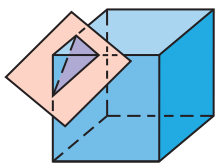
a.



b.



c.



SOLUTION

a. This cross section is a square.

b. This cross section is a pentagon.

c. This cross section is a triangle.

. . . . . .

The square, pentagon, and triangle cross sections of a cube are described in Example 3. Some other cross sections are the rectangle, trapezoid, and hexagon.

STUDENT HELP

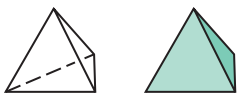
Study Tip
When sketching a cross section of a polyhedron, first sketch the solid. Then, locate the vertices of the cross section and draw the sides of the polygon.

GOAL 2 USING EULER'S THEOREM

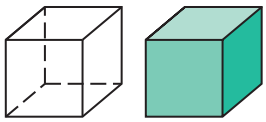
STUDENT HELP

Study Tip
Notice that four of the Platonic solids end in "hedron." *Hedron* is Greek for "side" or "face." A cube is sometimes called a *hexahedron*.

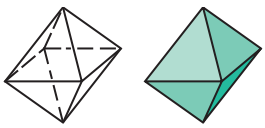
There are five regular polyhedra, called *Platonic solids*, after the Greek mathematician and philosopher Plato. The **Platonic solids** are a regular **tetrahedron** (4 faces), a cube (6 faces), a regular **octahedron** (8 faces), a regular **dodecahedron** (12 faces), and a regular **icosahedron** (20 faces).



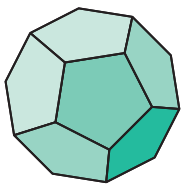
Regular tetrahedron
4 faces, 4 vertices, 6 edges



Cube
6 faces, 8 vertices, 12 edges



Regular octahedron
8 faces, 6 vertices, 12 edges



Regular dodecahedron
12 faces, 20 vertices, 30 edges

Regular icosahedron
20 faces, 12 vertices, 30 edges

Notice that the sum of the number of faces and vertices is two more than the number of edges in the solids above. This result was proved by the Swiss mathematician Leonhard Euler (1707–1783).

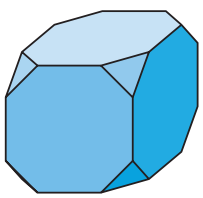
THEOREM

THEOREM 12.1 ***Euler's Theorem***

The number of faces (F), vertices (V), and edges (E) of a polyhedron are related by the formula $F + V = E + 2$.

EXAMPLE 4 *Using Euler's Theorem*

The solid has 14 faces; 8 triangles and 6 octagons. How many vertices does the solid have?



SOLUTION

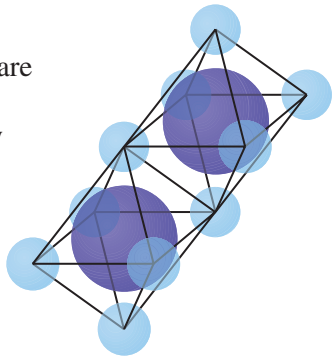
On their own, 8 triangles and 6 octagons have $8(3) + 6(8)$, or 72 edges. In the solid, each side is shared by exactly two polygons. So, the number of edges is one half of 72, or 36. Use Euler's Theorem to find the number of vertices.

$F + V = E + 2$	**Write Euler's Theorem.**
$14 + V = 36 + 2$	**Substitute.**
$V = 24$	**Solve for *V*.**

▶ The solid has 24 vertices.

EXAMPLE 5 *Finding the Number of Edges*

CHEMISTRY In molecules of sodium chloride, commonly known as table salt, chloride atoms are arranged like the vertices of regular octahedrons. In the crystal structure, the molecules share edges. How many sodium chloride molecules share the edges of one sodium chloride molecule?



SOLUTION

To find the number of molecules that share edges with a given molecule, you need to know the number of edges of the molecule.

You know that the molecules are shaped like regular octahedrons. So, they each have 8 faces and 6 vertices. You can use Euler's Theorem to find the number of edges, as shown below.

$F + V = E + 2$	**Write Euler's Theorem.**
$8 + 6 = E + 2$	**Substitute.**
$12 = E$	**Simplify.**

▶ So, 12 other molecules share the edges of the given molecule.

FOCUS ON APPLICATIONS

GEODESIC DOME The dome has the same underlying structure as a soccer ball, but the faces are subdivided into triangles.

APPLICATION LINK
www.mcdougallittell.com

EXAMPLE 6 *Finding the Number of Vertices*

SPORTS A soccer ball resembles a polyhedron with 32 faces; 20 are regular hexagons and 12 are regular pentagons. How many vertices does this polyhedron have?

SOLUTION

Each of the 20 hexagons has 6 sides and each of the 12 pentagons has 5 sides. Each edge of the soccer ball is shared by two polygons. Thus, the total number of edges is as follows:

$E = \frac{1}{2}(6 \cdot 20 + 5 \cdot 12)$	**Expression for number of edges**
$= \frac{1}{2}(180)$	**Simplify inside parentheses.**
$= 90$	**Multiply.**

Knowing the number of edges, 90, and the number of faces, 32, you can apply Euler's Theorem to determine the number of vertices.

$F + V = E + 2$	**Write Euler's Theorem.**
$32 + V = 90 + 2$	**Substitute.**
$V = 60$	**Simplify.**

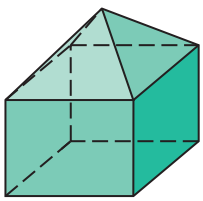
▶ So, the polyhedron has 60 vertices.

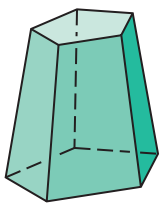
GUIDED PRACTICE

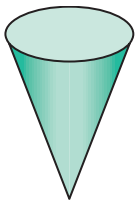
Vocabulary Check ✓ 1. Define *polyhedron* in your own words.

Concept Check ✓ 2. Is a regular octahedron convex? Are all the Platonic solids convex? Explain.

Skill Check ✓ **Decide whether the solid is a polyhedron. Explain.**

3.


4.


5.


Use Euler's Theorem to find the unknown number.

6. Faces: __?__
Vertices: 6
Edges: 12

7. Faces: 5
Vertices: __?__
Edges: 9

8. Faces: __?__
Vertices: 10
Edges: 15

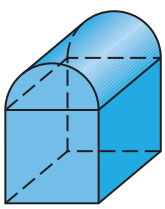
9. Faces: 20
Vertices: 12
Edges: __?__

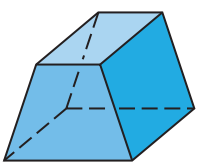
PRACTICE AND APPLICATIONS

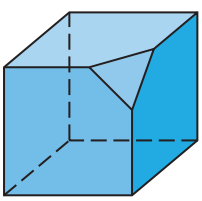
STUDENT HELP

Extra Practice to help you master skills is on p. 825.

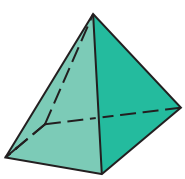
IDENTIFYING POLYHEDRA **Tell whether the solid is a polyhedron. Explain your reasoning.**

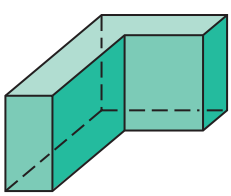
10.


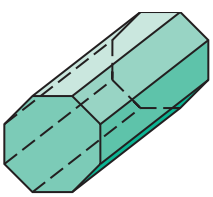
11.


12.


ANALYZING SOLIDS **Count the number of faces, vertices, and edges of the polyhedron.**

13.


14.


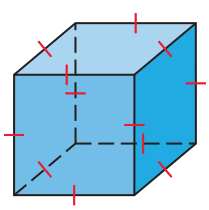
15.


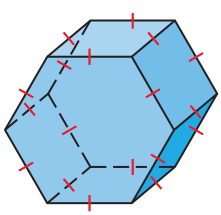
STUDENT HELP

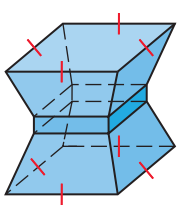
HOMEWORK HELP

Example 1: Exs. 10–15
Example 2: Exs. 16–24
Example 3: Exs. 25–35
Example 4: Exs. 36–52
Example 5: Ex. 53
Example 6: Exs. 47–52

ANALYZING POLYHEDRA **Decide whether the polyhedron is regular and/or convex. Explain.**

16.


17.


18.


LOGICAL REASONING **Determine whether the statement is *true* or *false*. Explain your reasoning.**

19. Every convex polyhedron is regular.

20. A polyhedron can have exactly 3 faces.

21. A cube is a regular polyhedron.

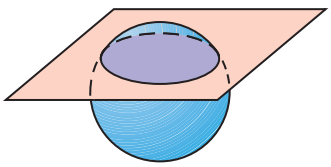
22. A polyhedron can have exactly 4 faces.

23. A cone is a regular polyhedron.

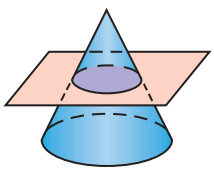
24. A polyhedron can have exactly 5 faces.

CROSS SECTIONS **Describe the cross section.**

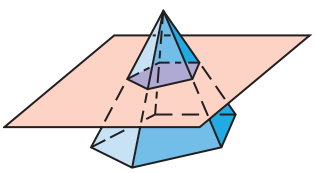
25.



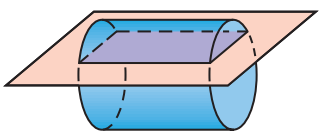
26.



27.

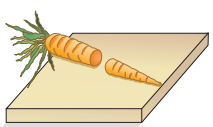


28.

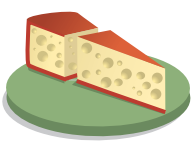


COOKING **Describe the shape that is formed by the cut made in the food shown.**

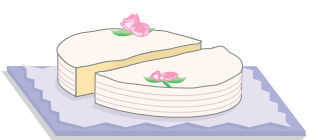
29. Carrot



30. Cheese



31. Cake



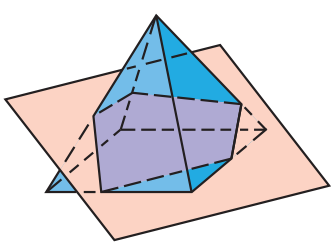
CRITICAL THINKING **In the diagram, the bottom face of the pyramid is a square.**

32. Name the cross section shown.

33. Can a plane intersect the pyramid at a point? If so, sketch the intersection.

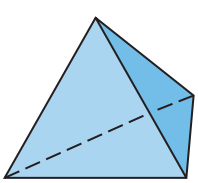
34. Describe the cross section when the pyramid is sliced by a plane parallel to its bottom face.

35. Is it possible to have an isosceles trapezoid as a cross section of this pyramid? If so, draw the cross section.

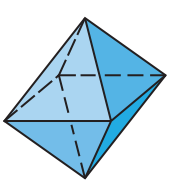


POLYHEDRONS **Name the regular polyhedron.**

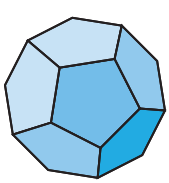
36.



37.



38.



FOCUS ON CAREERS

REAL LIFE **MINERALOGY** By studying the arrangement of atoms in a crystal, mineralogists are able to determine the chemical and physical properties of the crystal.

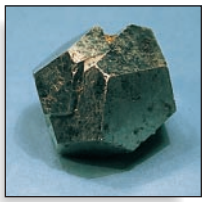
INTERNET **CAREER LINK** www.mcdougallittell.com

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 47–52.

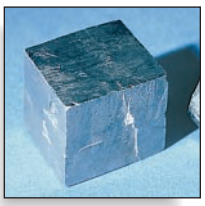
CRYSTALS In Exercises 39–41, name the Platonic solid that the crystal resembles.

39. Cobaltite



40. Fluorite

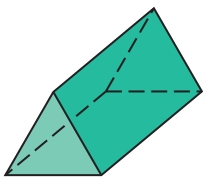
41. Pyrite



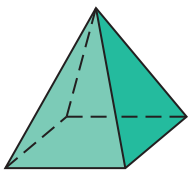
42. VISUAL THINKING Sketch a cube and describe the figure that results from connecting the centers of adjoining faces.

EULER'S THEOREM In Exercises 43–45, find the number of faces, edges, and vertices of the polyhedron and use them to verify Euler's Theorem.

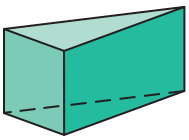
43.



44.



45.

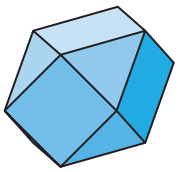


46. MAKING A TABLE Make a table of the number of faces, vertices, and edges for the Platonic solids. Use it to show Euler's Theorem is true for each solid.

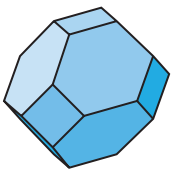
USING EULER'S THEOREM In Exercises 47–52, calculate the number of vertices of the solid using the given information.

47. 20 faces; all triangles

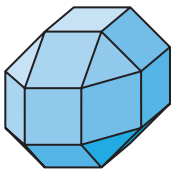
48. 14 faces; 8 triangles and 6 squares



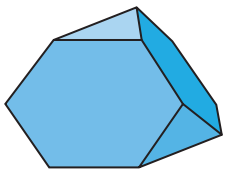
49. 14 faces; 8 hexagons and 6 squares



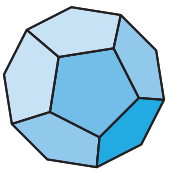
50. 26 faces; 18 squares and 8 triangles



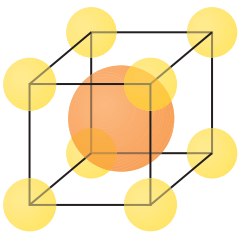
51. 8 faces; 4 hexagons and 4 triangles



52. 12 faces; all pentagons



53. SCIENCE CONNECTION In molecules of cesium chloride, chloride atoms are arranged like the vertices of cubes. In its crystal structure, the molecules share faces to form an array of cubes. How many cesium chloride molecules share the faces of a given cesium chloride molecule?

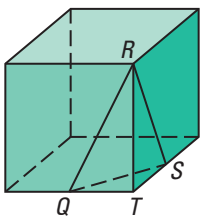


Test Preparation

54. MULTIPLE CHOICE A polyhedron has 18 edges and 12 vertices. How many faces does it have?

Ⓐ 4 Ⓑ 6 Ⓒ 8 Ⓓ 10 Ⓔ 12

55. MULTIPLE CHOICE In the diagram, Q and S are the midpoints of two edges of the cube. What is the length of $\overline{QS}$, if each edge of the cube has length h?



Ⓐ $\frac{h}{2}$ Ⓑ $\frac{h}{\sqrt{2}}$ Ⓒ $\frac{2h}{\sqrt{2}}$

Ⓓ $\sqrt{2}h$ Ⓔ $2h$

★ Challenge

SKETCHING CROSS SECTIONS Sketch the intersection of a cube and a plane so that the given shape is formed.

EXTRA CHALLENGE
www.mcdougallittell.com

56. An equilateral triangle

57. A regular hexagon

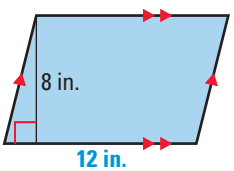
58. An isosceles trapezoid

59. A rectangle

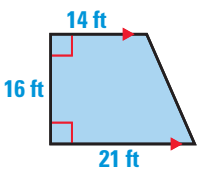
MIXED REVIEW

FINDING AREA OF QUADRILATERALS Find the area of the figure. (Review 6.7 for 12.2)

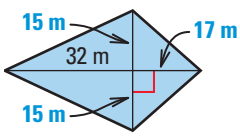
60.



61.



62.



FINDING AREA OF REGULAR POLYGONS Find the area of the regular polygon described. Round your answer to two decimal places. (Review 11.2 for 12.2)

63. An equilateral triangle with a perimeter of 48 meters and an apothem of 4.6 meters.

64. A regular octagon with a perimeter of 28 feet and an apothem of 4.22 feet.

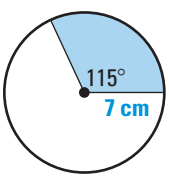
65. An equilateral triangle whose sides measure 8 centimeters.

66. A regular hexagon whose sides measure 4 feet.

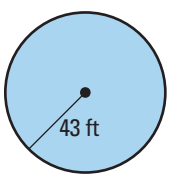
67. A regular dodecagon whose sides measure 16 inches.

FINDING AREA Find the area of the shaded region. Round your answer to two decimal places. (Review 11.5)

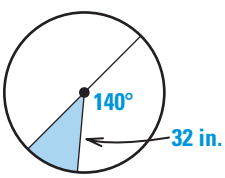
68.



69.



70.



12.2 Surface Area of Prisms and Cylinders

What you should learn

GOAL 1 Find the surface area of a prism.

GOAL 2 Find the surface area of a cylinder.

Why you should learn it

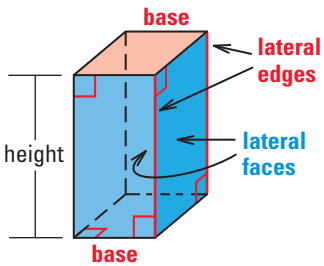
▼ You can find the surface area of **real-life** objects, such as the cylinder records used on phonographs during the late 1800s. See **Ex. 43**.

REAL LIFE

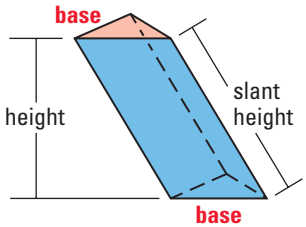
GOAL 1 FINDING THE SURFACE AREA OF A PRISM

A **prism** is a polyhedron with two congruent faces, called **bases**, that lie in parallel planes. The other faces, called **lateral faces**, are parallelograms formed by connecting the corresponding vertices of the bases. The segments connecting these vertices are *lateral edges*.

The *altitude* or *height* of a prism is the perpendicular distance between its bases. In a **right prism**, each lateral edge is perpendicular to both bases. Prisms that have lateral edges that are not perpendicular to the bases are **oblique prisms**. The length of the oblique lateral edges is the *slant height* of the prism.



Right rectangular prism



Oblique triangular prism

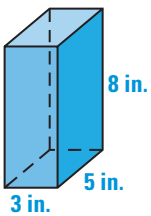
Prisms are classified by the shapes of their bases. For example, the figures above show one rectangular prism and one triangular prism. The **surface area** of a polyhedron is the sum of the areas of its faces. The **lateral area** of a polyhedron is the sum of the areas of its lateral faces.

EXAMPLE 1 *Finding the Surface Area of a Prism*

Find the surface area of a right rectangular prism with a height of 8 inches, a length of 3 inches, and a width of 5 inches.

SOLUTION

Begin by sketching the prism, as shown. The prism has 6 faces, two of each of the following:



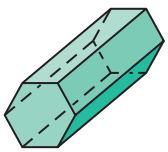
Faces	Dimensions	Area of faces
Left and right	8 in. by 5 in.	40 in.^{2}
Front and back	8 in. by 3 in.	24 in.^{2}
Top and bottom	3 in. by 5 in.	15 in.^{2}

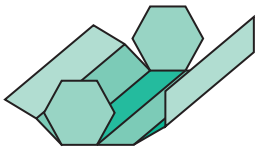
▶ The surface area of the prism is $S = 2(40) + 2(24) + 2(15) = 158$ in.^{2}

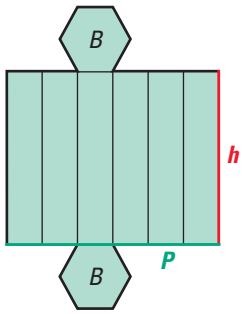
STUDENT HELP

Study Tip
When sketching prisms, first draw the two bases. Then connect the corresponding vertices of the bases.

Imagine that you cut some edges of a right hexagonal prism and unfolded it. The two-dimensional representation of all of the faces is called a **net**.







In the net of the prism, notice that the lateral area (the sum of the areas of the lateral faces) is equal to the perimeter of the base multiplied by the height.

THEOREM

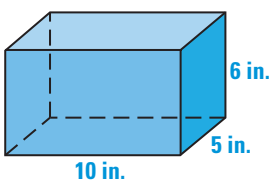
THEOREM 12.2 ***Surface Area of a Right Prism***

The surface area S of a right prism can be found using the formula $S = 2B + Ph$, where B is the area of a base, P is the perimeter of a base, and h is the height.

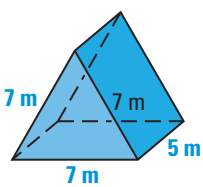
EXAMPLE 2 *Using Theorem 12.2*

Find the surface area of the right prism.

a.



b.



STUDENT HELP

Study Tip
The prism in part (a) has three pairs of parallel, congruent faces. Any pair can be called bases, whereas the prism in part (b) has only one pair of parallel, congruent faces that can be bases.

SOLUTION

a. Each base measures 5 inches by 10 inches with an area of

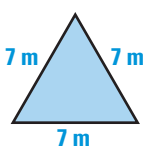
$$B = 5(10) = 50 \text{ in.}^2$$

The perimeter of the base is $P = 30$ in. and the height is $h = 6$ in.

▶ So, the surface area is

$$S = 2B + Ph = 2(50) + 30(6) = 280 \text{ in.}^2$$

b. Each base is an equilateral triangle with a side length, s, of 7 meters. Using the formula for the area of an equilateral triangle, the area of each base is



$$B = \frac{1}{4}\sqrt{3}\,(s^2) = \frac{1}{4}\sqrt{3}\,(7^2) = \frac{49}{4}\sqrt{3} \text{ m}^2.$$

The perimeter of each base is $P = 21$ m and the height is $h = 5$ m.

▶ So, the surface area is

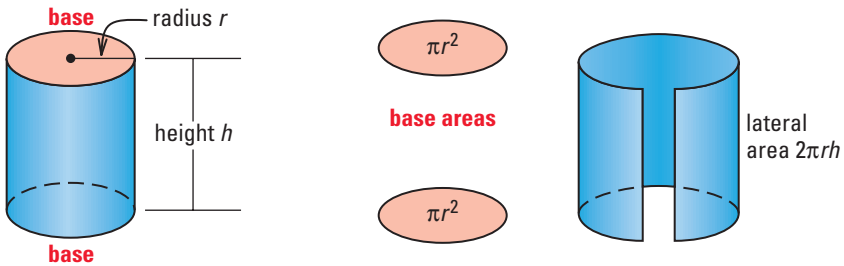
$$S = 2B + Ph = 2\left(\frac{49}{4}\sqrt{3}\right) + 21(5) \approx 147 \text{ m}^2.$$

STUDENT HELP

Look Back
For help with finding the area of an equilateral triangle, see p. 669.

GOAL 2 FINDING THE SURFACE AREA OF A CYLINDER

A **cylinder** is a solid with congruent circular bases that lie in parallel planes. The *altitude*, or *height*, of a cylinder is the perpendicular distance between its bases. The radius of the base is also called the *radius* of the cylinder. A cylinder is called a **right cylinder** if the segment joining the centers of the bases is perpendicular to the bases.



The **lateral area of a cylinder** is the area of its curved surface. The lateral area is equal to the product of the circumference and the height, which is $2\pi rh$. The entire **surface area of a cylinder** is equal to the sum of the lateral area and the areas of the two bases.

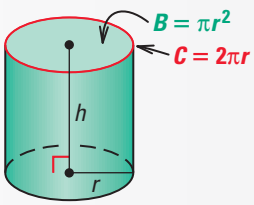
THEOREM

THEOREM 12.3 ***Surface Area of a Right Cylinder***

The surface area S of a right cylinder is

$$S = 2B + Ch = 2\pi r^2 + 2\pi rh,$$

where B is the area of a base, C is the circumference of a base, r is the radius of a base, and h is the height.

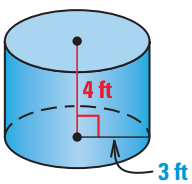


EXAMPLE 3 *Finding the Surface Area of a Cylinder*

STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

Find the surface area of the right cylinder.



SOLUTION

Each base has a radius of 3 feet, and the cylinder has a height of 4 feet.

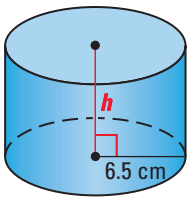
$S = 2\pi r^2 + 2\pi rh$	**Formula for surface area of cylinder**
$= 2\pi(3^2) + 2\pi(3)(4)$	**Substitute.**
$= 18\pi + 24\pi$	**Simplify.**
$= 42\pi$	**Add.**
≈ 131.95	**Use a calculator.**

▶ The surface area is about 132 square feet.

Using Algebra

EXAMPLE 4 *Finding the Height of a Cylinder*

Find the height of a cylinder which has a radius of 6.5 centimeters and a surface area of 592.19 square centimeters.



SOLUTION

Use the formula for the surface area of a cylinder and solve for the height h.

$S = 2\pi r^2 + 2\pi rh$	**Formula for surface area**
$592.19 = 2\pi(6.5)^2 + 2\pi(6.5)h$	**Substitute 6.5 for *r*.**
$592.19 = 84.5\pi + 13\pi h$	**Simplify.**
$592.19 - 84.5\pi = 13\pi h$	**Subtract 84.5π from each side.**
$326.73 \approx 13\pi h$	**Simplify.**
$8 \approx h$	**Divide each side by 13π.**

▶ The height is about 8 centimeters.

GUIDED PRACTICE

Vocabulary Check ✓

1. Describe the differences between a prism and a cylinder. Describe their similarities.

Concept Check ✓

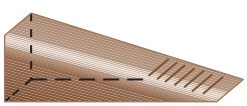
2. Sketch a triangular prism. Then sketch a net of the triangular prism. Describe how to find its lateral area and surface area.

Skill Check ✓

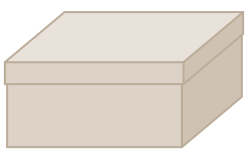
Give the mathematical name of the solid.

3. Soup can

4. Door stop



5. Shoe box



Use the diagram to find the measurement of the right rectangular prism.

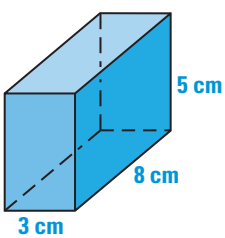
6. Perimeter of a base

7. Length of a lateral edge

8. Lateral area of the prism

9. Area of a base

10. Surface area of the prism



Make a sketch of the described solid.

11. Right rectangular prism with a 3.4 foot square base and a height of 5.9 feet

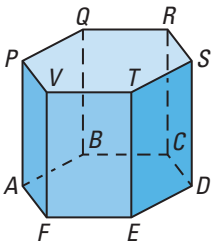
12. Right cylinder with a diameter of 14 meters and a height of 22 meters

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 825.

STUDYING PRISMS **Use the diagram at the right.**



13. Give the mathematical name of the solid.

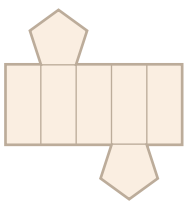
14. How many lateral faces does the solid have?

15. What kind of figure is each lateral face?

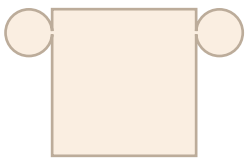
16. Name four lateral edges.

ANALYZING NETS **Name the solid that can be folded from the net.**

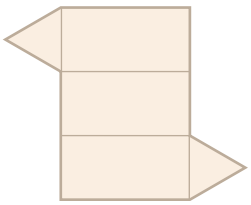
17.



18.

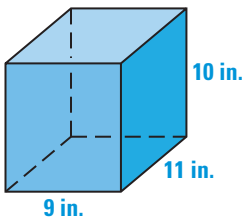


19.

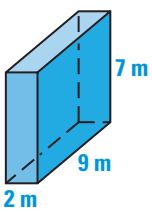


SURFACE AREA OF A PRISM **Find the surface area of the right prism. Round your result to two decimal places.**

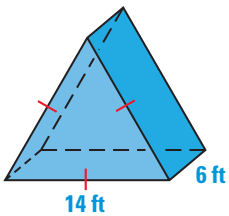
20.



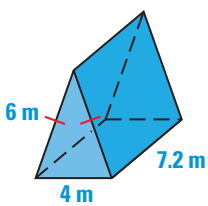
21.



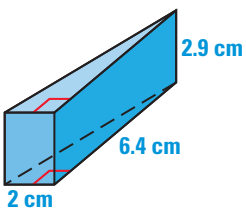
22.



23.



24.



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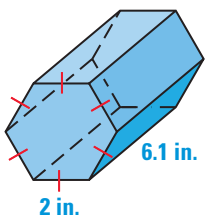
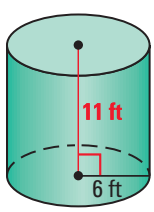
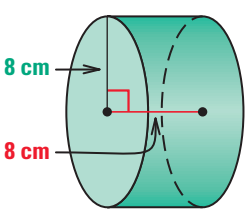


SURFACE AREA OF A CYLINDER **Find the surface area of the right cylinder. Round the result to two decimal places.**

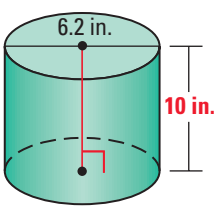
26.



27.



28.



VISUAL THINKING **Sketch the described solid and find its surface area.**

29. Right rectangular prism with a height of 10 feet, length of 3 feet, and width of 6 feet

30. Right regular hexagonal prism with all edges measuring 12 millimeters

31. Right cylinder with a diameter of 2.4 inches and a height of 6.1 inches

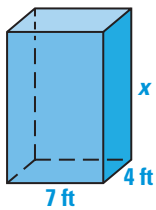
STUDENT HELP

HOMEWORK HELP

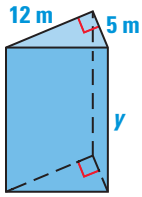
Example 1: Exs. 13–16, 20–25
Example 2: Exs. 20–25, 29–31, 35–37
Example 3: Exs. 26–28
Example 4: Exs. 32–34

USING ALGEBRA **Solve for the variable given the surface area S of the right prism or right cylinder. Round the result to two decimal places.**

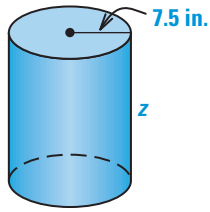
32. $S = 298 \text{ ft}^2$



33. $S = 870 \text{ m}^2$

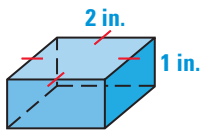


34. $S = 1202 \text{ in.}^2$

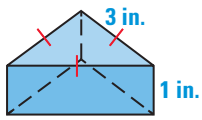


LOGICAL REASONING **Find the surface area of the right prism when the height is 1 inch, and then when the height is 2 inches. When the height doubles, does the surface area double?**

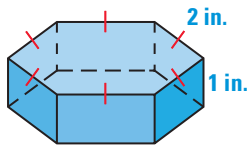
35.



36.

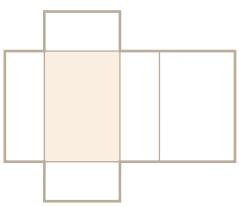


37.

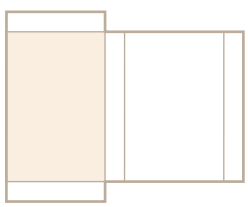


PACKAGING **In Exercises 38–40, sketch the box that results after the net has been folded. Use the shaded face as a base.**

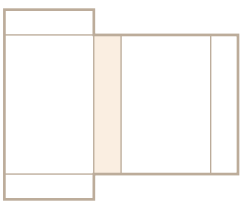
38.



39.



40.

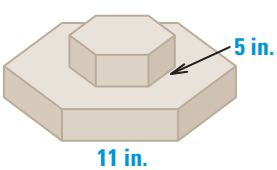


41. CRITICAL THINKING If you were to unfold a cardboard box, the cardboard would not match the net of the original solid. What sort of differences would there be? Why do these differences exist?

42. ARCHITECTURE A skyscraper is a rectangular prism with a height of 414 meters. The bases are squares with sides that are 64 meters. What is the surface area of the skyscraper (including both bases)?

43. WAX CYLINDER RECORDS The first versions of phonograph records were hollow wax cylinders. Grooves were cut into the lateral surface of the cylinder, and the cylinder was rotated on a phonograph to reproduce the sound. In the late 1800's, a standard sized cylinder was about 2 inches in diameter and 4 inches long. Find the exterior lateral area of the cylinder described.

44. CAKE DESIGN Two layers of a cake are right regular hexagonal prisms as shown in the diagram. Each layer is 3 inches high. Calculate the area of the cake that will be frosted. If one can of frosting will cover 130 square inches of cake, how many cans do you need? (*Hint:* The bottom of each layer will not be frosted and the entire top of the bottom layer will be frosted.)

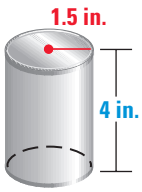


FOCUS ON CAREERS

ARCHITECTS use the surface area of a building to help them calculate the amount of building materials needed to cover the outside of a building.

Test Preparation

MULTI-STEP PROBLEM **Use the following information.**
A canned goods company manufactures cylindrical cans resembling the one at the right.



45. Find the surface area of the can.

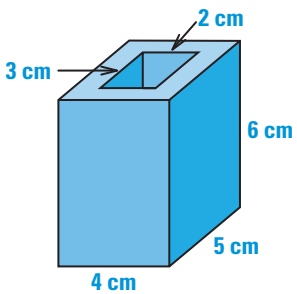
46. Find the surface area of a can whose radius and height are twice that of the can shown.

47. *Writing* Use the formula for the surface area of a right cylinder to explain why the answer in Exercise 46 is not twice the answer in Exercise 45.

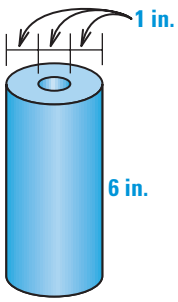
★ Challenge

FINDING SURFACE AREA **Find the surface area of the solid. Remember to include both lateral areas. Round the result to two decimal places.**

48.



49.

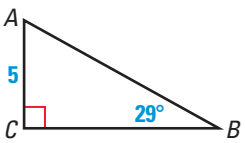


EXTRA CHALLENGE
www.mcdougallittell.com

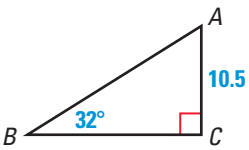
MIXED REVIEW

EVALUATING TRIANGLES **Solve the right triangle. Round your answers to two decimal places.** (Review 9.6)

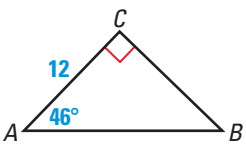
50.



51.

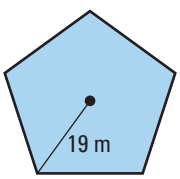


52.

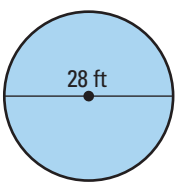


FINDING AREA **Find the area of the regular polygon or circle. Round the result to two decimal places.** (Review 11.2, 11.5 for 12.3)

53.



54.



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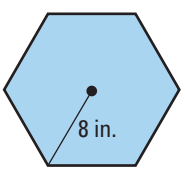
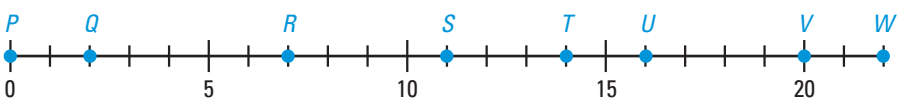


FINDING PROBABILITY **Find the probability that a point chosen at random on $\overline{PW}$ is on the given segment.** (Review 11.6)



56. $\overline{QS}$ **57.** $\overline{PU}$ **58.** $\overline{QU}$ **59.** $\overline{TW}$ **60.** $\overline{PV}$

12.3 Surface Area of Pyramids and Cones

What you should learn

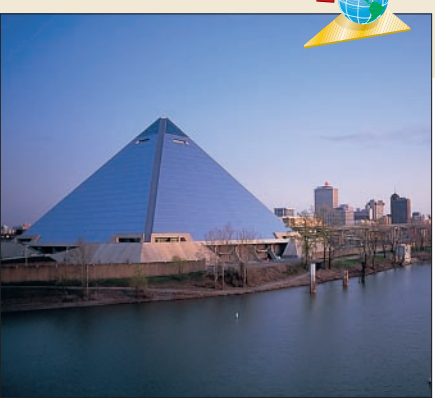
GOAL 1 Find the surface area of a pyramid.

GOAL 2 Find the surface area of a cone.

Why you should learn it

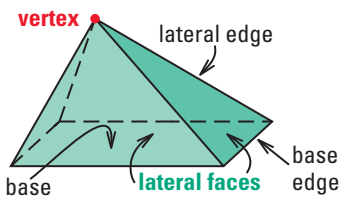
▼ To find the surface area of solids in **real life**, such as the Pyramid Arena in Memphis, Tennessee, shown below and in **Example 1**.

REAL LIFE

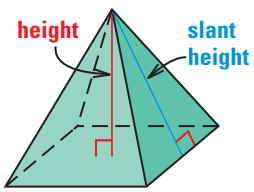


GOAL 1 FINDING THE SURFACE AREA OF A PYRAMID

A **pyramid** is a polyhedron in which the *base* is a polygon and the *lateral faces* are triangles with a common *vertex*. The intersection of two lateral faces is a *lateral edge*. The intersection of the base and a lateral face is a *base edge*. The *altitude*, or *height*, of the pyramid is the perpendicular distance between the base and the vertex.



Pyramid



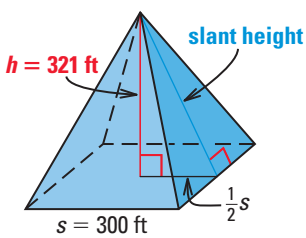
Regular pyramid

A **regular pyramid** has a regular polygon for a base and its height meets the base at its center. The *slant height* of a regular pyramid is the altitude of any lateral face. A nonregular pyramid does not have a slant height.

EXAMPLE 1 *Finding the Area of a Lateral Face*

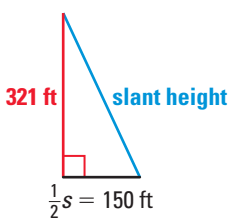
REAL LIFE

ARCHITECTURE The lateral faces of the Pyramid Arena in Memphis, Tennessee, are covered with steel panels. Use the diagram of the arena at the right to find the area of each lateral face of this regular pyramid.



SOLUTION

To find the slant height of the pyramid, use the Pythagorean Theorem.



$(\text{Slant height})^2 = h^2 + \left(\frac{1}{2}s\right)^2$	**Write formula.**
$(\text{Slant height})^2 = 321^2 + 150^2$	**Substitute.**
$(\text{Slant height})^2 = 125{,}541$	**Simplify.**
$\text{Slant height} = \sqrt{125{,}541}$	**Take the positive square root.**
$\text{Slant height} \approx 354.32$	**Use a calculator.**

▶ So, the area of each lateral face is $\frac{1}{2}$(base of lateral face)(slant height), or about $\frac{1}{2}(300)(354.32)$, which is about 53,148 square feet.

STUDENT HELP

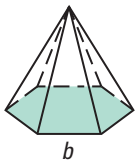
Study Tip

A *regular pyramid* is considered a regular polyhedron only if *all* its faces, including the base, are congruent. So, the only pyramid that is a regular polyhedron is the regular triangular pyramid, or *tetrahedron*. See page 721.

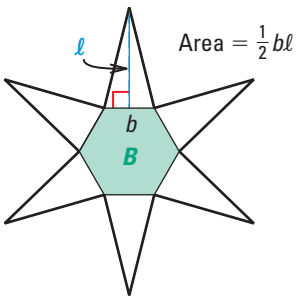
STUDENT HELP

Study Tip
When sketching the net of a pyramid, first sketch the base. Then sketch the lateral faces.

A regular hexagonal pyramid and its net are shown at the right. Let b represent the length of a base edge, and let ℓ represent the slant height of the pyramid.



The area of each lateral face is $\frac{1}{2}b\ell$ and the perimeter of the base is $P = 6b$. So, the surface area is as follows:



$S = (\text{Area of base}) + 6(\text{Area of lateral face})$

$S = B + 6\left(\frac{1}{2}b\ell\right)$ **Substitute.**

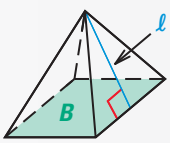
$S = B + \frac{1}{2}(6b)\ell$ **Rewrite $6\left(\frac{1}{2}b\ell\right)$ as $\frac{1}{2}(6b)\ell$.**

$S = B + \frac{1}{2}P\ell$ **Substitute P for $6b$.**

THEOREM

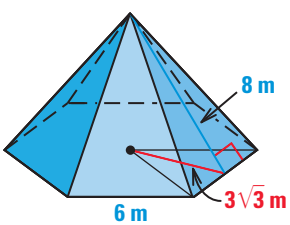
THEOREM 12.4 ***Surface Area of a Regular Pyramid***

The surface area S of a regular pyramid is $S = B + \frac{1}{2}P\ell$, where B is the area of the base, P is the perimeter of the base, and ℓ is the slant height.



EXAMPLE 2 *Finding the Surface Area of a Pyramid*

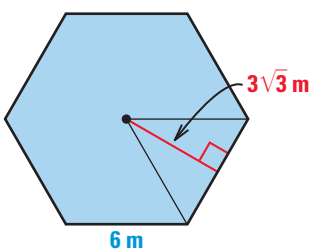
To find the surface area of the regular pyramid shown, start by finding the area of the base.



STUDENT HELP

Look Back
For help with finding the area of regular polygons see pp. 669–671.

Use the formula for the area of a regular polygon, $\frac{1}{2}(\text{apothem})(\text{perimeter})$. A diagram of the base is shown at the right. After substituting, the area of the base is $\frac{1}{2}(3\sqrt{3})(6 \cdot 6)$, or $54\sqrt{3}$ square meters.



Now you can find the surface area, using $54\sqrt{3}$ for the area of the base, B.

$S = B + \frac{1}{2}P\ell$ **Write formula.**

$= 54\sqrt{3} + \frac{1}{2}(36)(8)$ **Substitute.**

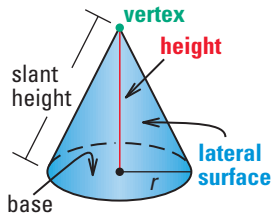
$= 54\sqrt{3} + 144$ **Simplify.**

≈ 237.5 **Use a calculator.**

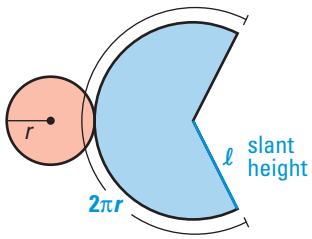
▶ So, the surface area is about 237.5 square meters.

GOAL 2 FINDING THE SURFACE AREA OF A CONE

A **circular cone**, or **cone**, has a circular *base* and a *vertex* that is not in the same plane as the base. The *altitude*, or *height*, is the perpendicular distance between the vertex and the base. In a **right cone**, the height meets the base at its center and the *slant height* is the distance between the vertex and a point on the base edge.



The **lateral surface** of a cone consists of all segments that connect the vertex with points on the base edge. When you cut along the slant height and lie the cone flat, you get the net shown at the right. In the net, the circular base has an area of πr^2 and the lateral surface is the sector of a circle. You can find the area of this sector by using a proportion, as shown below.



$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\text{Arc length}}{\text{Circumference of circle}}$$ **Set up proportion.**

$$\frac{\text{Area of sector}}{\pi\ell^2} = \frac{2\pi r}{2\pi\ell}$$ **Substitute.**

$$\text{Area of sector} = \pi\ell^2 \cdot \frac{2\pi r}{2\pi\ell}$$ **Multiply each side by $\pi\ell^2$.**

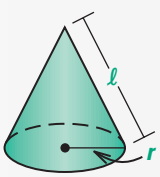
$$\text{Area of sector} = \pi r\ell$$ **Simplify.**

The surface area of a cone is the sum of the base area and the lateral area, $\pi r\ell$.

THEOREM

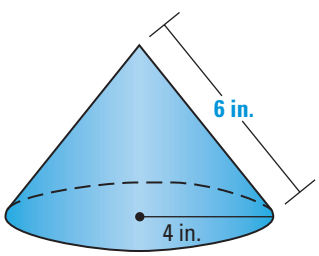
THEOREM 12.5 ***Surface Area of a Right Cone***

The surface area S of a right cone is $S = \pi r^2 + \pi r\ell$, where r is the radius of the base and ℓ is the slant height.



EXAMPLE 3 *Finding the Surface Area of a Right Cone*

To find the surface area of the right cone shown, use the formula for the surface area.



$S = \pi r^2 + \pi r\ell$ **Write formula.**

$= \pi 4^2 + \pi(4)(6)$ **Substitute.**

$= 16\pi + 24\pi$ **Simplify.**

$= 40\pi$ **Simplify.**

▶ The surface area is 40π square inches, or about 125.7 square inches.

GUIDED PRACTICE

Vocabulary Check ✓

1. Describe the differences between pyramids and cones. Describe their similarities.

Concept Check ✓

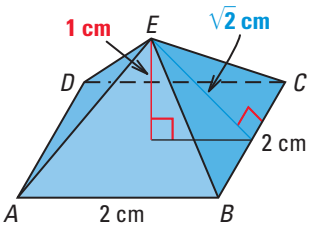
2. Can a pyramid have rectangles for lateral faces? Explain.

Skill Check ✓

Match the expression with the correct measurement.

3. Area of base
4. Height
5. Slant height
6. Lateral area
7. Surface area

A. $4\sqrt{2}$ cm^2
B. $\sqrt{2}$ cm
C. 4 cm^2
D. $(4 + 4\sqrt{2})$ cm^2
E. 1 cm



In Exercises 8–11, sketch a right cone with r = 3 ft and h = 7 ft.

8. Find the area of the base.
9. Find the slant height.
10. Find the lateral area.
11. Find the surface area.

Find the surface area of the regular pyramid described.

12. The base area is 9 square meters, the perimeter of the base is 12 meters, and the slant height is 2.5 meters.
13. The base area is $25\sqrt{3}$ square inches, the perimeter of the base is 30 inches, and the slant height is 12 inches.

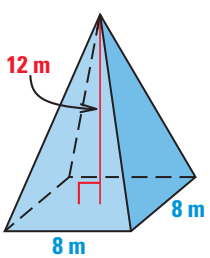
PRACTICE AND APPLICATIONS

STUDENT HELP

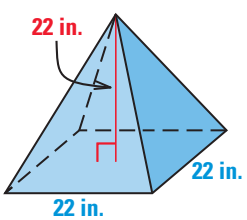
Extra Practice to help you master skills is on p. 825.

AREA OF A LATERAL FACE **Find the area of a lateral face of the regular pyramid. Round the result to one decimal place.**

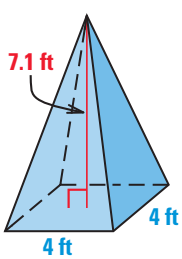
14.



15.

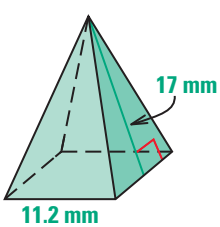


16.

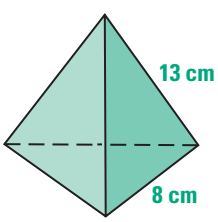


SURFACE AREA OF A PYRAMID **Find the surface area of the regular pyramid.**

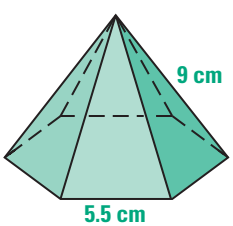
17.



18.



19.



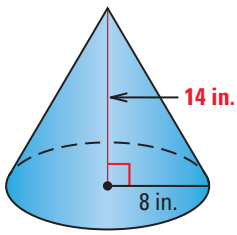
STUDENT HELP

HOMEWORK HELP

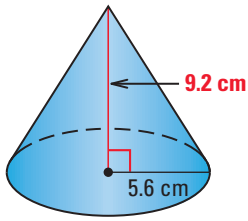
Example 1: Exs. 14–16
Example 2: Exs. 17–19
Example 3: Exs. 20–25

FINDING SLANT HEIGHT **Find the slant height of the right cone.**

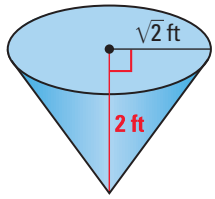
20.



21.

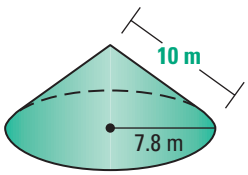


22.

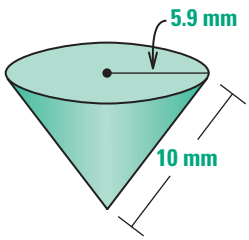


SURFACE AREA OF A CONE **Find the surface area of the right cone. Leave your answers in terms of π.**

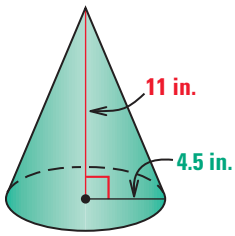
23.



24.

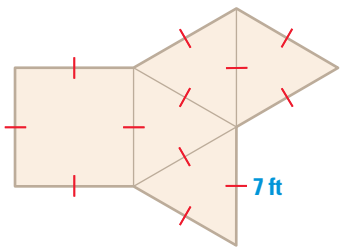


25.

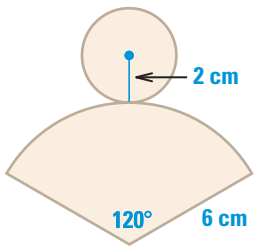


USING NETS **Name the figure that is represented by the net. Then find its surface area. Round the result to one decimal place.**

26.



27.



VISUAL THINKING **Sketch the described solid and find its surface area. Round the result to one decimal place.**

28. A regular pyramid has a triangular base with a base edge of 8 centimeters, a height of 12 centimeters, and a slant height of 12.2 centimeters.

29. A regular pyramid has a hexagonal base with a base edge of 3 meters, a height of 5.8 meters, and a slant height of 6.2 meters.

30. A right cone has a diameter of 11 feet and a slant height of 7.2 feet.

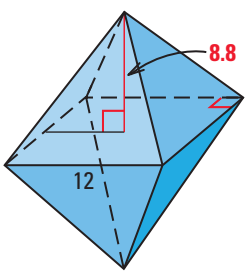
31. A right cone has a radius of 9 inches and a height of 12 inches.

STUDENT HELP

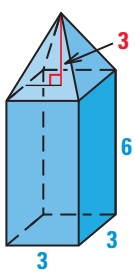
HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 32–34.

COMPOSITE SOLIDS **Find the surface area of the solid. The pyramids are regular and the prisms, cylinders, and cones are right. Round the result to one decimal place.**

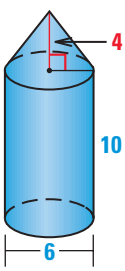
32.



33.

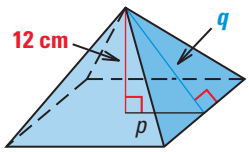


34.

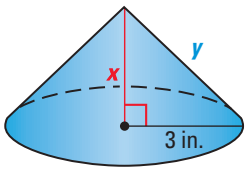


USING ALGEBRA **In Exercises 35–37, find the missing measurements of the solid. The pyramids are regular and the cones are right.**

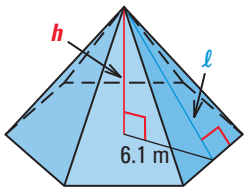
35. $P = 72$ cm



36. $S = 75.4$ in.^{2}



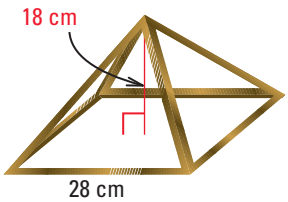
37. $S = 333$ m^2, $P = 42$ m



FOCUS ON APPLICATIONS

LAMPSHADES Many stained-glass lampshades are shaped like cones or pyramids. These shapes help direct the light down.

38. **LAMPSHADES** Some stained-glass lampshades are made out of decorative pieces of glass. Estimate the amount of glass needed to make the lampshade shown at the right by calculating the lateral area of the pyramid formed by the framing. The pyramid has a square base.



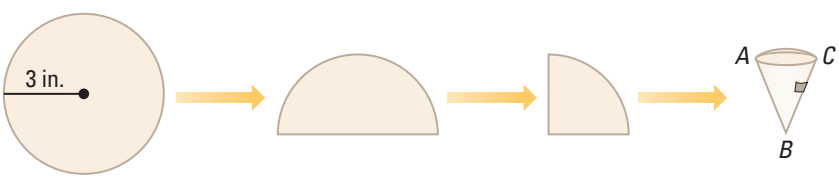
39. **PYRAMIDS** The three pyramids of Giza, Egypt, were built as regular square pyramids. The pyramid in the middle of the photo is Chephren's Pyramid and when it was built its base edge was $707\frac{3}{4}$ feet, and it had a height of 471 feet. Find the surface area of Chephren's Pyramid, including its base, when it was built.

40. **DATA COLLECTION** Find the dimensions of Chephren's Pyramid today and calculate its surface area. Compare this surface area with the surface area you found in Exercise 39.

41. **SQUIRREL BARRIER** Some bird feeders have a metal cone that prevents squirrels from reaching the bird seed, as shown. You are planning to manufacture this metal cone. The slant height of the cone is 12 inches and the radius is 8 inches. Approximate the amount of sheet metal you need.

42. **CRITICAL THINKING** A regular hexagonal pyramid with a base edge of 9 feet and a height of 12 feet is inscribed in a right cone. Find the lateral area of the cone.

43. **PAPER CUP** To make a paper drinking cup, start with a circular piece of paper that has a 3 inch radius, then follow the steps below. How does the surface area of the cup compare to the original paper circle? Find $m\angle ABC$.



Test Preparation

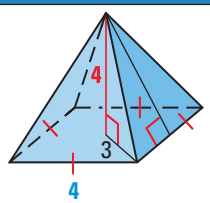
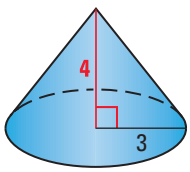
QUANTITATIVE COMPARISON **Choose the statement that is true about the given quantities.**

(A) The quantity in column A is greater.

(B) The quantity in column B is greater.

(C) The two quantities are equal.

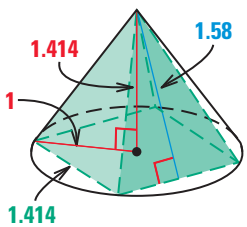
(D) The relationship cannot be determined from the given information.

	Column A	Column B
		
44.	Area of base	Area of base
45.	Lateral edge length	Slant height
46.	Lateral area	Lateral area

★ Challenge

INSCRIBED PYRAMIDS **Each of three regular pyramids are inscribed in a right cone whose radius is 1 unit and height is $\sqrt{2}$ units. The dimensions of each pyramid are listed in the table and the square pyramid is shown.**

Base	Base edge	Slant height
Square	1.414	1.58
Hexagon	1	1.65
Octagon	0.765	1.68



47. Find the surface area of the cone.

48. Find the surface area of each of the three pyramids.

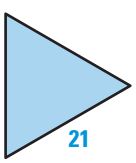
49. What happens to the surface area as the number of sides of the base increases? If the number of sides continues to increase, what number will the surface area approach?

EXTRA CHALLENGE
www.mcdougallittell.com

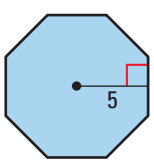
MIXED REVIEW

FINDING AREA **In Exercises 50–52, find the area of the regular polygon. Round your result to two decimal places.** (Review 11.2 for 12.4)

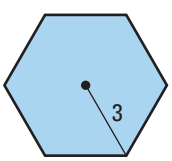
50.



51.



52.



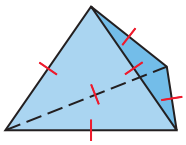
53. **AREA OF A SEMICIRCLE** A semicircle has an area of 190 square inches. Find the approximate length of the radius. (Review 11.5 for 12.4)

QUIZ 1

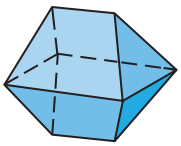
Self-Test for Lessons 12.1–12.3

State whether the polyhedron is regular and/or convex. Then calculate the number of vertices of the solid using the given information. (Lesson 12.1)

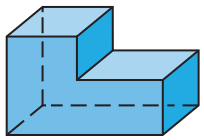
1. 4 faces; all triangles



2. 8 faces; 4 triangles and 4 trapezoids

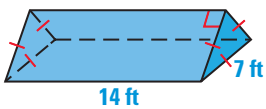


3. 8 faces; 2 hexagons and 6 rectangles

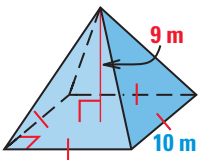


Find the surface area of the solid. Round your result to two decimal places. (Lesson 12.2 and 12.3)

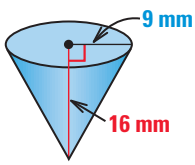
4.



5.



6.



MATH & History — History of Containers

APPLICATION LINK
www.mcdougallittell.com

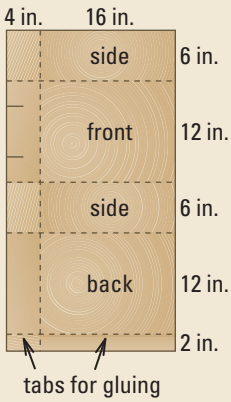
THEN

THROUGHOUT HISTORY, people have created containers for items that were important to store, such as liquids and grains. In ancient civilizations, large jars called *amphorae* were used to store water and other liquids.

NOW

TODAY, containers are no longer used just for the bare necessities. People use containers of many shapes and sizes to store a variety of objects.

1. How much paper is required to construct a paper grocery bag using the pattern at the right?
2. The sections on the left side of the pattern are folded to become the rectangular base of the bag. Find the dimensions of the base. Then find the surface area of the completed bag.



c. 525 B.C.

Amphorae are used in Ancient Greece to store water and oils.

1810

Tin containers are first used to package food.

1870

Margaret Knight patents machine to make paper bags.

1990s

Water bottles come in all shapes and sizes.

12.4 Volume of Prisms and Cylinders

What you should learn

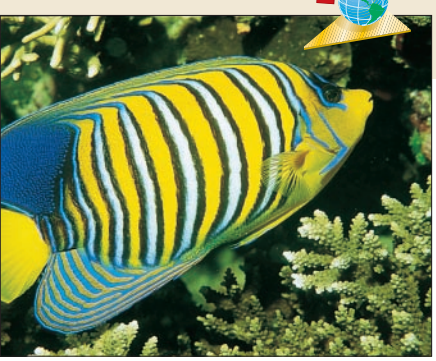
GOAL 1 Use volume postulates.

GOAL 2 Find the volume of prisms and cylinders in **real life**, such as the concrete block in **Example 4**.

Why you should learn it

▼ Learning to find the volumes of prisms and cylinders is important in **real life**, such as in finding the volume of a fish tank in **Exs. 7–9, and 46**.

REAL LIFE



GOAL 1 EXPLORING VOLUME

The **volume of a solid** is the number of cubic units contained in its interior. Volume is measured in cubic units, such as cubic meters (m^3).

VOLUME POSTULATES

POSTULATE 27 ***Volume of a Cube***

The volume of a cube is the cube of the length of its side, or $V = s^3$.

POSTULATE 28 ***Volume Congruence Postulate***

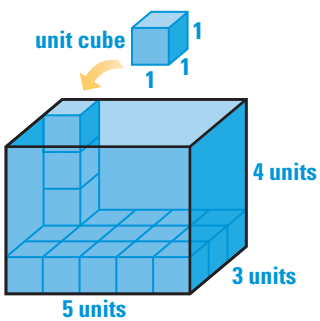
If two polyhedra are congruent, then they have the same volume.

POSTULATE 29 ***Volume Addition Postulate***

The volume of a solid is the sum of the volumes of all its nonoverlapping parts.

EXAMPLE 1 *Finding the Volume of a Rectangular Prism*

The box shown is 5 units long, 3 units wide, and 4 units high. How many unit cubes will fit in the box? What is the volume of the box?



SOLUTION

The base of the box is 5 units by 3 units. This means 5 • 3, or 15 unit cubes, will cover the base.

Three more layers of 15 cubes each can be placed on top of the lower layer to fill the box. Because the box contains 4 layers with 15 cubes in each layer, the box contains a total of 4 • 15, or 60 unit cubes.

▶ Because the box is completely filled by the 60 cubes and each cube has a volume of 1 cubic unit, it follows that the volume of the box is 60 • 1, or 60 cubic units.

In Example 1, the area of the base, 15 square units, multiplied by the height, 4 units, yields the volume of the box, 60 cubic units. So, the volume of the prism can be found by multiplying the area of the base by the height. This method can also be used to find the volume of a cylinder.

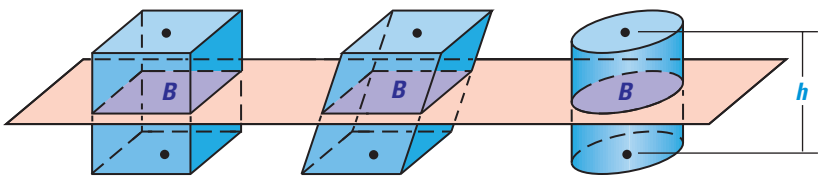
GOAL 2 FINDING VOLUMES OF PRISMS AND CYLINDERS

THEOREM

THEOREM 12.6 ***Cavalieri's Principle***

If two solids have the same height and the same cross-sectional area at every level, then they have the same volume.

Theorem 12.6 is named after mathematician Bonaventura Cavalieri (1598–1647). To see how it can be applied, consider the solids below. All three have cross sections with equal areas, B, and all three have equal heights, h. By Cavalieri's Principle, it follows that each solid has the same volume.



VOLUME THEOREMS

THEOREM 12.7 ***Volume of a Prism***

The volume V of a prism is $V = Bh$, where B is the area of a base and h is the height.

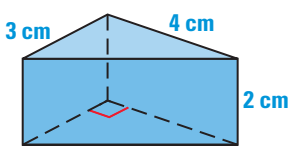
THEOREM 12.8 ***Volume of a Cylinder***

The volume V of a cylinder is $V = Bh = \pi r^2 h$, where B is the area of a base, h is the height, and r is the radius of a base.

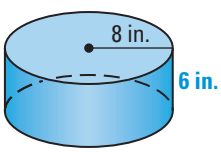
EXAMPLE 2 *Finding Volumes*

Find the volume of the right prism and the right cylinder.

a.



b.



SOLUTION

a. The area B of the base is $\frac{1}{2}(3)(4)$, or 6 cm^2. Use $h = 2$ to find the volume.

$$V = Bh = 6(2) = 12 \text{ cm}^3$$

b. The area B of the base is $\pi \cdot 8^2$, or 64π in.^{2} Use $h = 6$ to find the volume.

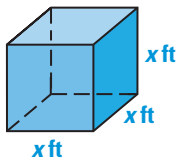
$$V = Bh = 64\pi(6) = 384\pi \approx 1206.37 \text{ in.}^3$$

Using Algebra

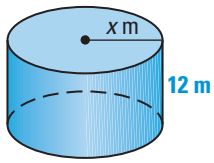
EXAMPLE 3 *Using Volumes*

Use the measurements given to solve for x.

a. Cube, $V = 100\ \text{ft}^3$



b. Right cylinder, $V = 4561\ \text{m}^3$



STUDENT HELP

INTERNET KEYSTROKE HELP

If your calculator does not have a cube root key, you can raise a number to the $\frac{1}{3}$ to find its cube root. For example, the cube root of 8 can be found as follows:

8 ^ (1 ÷ 3)

SOLUTION

a. A side length of the cube is x feet.

$V = s^3$	**Formula for volume of cube**
$100 = x^3$	**Substitute.**
$4.64 \approx x$	**Take the cube root.**

▶ So, the height, width, and length of the cube are about 4.64 feet.

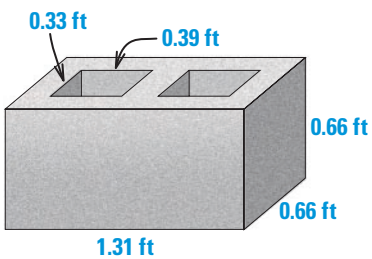
b. The area of the base is πx^2 square meters.

$V = Bh$	**Formula for volume of cylinder**
$4561 = \pi x^2(12)$	**Substitute.**
$4561 = 12\pi x^2$	**Rewrite.**
$\frac{4561}{12\pi} = x^2$	**Divide each side by 12π.**
$11 \approx x$	**Find the positive square root.**

▶ So, the radius of the cylinder is about 11 meters.

EXAMPLE 4 *Using Volumes in Real Life*

CONSTRUCTION Concrete weighs 145 pounds per cubic foot. To find the weight of the concrete block shown, you need to find its volume. The area of the base can be found as follows:



$$B = \boxed{\text{Area of large rectangle}} - 2 \cdot \boxed{\text{Area of small rectangle}}$$

$$= (1.31)(0.66) - 2(0.33)(0.39)$$

$$\approx 0.61\ \text{ft}^2$$

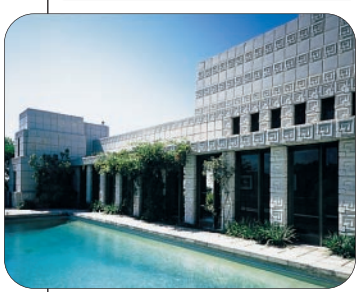
Using the formula for the volume of a prism, the volume is

$$V = Bh \approx 0.61(0.66) \approx 0.40\ \text{ft}^3.$$

▶ To find the weight of the block, multiply the pounds per cubic foot, 145 lb/ft^3, by the number of cubic feet, 0.40 ft^3.

$$\text{Weight} = \frac{145\ \text{lb}}{1\ \cancel{\text{ft}^3}} \cdot 0.4\ \cancel{\text{ft}^3} \approx 58\ \text{lb}$$

FOCUS ON APPLICATIONS



REAL LIFE **CONSTRUCTION** The Ennis-Brown House, shown above, was designed by Frank Lloyd Wright. It was built using concrete blocks.

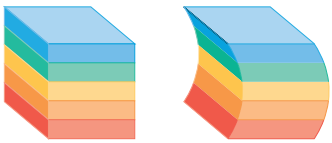
GUIDED PRACTICE

Vocabulary Check ✓

1. Surface area is measured in __?__ and volume is measured in __?__.

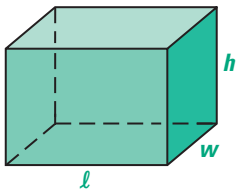
Concept Check ✓

2. Each stack of memo papers shown contains 500 sheets of paper. Explain why the stacks have the same volume. Then calculate the volume, given that each sheet of paper is 3 inches by 3 inches by 0.01 inches.



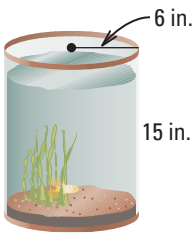
Skill Check ✓ **Use the diagram to complete the table.**

	ℓ	w	h	Volume
3.	17	3	5	?
4.	?	8	10	160
5.	4.8	6.1	?	161.04
6.	$6t$	?	$3t$	$54t^3$

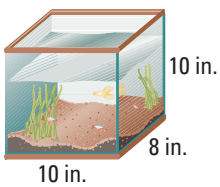


FISH TANKS **Find the volume of the tank.**

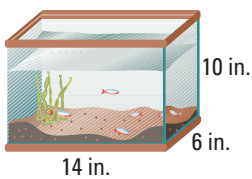
7.



8.



9.



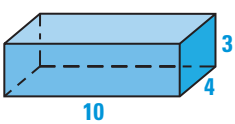
PRACTICE AND APPLICATIONS

STUDENT HELP

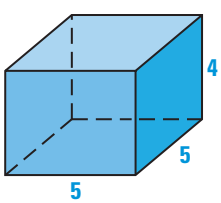
Extra Practice to help you master skills is on p. 826.

USING UNIT CUBES **Find the number of unit cubes that will fit in the box. Explain your reasoning.**

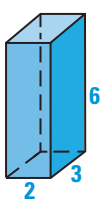
10.



11.



12.



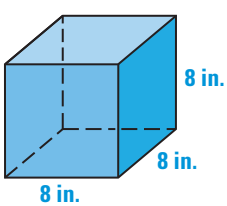
STUDENT HELP

HOMEWORK HELP

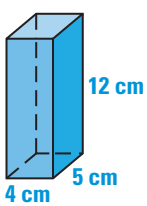
Example 1: Exs. 10–12
Example 2: Exs. 13–27
Example 3: Exs. 28–33
Example 4: Exs. 34–37

VOLUME OF A PRISM **Find the volume of the right prism.**

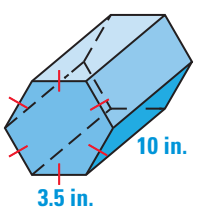
13.



14.

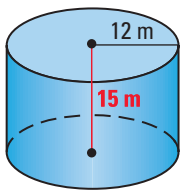


15.

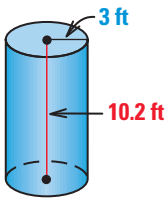


VOLUME OF A CYLINDER **Find the volume of the right cylinder. Round the result to two decimal places.**

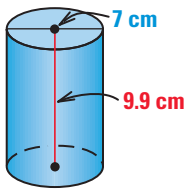
16.



17.



18.



VISUAL THINKING **Make a sketch of the solid and find its volume. Round the result to two decimal places.**

19. A prism has a square base with 4 meter sides and a height of 15 meters.

20. A pentagonal prism has a base area of 24 square feet and a height of 3 feet.

21. A prism has a height of 11.2 centimeters and an equilateral triangle for a base, where each base edge measures 8 centimeters.

22. A cylinder has a radius of 4 meters and a height of 8 meters.

23. A cylinder has a radius of 21.4 feet and a height of 33.7 feet.

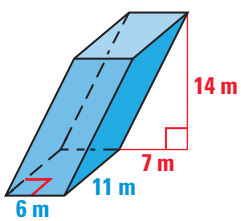
24. A cylinder has a diameter of 15 inches and a height of 26 inches.

STUDENT HELP

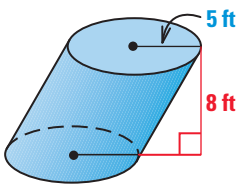
HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 25–27.

VOLUMES OF OBLIQUE SOLIDS **Use Cavalieri's Principle to find the volume of the oblique prism or cylinder.**

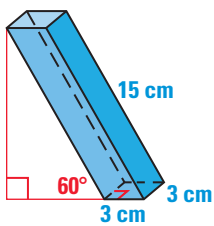
25.



26.

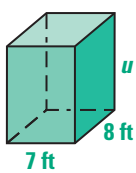


27.

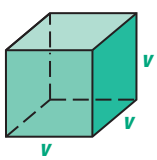


USING ALGEBRA **Solve for the variable using the given measurements. The prisms and the cylinders are right.**

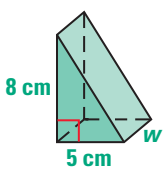
28. Volume = 560 ft^3



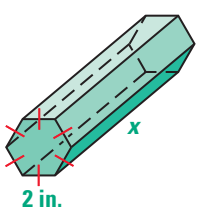
29. Volume = 2700 yd^3



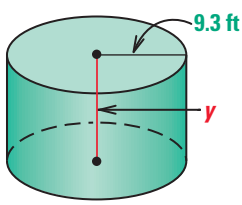
30. Volume = 80 cm^3



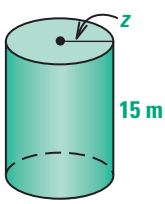
31. Volume = 72.66 $in.^3$



32. Volume = 3000 ft^3



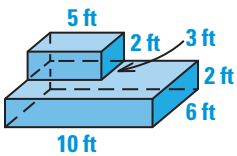
33. Volume = 1696.5 m^3



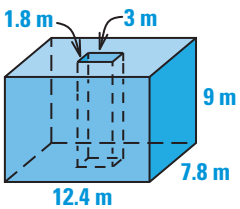
34. **CONCRETE BLOCK** In Example 4 on page 745, find the volume of the entire block and subtract the volume of the two rectangular prisms. How does your answer compare with the volume found in Example 4?

FINDING VOLUME Find the volume of the entire solid. The prisms and cylinders are right.

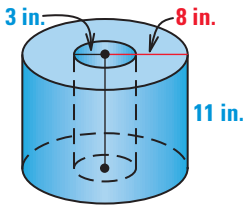
35.



36.



37.



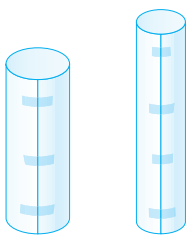
CONCRETE In Exercises 38–40, determine the number of yards of concrete you need for the given project. To builders, a "yard" of concrete means a cubic yard. (A cubic yard is equal to (36 in.)^3, or 46,656 in.^{3}.)

38. A driveway that is 30 feet long, 18 feet wide, and 4 inches thick

39. A tennis court that is 100 feet long, 50 feet wide, and 6 inches thick

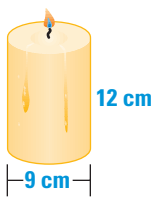
40. A circular patio that has a radius of 24 feet and is 8 inches thick

41. **LOGICAL REASONING** Take two sheets of paper that measure $8\frac{1}{2}$ inches by 11 inches and form two cylinders; one with the height as $8\frac{1}{2}$ inches and one with the height as 11 inches. Do the cylinders have the same volume? Explain.

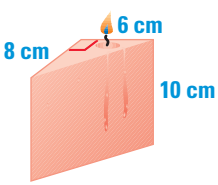


CANDLES In Exercises 42–44, you are melting a block of wax to make candles. How many candles of the given shape can be made using a block that measures 10 cm by 9 cm by 20 cm? The prisms and cylinder are right.

42.



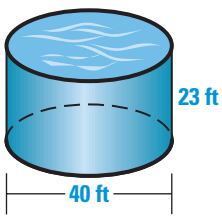
43.



44.

45. **CANNED GOODS** Find the volume and surface area of a prism with a height of 4 inches and a 3 inch by 3 inch square base. Compare the results with the volume and surface area of a cylinder with a height of 5.1 inches and a diameter of 3 inches. Use your results to explain why canned goods are usually packed in cylindrical containers.

AQUARIUM TANK The Caribbean Coral Reef Tank at the New England Aquarium is a cylindrical tank that is 23 feet deep and 40 feet in diameter, as shown.

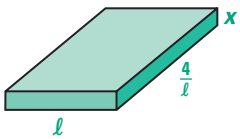


46. How many gallons of water are needed to fill the tank? (One gallon of water equals 0.1337 cubic foot.)

47. Determine the weight of the water in the tank. (One gallon of salt water weighs about 8.56 pounds.)

Test Preparation

48. MULTIPLE CHOICE If the volume of the rectangular prism at the right is 1, what does x equal?



Ⓐ $\frac{1}{4}$ Ⓑ $\frac{\ell}{4}$ Ⓒ ℓ

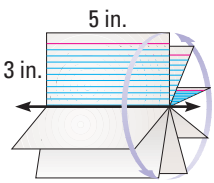
Ⓓ 4 Ⓔ 4ℓ

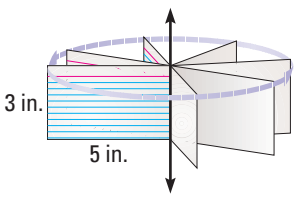
49. MULTIPLE CHOICE What is the volume of a cylinder with a radius of 6 and a height of 10?

Ⓐ 60π Ⓑ 90π Ⓒ 120π Ⓓ 180π Ⓔ 360π

★ Challenge

50. Suppose that a 3 inch by 5 inch index card is rotated around a horizontal line and a vertical line to produce two different solids, as shown. Which solid has a greater volume? Explain your reasoning.





EXTRA CHALLENGE
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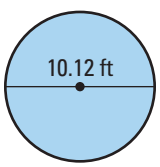
MIXED REVIEW

USING RATIOS Find the measures of the angles in the triangle whose angles are in the given extended ratio. (Review 8.1)

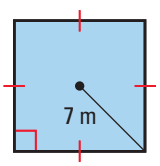
51. 2:5:5 **52.** 1:2:3 **53.** 3:4:5

FINDING AREA In Exercises 54–56, find the area of the figure. Round your result to two decimal places. (Review 11.2, 11.5 for 12.5)

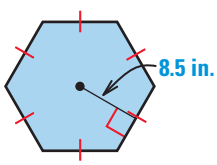
54.



55.



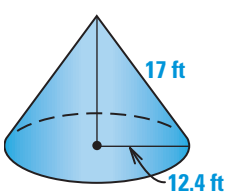
56.



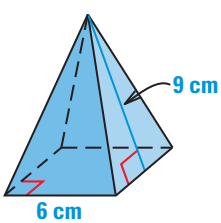
57. SURFACE AREA OF A PRISM A right rectangular prism has a height of 13 inches, a length of 1 foot, and a width of 3 inches. Sketch the prism and find its surface area. (Review 12.2)

SURFACE AREA Find the surface area of the solid. The cone is right and the pyramids are regular. (Review 12.3)

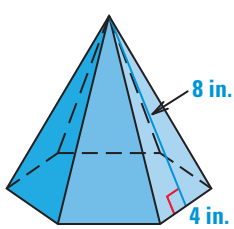
58.



59.



60.



12.5 Volume of Pyramids and Cones

What you should learn

GOAL 1 Find the volume of pyramids and cones.

GOAL 2 Find the volume of pyramids and cones in **real life**, such as the nautical prism in **Example 4**.

Why you should learn it

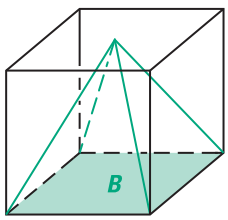
▼ Learning to find volumes of pyramids and cones is important in **real life**, such as in finding the volume of a volcano shown below and in **Ex. 34**.

REAL LIFE

Mount St. Helens

GOAL 1 FINDING VOLUMES OF PYRAMIDS AND CONES

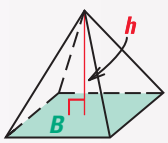
In Lesson 12.4, you learned that the volume of a prism is equal to Bh, where B is the area of the base and h is the height. From the figure at the right, it is clear that the volume of the pyramid with the same base area B and the same height h must be less than the volume of the prism. The volume of the pyramid is one third the volume of the prism.



THEOREMS

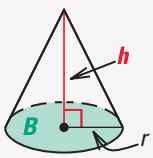
THEOREM 12.9 ***Volume of a Pyramid***

The volume V of a pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height.



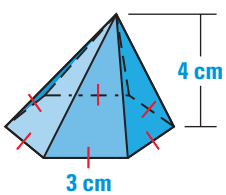
THEOREM 12.10 ***Volume of a Cone***

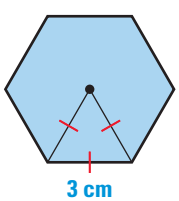
The volume V of a cone is $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$, where B is the area of the base, h is the height, and r is the radius of the base.



EXAMPLE 1 *Finding the Volume of a Pyramid*

Find the volume of the pyramid with the regular base.





SOLUTION

The base can be divided into six equilateral triangles. Using the formula for the area of an equilateral triangle, $\frac{1}{4}\sqrt{3} \cdot s^2$, the area of the base B can be found as follows:

$$6 \cdot \frac{1}{4}\sqrt{3} \cdot s^2 = 6 \cdot \frac{1}{4}\sqrt{3} \cdot 3^2 = \frac{27}{2}\sqrt{3}\ \text{cm}^2.$$

Use Theorem 12.9 to find the volume of the pyramid.

$V = \frac{1}{3}Bh$ — **Formula for volume of pyramid**

$= \frac{1}{3}\left(\frac{27}{2}\sqrt{3}\right)(4)$ — **Substitute.**

$= 18\sqrt{3}$ — **Simplify.**

▶ So, the volume of the pyramid is $18\sqrt{3}$, or about 31.2 cubic centimeters.

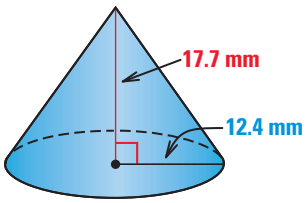
STUDENT HELP

Study Tip
The formulas given in Theorems 12.9 and 12.10 apply to all pyramids and cones, whether right or oblique. This follows from Cavalieri's Principle, stated in Lesson 12.4.

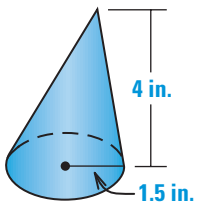
EXAMPLE 2 *Finding the Volume of a Cone*

Find the volume of each cone.

a. Right circular cone



b. Oblique circular cone



SOLUTION

a. Use the formula for the volume of a cone.

$V = \frac{1}{3}Bh$	**Formula for volume of cone**
$= \frac{1}{3}(\pi r^2)h$	**Base area equals πr^2.**
$= \frac{1}{3}(\pi 12.4^2)(17.7)$	**Substitute.**
$\approx 907.18\pi$	**Simplify.**

▶ So, the volume of the cone is about 907.18π, or 2850 cubic millimeters.

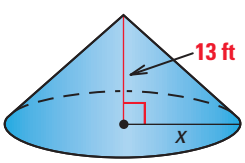
b. Use the formula for the volume of a cone.

$V = \frac{1}{3}Bh$	**Formula for volume of cone**
$= \frac{1}{3}(\pi r^2)h$	**Base area equals πr^2.**
$= \frac{1}{3}(\pi 1.5^2)(4)$	**Substitute.**
$= 3\pi$	**Simplify.**

▶ So, the volume of the cone is 3π, or about 9.42 cubic inches.

EXAMPLE 3 *Using the Volume of a Cone*

Use the given measurements to solve for x.



Volume = 2614 ft^3

SOLUTION

$V = \frac{1}{3}\pi r^2 h$	**Formula for volume**
$2614 = \frac{1}{3}(\pi x^2)(13)$	**Substitute.**
$7842 = 13\pi x^2$	**Multiply each side by 3.**
$192 \approx x^2$	**Divide each side by 13π.**
$13.86 \approx x$	**Find positive square root.**

▶ So, the radius of the cone is about 13.86 feet.

STUDENT HELP

Study Tip
To eliminate the fraction in an equation, you can multiply each side by the reciprocal of the fraction. This was done in Example 3.

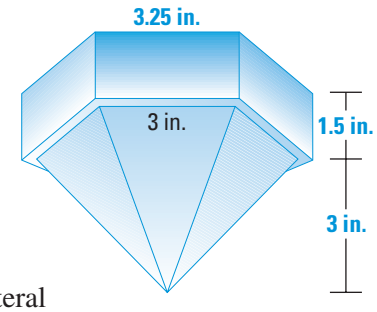
FOCUS ON APPLICATIONS

NAUTICAL PRISMS Before electricity, nautical prisms were placed in the decks of sailing ships. By placing the hexagonal face flush with the deck, the prisms would draw light to the lower regions of the ship.

GOAL 2 USING VOLUME IN REAL-LIFE PROBLEMS

EXAMPLE 4 *Finding the Volume of a Solid*

NAUTICAL PRISMS A nautical prism is a solid piece of glass, as shown. Find its volume.



SOLUTION

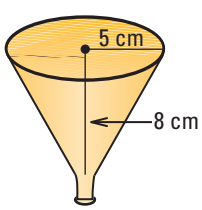
To find the volume of the entire solid, add the volumes of the prism and the pyramid. The bases of the prism and the pyramid are regular hexagons made up of six equilateral triangles. To find the area of each base, B, multiply the area of one of the equilateral triangles by 6, or $6\left(\frac{\sqrt{3}}{4}s^2\right)$, where s is the base edge.

Volume of prism	$= 6\left(\frac{\sqrt{3}}{4}s^2\right)h$	**Formula for volume of prism**
	$= 6\left(\frac{\sqrt{3}}{4}(3.25)^2\right)(1.5)$	**Substitute.**
	≈ 41.16	**Use a calculator.**
Volume of pyramid	$= \frac{1}{3} \cdot 6\left(\frac{\sqrt{3}}{4}s^2\right)h$	**Formula for volume of pyramid**
	$= \frac{1}{3} \cdot 6\left(\frac{\sqrt{3}}{4} \cdot 3^2\right)(3)$	**Substitute.**
	≈ 23.38	**Use a calculator.**

▶ The volume of the nautical prism is **41.16** + **23.38** or 64.54 cubic inches.

EXAMPLE 5 *Using the Volume of a Cone*

AUTOMOBILES If oil is being poured into the funnel at a rate of 147 milliliters per second and flows out of the funnel at a rate of 42 milliliters per second, estimate the time it will take for the funnel to overflow. $(1 \text{ mL} = 1 \text{ cm}^3)$



SOLUTION

First, find the approximate volume of the funnel.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(5^2)(8) \approx 209 \text{ cm}^3 = 209 \text{ mL}$$

The rate of accumulation of oil in the funnel is $147 - 42 = 105$ mL/s. To find the time it will take for the oil to fill the funnel, divide the volume of the funnel by the rate of accumulation of oil in the funnel as follows:

$$209 \text{ mL} \div \frac{105 \text{ mL}}{1 \text{ s}} = 209 \cancel{\text{mL}} \times \frac{1 \text{ s}}{105 \cancel{\text{mL}}} \approx 2 \text{ s}$$

▶ The funnel will overflow after about 2 seconds.

GUIDED PRACTICE

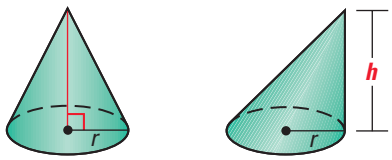
Vocabulary Check ✓

1. The volume of a cone with radius r and height h is $\frac{1}{3}$ the volume of a __?__ with radius r and height h.

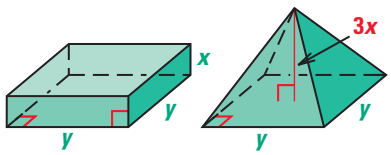
Concept Check ✓

Do the two solids have the same volume? Explain your answer.

2.



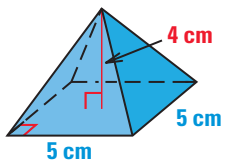
3.



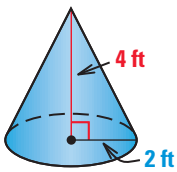
Skill Check ✓

In Exercises 4–6, find (a) the area of the base of the solid and (b) the volume of the solid.

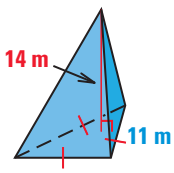
4.



5.



6.



7. **CRITICAL THINKING** You are given the radius and the slant height of a right cone. Explain how you can find the height of the cone.

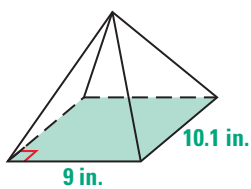
PRACTICE AND APPLICATIONS

STUDENT HELP

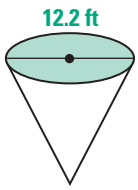
Extra Practice to help you master skills is on p. 826.

FINDING BASE AREAS **Find the area of the base of the solid.**

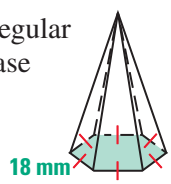
8.



9.

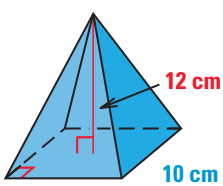


10. Regular base

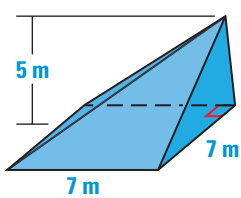


VOLUME OF A PYRAMID **Find the volume of the pyramid. Each pyramid has a regular polygon for a base.**

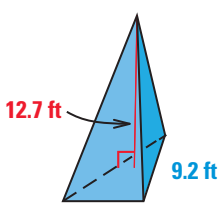
11.



12.



13.



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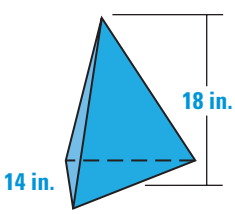
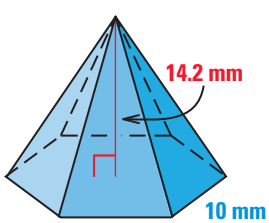


15.



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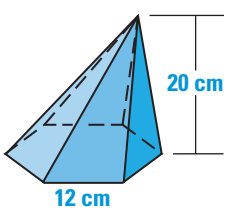


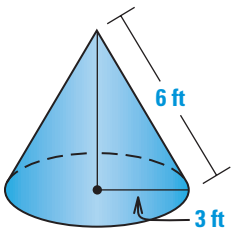
STUDENT HELP

HOMEWORK HELP

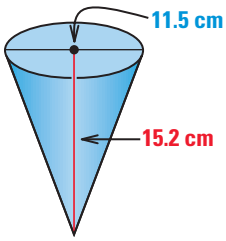
Example 1: Exs. 11–16
Example 2: Exs. 17–19
Example 3: Exs. 20–22
Example 4: Exs. 23–28
Example 5: Ex. 29

VOLUME OF A CONE **Find the volume of the cone. Round your result to two decimal places.**

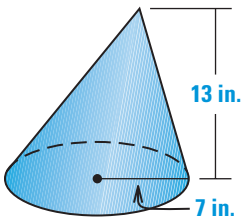
17.



18.

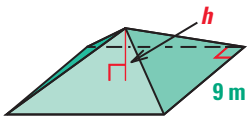


19.

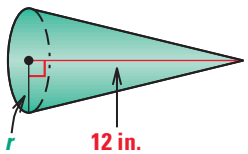


USING ALGEBRA **Solve for the variable using the given information.**

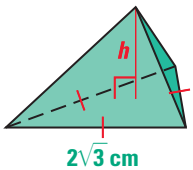
20. Volume $= 270\ \text{m}^3$



21. Volume $= 100\pi\ \text{in.}^3$



22. Volume $= 5\sqrt{3}\ \text{cm}^3$

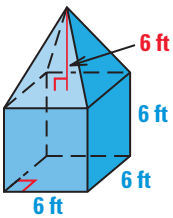


STUDENT HELP

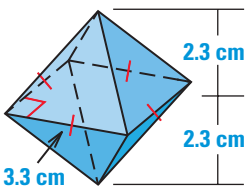
HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 23–25.

COMPOSITE SOLIDS **Find the volume of the solid. The prisms, pyramids, and cones are right. Round the result to two decimal places.**

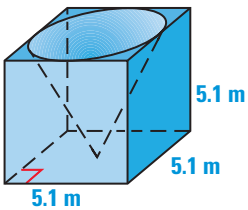
23.



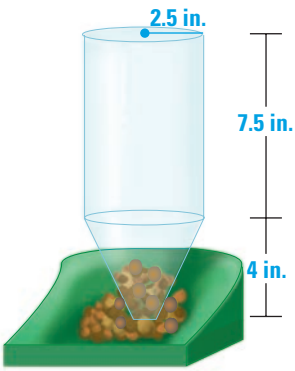
24.



25.



AUTOMATIC FEEDER **In Exercises 26 and 27, use the diagram of the automatic pet feeder. (1 cup = 14.4 in.^{3})**



26. Calculate the amount of food that can be placed in the feeder.

27. If a cat eats half of a cup of food, twice per day, will the feeder hold enough food for three days?

28. **ANCIENT CONSTRUCTION** Early civilizations in the Andes Mountains in Peru used cone-shaped adobe bricks to build homes. Find the volume of an adobe brick with a diameter of 8.3 centimeters and a slant height of 10.1 centimeters. Then calculate the amount of space 27 of these bricks would occupy in a mud mortar wall.

29. **SCIENCE CONNECTION** During a chemistry lab, you use a funnel to pour a solvent into a flask. The radius of the funnel is 5 centimeters and its height is 10 centimeters. If the solvent is being poured into the funnel at a rate of 80 milliliters per second and the solvent flows out of the funnel at a rate of 65 milliliters per second, how long will it be before the funnel overflows? $(1\ \text{mL} = 1\ \text{cm}^3)$

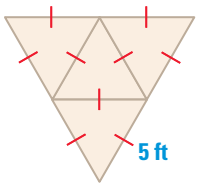
FOCUS ON CAREERS

VOLCANOLOGY Volcanologists collect and interpret data about volcanoes to help them predict when a volcano will erupt.

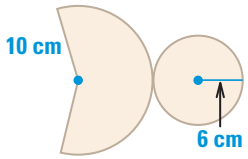
CAREER LINK
www.mcdougallittell.com

USING NETS **In Exercises 30–32, use the net to sketch the solid. Then find the volume of the solid. Round the result to two decimal places.**

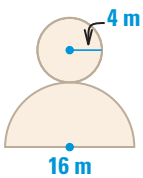
30.



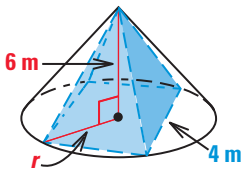
31.



32.



33. FINDING VOLUME In the diagram at the right, a regular square pyramid with a base edge of 4 meters is inscribed in a cone with a height of 6 meters. Use the dimensions of the pyramid to find the volume of the cone.



34. VOLCANOES Before 1980, Mount St. Helens was cone shaped with a height of about 1.83 miles and a base radius of about 3 miles. In 1980, Mount St. Helens erupted. The tip of the cone was destroyed, as shown, reducing the volume by 0.043 cubic mile. The cone-shaped tip that was destroyed had a radius of about 0.4 mile. How tall is the volcano today? (*Hint:* Find the height of the destroyed cone-shaped tip.)

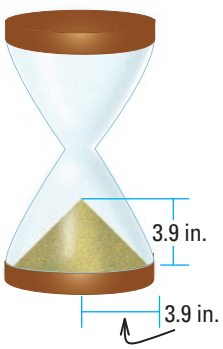
Test Preparation

MULTI-STEP PROBLEM **Use the diagram of the hourglass below.**

35. Find the volume of the cone-shaped pile of sand.

36. The sand falls through the opening at a rate of one cubic inch per minute. Is the hourglass a true "hour"-glass? Explain. (1 hr = 60 min)

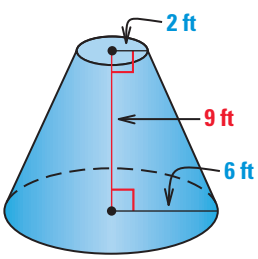
37. *Writing* The sand in the hourglass falls into a conical shape with a one-to-one ratio between the radius and the height. Without doing the calculations, explain how to find the radius and height of the pile of sand that has accumulated after 30 minutes.



★ Challenge

FRUSTUMS **A *frustum* of a cone is the part of the cone that lies between the base and a plane parallel to the base, as shown. Use the information below to complete Exercises 38 and 39.**

One method for calculating the volume of a frustum is to add the areas of the two bases to their geometric mean, then multiply the result by $\frac{1}{3}$ the height.



38. Use the measurements in the diagram to calculate the volume of the frustum.

39. Write a formula for the volume of a frustum that has bases with radii r_1 and r_2 and a height h.

STUDENT HELP

Look Back
For help with finding geometric means, see p. 466.

MIXED REVIEW

FINDING ANGLE MEASURES **Find the measure of each interior and exterior angle of a regular polygon with the given number of sides.** (Review 11.1)

40. 9 **41.** 10 **42.** 19

43. 22 **44.** 25 **45.** 30

FINDING THE AREA OF A CIRCLE **Find the area of the described circle.** (Review 11.5 for 12.6)

46. The diameter of the circle is 25 inches.

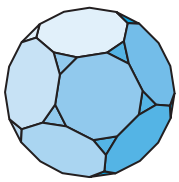
47. The radius of the circle is 16.3 centimeters.

48. The circumference of the circle is 48π feet.

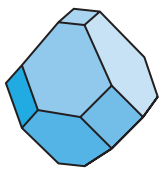
49. The length of a 36° arc of the circle is 2π meters.

USING EULER'S THEOREM **Calculate the number of vertices of the solid using the given information.** (Review 12.1)

50. 32 faces; 12 octagons and 20 triangles



51. 14 faces; 6 squares and 8 hexagons

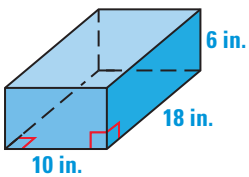


QUIZ 2

Self-Test for Lessons 12.4 and 12.5

In Exercises 1–6, find the volume of the solid. (Lessons 12.4 and 12.5)

1.



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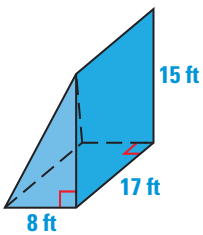

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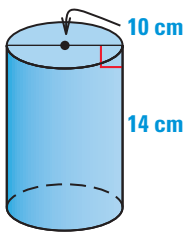

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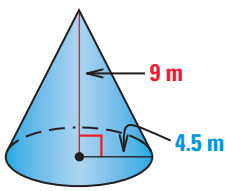

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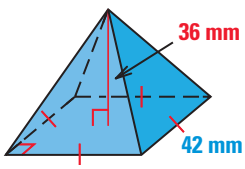

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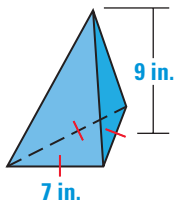
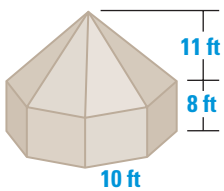


7. **STORAGE BUILDING** A road-salt storage building is composed of a regular octagonal pyramid and a regular octagonal prism as shown. Find the volume of salt that the building can hold. (Lesson 12.5)



12.6 Surface Area and Volume of Spheres

What you should learn

GOAL 1 Find the surface area of a sphere.

GOAL 2 Find the volume of a sphere in **real life**, such as the ball bearing in **Example 4**.

Why you should learn it

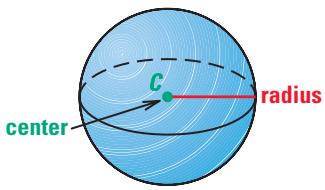
▼ You can find the surface area and volume of **real-life** spherical objects, such as the planets and moons in **Exs. 18 and 19**.

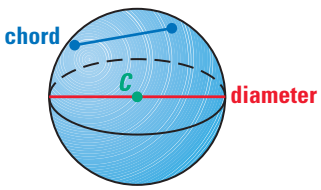
REAL LIFE

Saturn

GOAL 1 FINDING THE SURFACE AREA OF A SPHERE

In Lesson 10.7, a circle was described as the locus of points in a plane that are a given distance from a point. A **sphere** is the locus of points in *space* that are a given distance from a point. The point is called the **center of the sphere**. A **radius of a sphere** is a segment from the center to a point on the sphere.





A **chord of a sphere** is a segment whose endpoints are on the sphere. A **diameter** is a chord that contains the center. As with circles, the terms radius and diameter also represent distances, and the diameter is twice the radius.

THEOREM

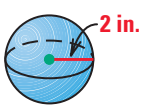
THEOREM 12.11 ***Surface Area of a Sphere***

The surface area S of a sphere with radius r is $S = 4\pi r^2$.

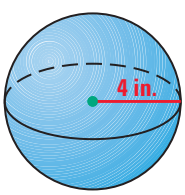
EXAMPLE 1 *Finding the Surface Area of a Sphere*

Find the surface area. When the radius doubles, does the surface area double?

a.



b.



SOLUTION

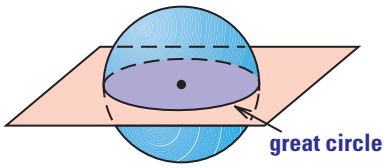
a. $S = 4\pi r^2 = 4\pi(2)^2 = 16\pi$ in.^{2}

b. $S = 4\pi r^2 = 4\pi(4)^2 = 64\pi$ in.^{2}

The surface area of the sphere in part (b) is four times greater than the surface area of the sphere in part (a) because $16\pi \cdot 4 = 64\pi$.

▶ So, when the radius of a sphere doubles, the surface area does *not* double.

If a plane intersects a sphere, the intersection is either a single point or a circle. If the plane contains the center of the sphere, then the intersection is a **great circle** of the sphere. Every great circle of a sphere separates a sphere into two congruent halves called **hemispheres**.

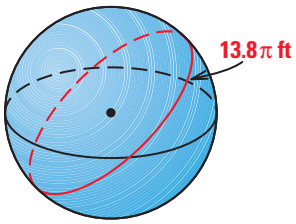


EXAMPLE 2 *Using a Great Circle*

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

The circumference of a great circle of a sphere is 13.8π feet. What is the surface area of the sphere?



SOLUTION

Begin by finding the radius of the sphere.

$C = 2\pi r$	**Formula for circumference of circle**
$13.8\pi = 2\pi r$	**Substitute 13.8π for C.**
$6.9 = r$	**Divide each side by 2π.**

Using a radius of 6.9 feet, the surface area is

$$S = 4\pi r^2 = 4\pi(6.9)^2 = 190.44\pi \text{ ft}^2.$$

▶ So, the surface area of the sphere is 190.44π, or about 598 ft^2.

EXAMPLE 3 *Finding the Surface Area of a Sphere*

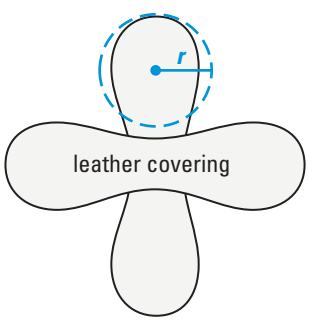
REAL LIFE

BASEBALL A baseball and its leather covering are shown. The baseball has a radius of about 1.45 inches.

a. Estimate the amount of leather used to cover the baseball.

b. The surface of a baseball is sewn from two congruent shapes, each of which resembles two joined circles. How does this relate to the formula for the surface area of a sphere?

$r = 1.45$ in.



SOLUTION

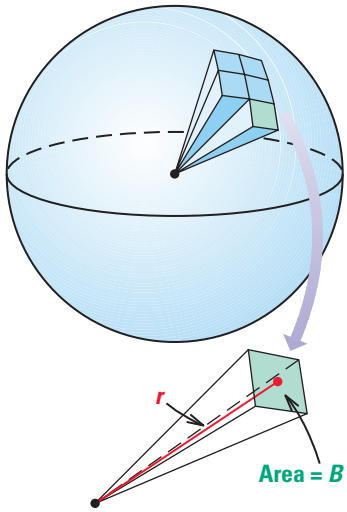
a. Because the radius r is about 1.45 inches, the surface area is as follows:

$S = 4\pi r^2$	**Formula for surface area of sphere**
$\approx 4\pi(1.45)^2$	**Substitute 1.45 for r.**
$\approx 26.4 \text{ in.}^2$	**Use a calculator.**

b. Because the covering has two pieces, each resembling two joined circles, then the entire covering consists of four circles with radius r. The area of a circle of radius r is $A = \pi r^2$. So, the area of the covering can be approximated by $4\pi r^2$. This is the same as the formula for the surface area of a sphere.

GOAL 2 FINDING THE VOLUME OF A SPHERE

Imagine that the interior of a sphere with radius r is approximated by n pyramids, each with a base area of B and a height of r, as shown. The volume of each pyramid is $\frac{1}{3}Br$ and the sum of the base areas is nB. The surface area of the sphere is approximately equal to nB, or $4\pi r^2$. So, you can approximate the volume V of the sphere as follows.



$$V \approx n\frac{1}{3}Br \qquad \text{Each pyramid has a volume of } \tfrac{1}{3}Br.$$

$$= \frac{1}{3}(nB)r \qquad \text{Regroup factors.}$$

$$\approx \frac{1}{3}(4\pi r^2)r \qquad \text{Substitute } 4\pi r^2 \text{ for } nB.$$

$$= \frac{4}{3}\pi r^3 \qquad \text{Simplify.}$$

STUDENT HELP

Study Tip
If you understand how a formula is derived, then it will be easier for you to remember the formula.

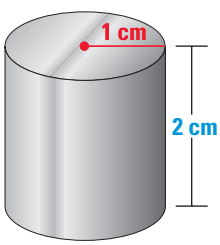
THEOREM

THEOREM 12.12 ***Volume of a Sphere***

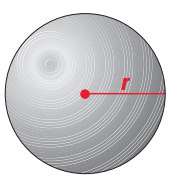
The volume V of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.

EXAMPLE 4 *Finding the Volume of a Sphere*

BALL BEARINGS To make a steel ball bearing, a cylindrical *slug* is heated and pressed into a spherical shape with the same volume. Find the radius of the ball bearing below.



slug



ball bearing

SOLUTION

To find the volume of the slug, use the formula for the volume of a cylinder.

$$V = \pi r^2 h = \pi(1^2)(2) = 2\pi \text{ cm}^3$$

To find the radius of the ball bearing, use the formula for the volume of a sphere and solve for r.

$$V = \frac{4}{3}\pi r^3 \qquad \text{Formula for volume of sphere}$$

$$2\pi = \frac{4}{3}\pi r^3 \qquad \text{Substitute } 2\pi \text{ for } V.$$

$$6\pi = 4\pi r^3 \qquad \text{Multiply each side by 3.}$$

$$1.5 = r^3 \qquad \text{Divide each side by } 4\pi.$$

$$1.14 \approx r \qquad \text{Use a calculator to take the cube root.}$$

▶ So, the radius of the ball bearing is about 1.14 centimeters.

FOCUS ON PEOPLE

IN-LINE SKATING Ball bearings help the wheels of an in-line skate turn smoothly. The two brothers above, Scott and Brennan Olson, pioneered the design of today's in-line skates.

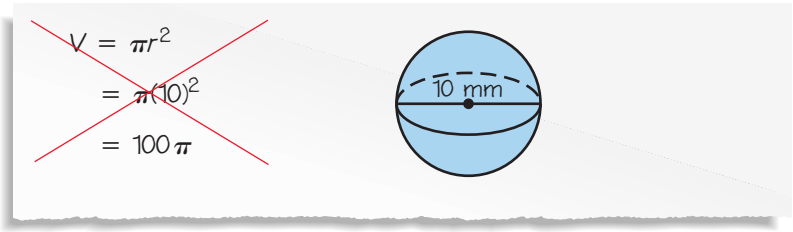
GUIDED PRACTICE

Vocabulary Check ✓

1. The locus of points in space that are __?__ from a __?__ is called a sphere.

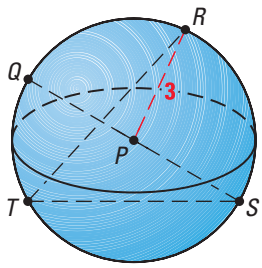
Concept Check ✓

2. **ERROR ANALYSIS** Melanie is asked to find the volume of a sphere with a diameter of 10 millimeters. Explain her error(s).



Skill Check ✓

In Exercises 3–8, use the diagram of the sphere, whose center is *P*.



3. Name a chord of the sphere.
4. Name a segment that is a radius of the sphere.
5. Name a segment that is a diameter of the sphere.
6. Find the circumference of the great circle that contains Q and S.
7. Find the surface area of the sphere.
8. Find the volume of the sphere.
9. **CHEMISTRY** A helium atom is approximately spherical with a radius of about 0.5×10^{-8} centimeter. What is the volume of a helium atom?

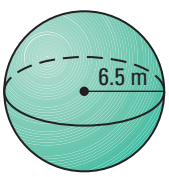
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 826.

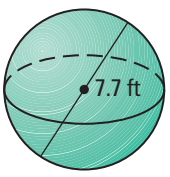
FINDING SURFACE AREA **Find the surface area of the sphere. Round your result to two decimal places.**

10.

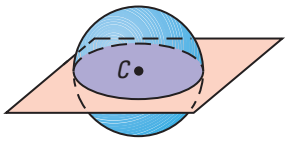


11.

12.



USING A GREAT CIRCLE **In Exercises 13–16, use the sphere below. The center of the sphere is *C* and its circumference is 7.4π inches.**



13. What is half of the sphere called?
14. Find the radius of the sphere.
15. Find the diameter of the sphere.
16. Find the surface area of half of the sphere.
17. **SPORTS** The diameter of a softball is 3.8 inches. Estimate the amount of leather used to cover the softball.

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 10–12
Example 2: Exs. 13–16
Example 3: Ex. 17
Example 4: Exs. 20–22, 41–43

FOCUS ON APPLICATIONS

PLANETS Jupiter is the largest planet in our solar system. It has a diameter of 88,730 miles, or 142,800 kilometers.

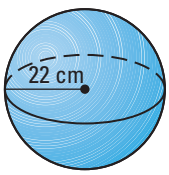
APPLICATION LINK
www.mcdougallittell.com

18. PLANETS The circumference of Earth at the equator (great circle of Earth) is 24,903 miles. The diameter of the moon is 2155 miles. Find the surface area of Earth and of the moon to the nearest hundred. How does the surface area of the moon compare to the surface area of Earth?

19. DATA COLLECTION Research to find the diameters of Neptune and its two moons, Triton and Nereid. Use the diameters to find the surface area of each.

FINDING VOLUME **Find the volume of the sphere. Round your result to two decimal places.**

20.



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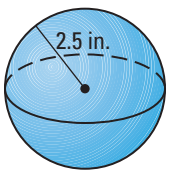

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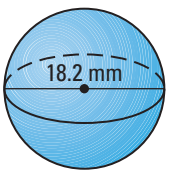
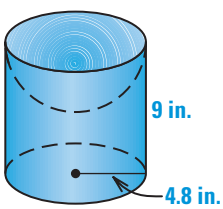


USING A TABLE **Copy and complete the table below. Leave your answers in terms of π.**

	Radius of sphere	Circumference of great circle	Surface area of sphere	Volume of sphere
23.	7 mm	?	?	?
24.	?	?	144π in.^{2}	?
25.	?	10π cm	?	?
26.	?	?	?	$\frac{4000\pi}{3}$ m^3

COMPOSITE SOLIDS **Find (a) the surface area of the solid and (b) the volume of the solid. The cylinders and cones are right. Round your results to two decimal places.**

27.



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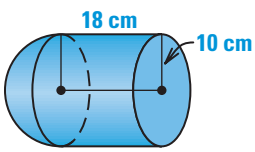

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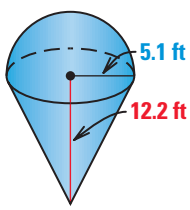


TECHNOLOGY **In Exercises 30–33, consider five spheres whose radii are 1 meter, 2 meters, 3 meters, 4 meters, and 5 meters.**

30. Find the volume and surface area of each sphere. Leave your results in terms of π.

31. Use your answers to Exercise 30 to find the ratio of the volume to the surface area, $\frac{V}{S}$, for each sphere.

32. Use a graphing calculator to plot the graph of $\frac{V}{S}$ as a function of the radius. What do you notice?

33. *Writing* If the radius of a sphere triples, does its surface area triple? Explain your reasoning.

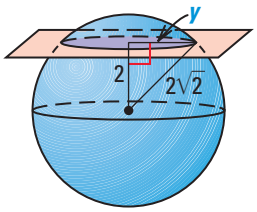
34. VISUAL THINKING A sphere with radius r is inscribed in a cylinder with height $2r$. Make a sketch and find the volume of the cylinder in terms of r.

STUDENT HELP

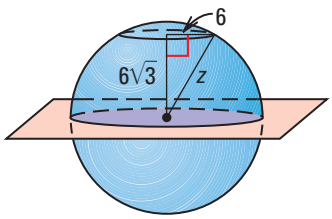
HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 35 and 36.

USING ALGEBRA In Exercises 35 and 36, solve for the variable. Then find the area of the intersection of the sphere and the plane.

35.

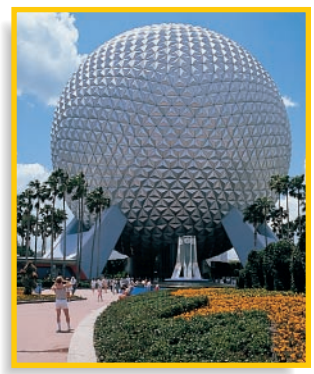


36.



37. CRITICAL THINKING Sketch the intersection of a sphere and a plane that does not pass through the center of the sphere. If you know the circumference of the circle formed by the intersection, can you find the surface area of the sphere? Explain.

SPHERES IN ARCHITECTURE The spherical building below has a diameter of 165 feet.



38. Find the surface area of a sphere with a diameter of 165 feet. Looking at the surface of the building, do you think its surface area is the same? Explain.

39. The surface of the building consists of 1000 (nonregular) triangular pyramids. If the lateral area of each pyramid is about 267.3 square feet, estimate the actual surface area of the building.

40. Estimate the volume of the building using the formula for the volume of a sphere.

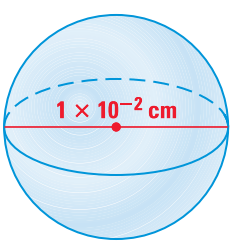
BALL BEARINGS In Exercises 41–43, refer to the description of how ball bearings are made in Example 4 on page 761.

41. Find the radius of a steel ball bearing made from a cylindrical slug with a radius of 3 centimeters and a height of 6 centimeters.

42. Find the radius of a steel ball bearing made from a cylindrical slug with a radius of 2.57 centimeters and a height of 4.8 centimeters.

43. If a steel ball bearing has a radius of 5 centimeters, and the radius of the cylindrical slug it was made from was 4 centimeters, then what was the height of the cylindrical slug?

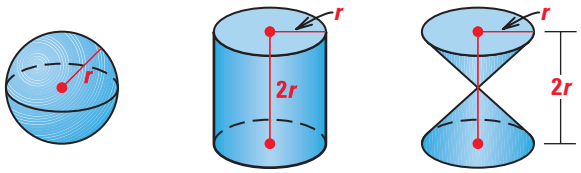
44. COMPOSITION OF ICE CREAM In making ice cream, a mix of solids, sugar, and water is frozen. Air bubbles are whipped into the mix as it freezes. The air bubbles are about 1×10^{-2} centimeter in diameter. If one quart, 946.34 cubic centimeters, of ice cream has about 1.446×10^{9} air bubbles, what percent of the ice cream is air? (*Hint:* Start by finding the volume of one air bubble.)



Air bubble

Test Preparation

MULTI-STEP PROBLEM **Use the solids below.**



45. Write an expression for the volume of the sphere in terms of r.

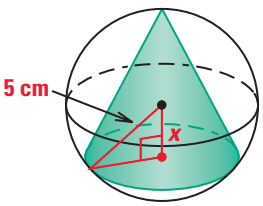
46. Write an expression for the volume of the cylinder in terms of r.

47. Write an expression for the volume of the solid composed of two cones in terms of r.

48. Compare the volumes of the cylinder and the cones to the volume of the sphere. What do you notice?

★ Challenge

49. A cone is inscribed in a sphere with a radius of 5 centimeters, as shown. The distance from the center of the sphere to the center of the base of the cone is x. Write an expression for the volume of the cone in terms of x. (*Hint:* Use the radius of the sphere as part of the height of the cone.)



EXTRA CHALLENGE
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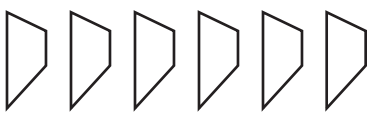
MIXED REVIEW

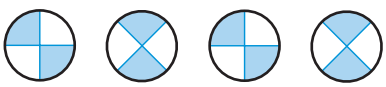
CLASSIFYING PATTERNS **Name the isometries that map the frieze pattern onto itself.** (Review 7.6)

50.

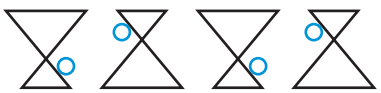


51.

52.

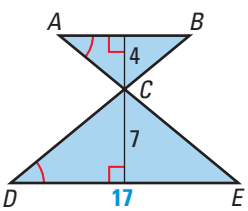


53.

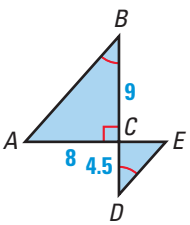


FINDING AREA **In Exercises 54–56, determine whether △*ABC* is similar to △*EDC*. If so, then find the area of △*ABC*.** (Review 8.4, 11.3 for 12.7)

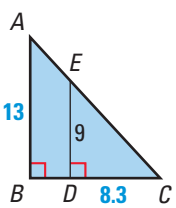
54.



55.



56.



57. MEASURING CIRCLES The tire at the right has an outside diameter of 26.5 inches. How many revolutions does the tire make when traveling 100 feet? (Review 11.4)